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A stochastic theory for deep bed filtration accounting for dispersion and size distributions

A.A. Shapiro [a],*, P.G. Bedrikovetsky [b]

[a] *Center for Phase Equilibria and Separation Processes IVC-SEP, Department of Chemical Engineering, Technical University of Denmark, DTU b. 229, DK 2800 Lyngby, Denmark*

[b] *Australian School of Petroleum, The University of Adelaide, SA 5005, Australia*

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ABSTRACT

We develop a stochastic theory for filtration of suspensions in porous media. The theory takes into account particle and pore size distributions, as well as the random character of the particle motion, which is described in the framework of the theory of continuous-time random walks (CTRW). In the limit of the infinitely many small walk steps we derive a system of governing equations for the evolution of the particle and pore size distributions. We consider the case of concentrated suspensions, where plugging the pores by particles may change porosity and other parameters of the porous medium. A procedure for averaging of the derived system of equations is developed for polydisperse suspensions with several distinctive particle sizes. A numerical method for solution of the flow equations is proposed. Sample calculations are applied to compare the roles of the particle size distribution and of the particle flight dispersion on the deposition profiles. It is demonstrated that the temporal flight dispersion is the most likely mechanism forming the experimentally observed hyperexponential character of the deposition profiles.

1. Introduction

Flows of suspensions in porous media are important for a large variety of applications [1,2] and therefore have been actively studied. The modeling studies may roughly be classified into studies on the micro-level, of the individual interactions of the particles and the pores (see review in Ref. [3] and analysis in Ref. [4]), and those on the macro-level. On the large scale, the suspension flow with particle deposition is usually modeled by the advection-dispersion equation with deposition [5,6] or its various modifications [7–9]. A few studies, based on diverse approaches, consider the interaction between the macro- and microscales [10–17].

A necessity for multiscale studies is caused, in particular, by the insufficiency of the standard model of filtration [5,6] to explain some experimental observations. The standard theory predicts that, under pulse injection of a tracer or a suspension in a porous medium, the maximum of the pulse will move with the velocity of the flow; the wash-out will be symmetric with regard to maximum; and the "tails" will vanish rather rapidly with the distance from the maximum. This contradicts a number of experimental observations [13,18,19]: in many cases, the forward tail of the distribution is much more massive than its backward part, and the maximum of the concentration moves much slower than the carrying liquid. Another prediction is related to the injection of a finite portion of the suspended particles followed by the pure liquid. The standard theory predicts that under such injection the deposition profile will be exponential. Meanwhile, hyperexponential profiles

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* Corresponding author. Tel.: +45 4525 2881; fax: +45 4588 2258.
E-mail addresses: ash@kt.dtu.dk (A.A. Shapiro), pavel@asp.adelaide.edu.au (P.G. Bedrikovetsky).

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are often observed: much more particles than expected stay close to the inlet of a porous medium, and simultaneously many particles propagate further than expected [3,13,20].

Modeling of deep bed filtration as a stochastic process has been applied in a number of different ways [13,14,16,18–22]. Randomness is mainly related to the particle size and shape distribution; stochastic heterogeneity of the pore space and surface; and stochastic structure of the flow. Stochastic pore geometry and particle size distributions were first explored in the pore network or percolation models [10,16]. An approach based on the population balance and Boltzmann-type equations was developed in Refs. [23,24]. The random character of the flow on the micro-level was reproduced in the multiscale numerical approach, where a random particle transfer equation was solved on the pore level, and then upscaled to the macro-level [11,12].

Recently a new approach to the tracer and suspension flows in porous media emerged in the framework of the Continuous Time Random Walk (CTRW) theory [18,25]. The CTRW has sufficed to explain many of the unusual experimental observations [18,19]. In our works [26,27] we have shown that, in the limit of infinitely many small time and spatial steps, where the particle flight becomes small compared to the macroscopic scale of the problem, the CTRW approach is reduced to an elliptic transport equation. Compared to the standard model, this equation contains an additional term $D_t\partial^2 c/\partial t^2$ (where c is the particle concentration), describing dispersion of the particle time step, just like the standard dispersion term $D\partial^2 c/\partial x^2$ describes the spatial dispersion of the step. It was shown [27] that the solutions of the elliptic equation exhibit behavior similar to that previously predicted by the CTRW and are able to reproduce experimental observations about pulse injection, at least, qualitatively.

To the best of our knowledge, the models developed in the framework of the CTRW have not taken into account the particle size distribution. Meanwhile, this is a factor of a considerable practical importance (i.e., particles of different sizes are filtered in different ways, and the smaller particles may penetrate deeper into a porous medium). It was shown that the particle size distribution may result in hyperexponential deposition profiles [13,20].

In the present paper we develop a complete stochastic theory of filtration, accounting for the dispersion of the particle flight, as well as the particle and pore size distributions. We consider the case of moderately concentrated suspensions, where the particles filter independently (which is not always the case, cf, Ref. [16]), but their deposition affects porosity and permeability. A rigorous derivation of the governing equations for the particle and pore size distributions is based on the theory of stochastic Markov semigroups [28]. It is shown that the system of governing equations allows for averaging and reduction to a system of elliptic equations for concentrations of the particles of different sizes coupled with ordinary differential equations describing the pore plugging. A numerical method for solution of the obtained equations is developed and tested on a characteristic example. Comparison of the contributions of the polydispersity of the suspension and flow dispersion to hyperexponentiality of the deposition profiles is presented.

In the present paper, we do not provide a detailed treatment of the physics of flow on the microlevel and particle retention. Our goal is to develop a consistent way for transfer from the micro- to macro-level, which would make it possible to obtain a macroscopic system of equations for many different microscale transport mechanisms. That is why every time where we discuss the flow on the level of separate particles and pores we select the simplest possible assumptions. In particular, the only considered mechanism of particle deposition is size exclusion. Extensions and generalizations onto more complex particle–pore interaction mechanisms are subjects for separate work.

The paper is organized as follows. Section 2 presents the basic stochastic characteristics of the suspension and of the porous medium. In Section 3 the governing equations for particle and pore size distributions are derived. A reader, who is not interested in details of the derivations, may only refer to Sections 3.5, 3.6, where the results are summarized. Section 4 describes a procedure for averaging of the system of governing equations, first, for a monodisperse (similar to Ref. [24]), and, further, for a polydisperse suspension. Section 5 discusses a characteristic example of the bidisperse suspension in a monodisperse porous medium. A system of governing equations for such a system is strongly simplified and is brought to a dimensionless form. Characteristic dimensionless parameters governing the different effects are analyzed. A numerical method for solution of the resulting system is developed. Finally, sample calculations are presented, comparing contributions of polydispersity and temporal dispersion to the deposition profiles. The results are summarized in Conclusions, and the auxiliary and cumbersome details of the computations are reported in the Appendices.

2. Basic relations and definitions

2.1. Particle and pore size distributions

We consider deep bed filtration of a suspension of particles in a carrying incompressible liquid flowing in direction x. The particles are characterized by the different radii r_s. In the classical filtration theory [5,6] the equations are derived with regard to the particle concentration per unit porous volume, $c(x,t)$. A more detailed characteristic [23] is the distribution $C(r_s,x,t)$ of the particles by sizes r_s, and the corresponding distribution density $f(r_s,x,t)$. By definition,

$$c(x,t)=\int C(r_s,x,t)\mathrm{d}r_s;\qquad f(r_s,x,t)=C(r_s,x,t)/c(x,t). \tag{1}$$

The particles move in a porous medium, which may be visualized as a multigraph [29]: an entangled network of pores/capillaries distributed by effective radii r_p and lengths l. The distribution of the pores by radii r_p was considered in

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Ref. [23], while distribution by lengths was introduced in Ref. [27] in connection to the particle flight. Here we consider the combined distribution $H(l, r_p, x, t)$. This distribution, as well as all the other distributions in this paper, is considered for a representative volume containing a large number of pores. It varies with time, affected by the particle deposition. The total number of pores $h(x, t)$ and the density $f(l, r_p, x, t)$ are defined as

$$h(x,t) = \int H(l, r_p, x, t)\mathrm{d}l\mathrm{d}r_p; \qquad f(l, r_p, x, t) = H(l, r_p, x, t)/h(x, t). \tag{2}$$

As discussed in Ref. [23], values of r_s and r_p may not only be understood as actual (or effective) sizes, but as any kind of geometric or physical parameters affecting the particle–pore interaction. For the case of geometrical capturing considered here, they may also be treated as parameters of the particle shape, if such shapes are irregular. On the contrary, by l we mean exactly actual travelling distances of the particles in the pores or, rather, their projections on the flow direction x.

The different conditional probability densities may be formed on the basis of the introduced distributions. We will denote by $f(A|B)$ the distribution density of A under condition B, where A, B are some sets of distributed parameters. For example,

$$f\left(l|r_p, x, t\right) = f(l, r_p, x, t)/f(r_p, x, t); \qquad f(r_p, x, t) = \int f(l, r_p, x, t)\mathrm{d}l. \tag{3}$$

As usual, the words "under condition B" mean "under known and fixed parameters B". Therefore, we understand the distributions at point x at the moment t as the distributions "under condition" x, t.

2.2. Particle flight distributions

We assume that the particles move in the porous space and are captured independently of each other. Motion of the particles in the separate pores is determined by the distribution of the particle flights. A particle of size r_s passes a pore of size r_p and length l during time interval τ. The particle may go through the pore or be captured in it; let p be the probability of passing the pore, while $1 - p$ is the probability of being captured. The distribution density of one particle flight is, thus, $f(l, p, \tau|r_p, r_s, x, t)$. It is easy to show that (cf. Eq. (3))

$$f(l, p, \tau|r_p, r_s, x, t) = f(p, \tau|r_s, l, r_p, x, t)f(l|r_p, x, t). \tag{4}$$

The distribution $f(l|r_p, x, t)$ is to be found, while $f(p, \tau|r_s, l, r_p, x, t)$ in many cases may be considered to be known in advance. Indeed, the bprobability of capturing and the time of flight are usually defined by the particle and pore sizes (or other physical parameters), as well as the local flow characteristics [3,6]. For example, in the case of no deposition, the distribution of local velocities in the pores u may be assumed to be proportional to the macroscale hydrodynamic interstitial velocity U/ϕ, where $U(x, t)$ is the superficial flow velocity of the carrying liquid, and ϕ is the porosity open for the particles (this is the simplest possible relation between the interstitial and superficial velocities, omitting effects like channeling). In other words, there is a universal distribution

$$f_U(\eta|r_s, l, r_p), \quad \eta = u\phi(x, t)/U(x, t).$$

The distribution of the flight times is then given by

$$f(\tau|r_s, l, r_p, x, t) = f_U(\phi l/U\tau|r_s, l, r_p)\frac{\phi l}{U\tau^2}. \tag{5}$$

In many cases (although not for the most general derivation) we will assume that the distribution (5) is also valid for flows with deposition.

Another significant simplification may be achieved by allowing a strong assumption that the capturing probability is independent of the flow kinetics, that is, distribution $f(p|r_s, l., r_p, \tau)$ is independent of τ. Combining with (5), we obtain:

$$f(p, \tau|r_s, l, r_p, x, t) = f_U(\phi l/U\tau|r_s, l, r_p)\frac{\phi l}{U\tau^2}f(p|r_s, l., r_p) \tag{6}$$

and, finally,

$$f(l, p, \tau|r_p, r_s, x, t) = f_U(\phi l/U\tau|r_s, l, r_p)\frac{\phi l}{U\tau^2}f(p|r_s, l., r_p)f(l|r_p, x, t). \tag{7}$$

Eq. (7) will be applied as a constituting dependence for $f(l, p, \tau|r_p, r_s, x, t)$ in all the examples, but not in the basic derivation.

We will also be interested in the distribution of the particle flights regardless pore sizes r_p. This distribution is given by

$$f(l, p, \tau|r_s, x, t) = \int f(l, p, \tau|r_s, r_p, x, t)f(r_p, x, t)\mathrm{d}r_p,$$

or, taking in account Eq. (4),

$$f(l, p, \tau|r_s, x, t) = \int f(p, \tau|r_s, l, r_p, x, t)f(l, r_p, x, t)\mathrm{d}r_p. \tag{8}$$

Again, the first distribution under the integral in many cases may be assumed to be known a priori, while the second is to be found.

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2.3. Integral characteristics of the particle flight

We will need the following first and second moments of the particle flight:

$$\begin{aligned}
L_s(r_s, x, t) &= \int plf(l, p, \tau|r_s, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\tau \\
T_s(r_s, x, t) &= \int p\tau f(l, p, \tau|r_s, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\tau \\
L_{2s}(r_s, x, t) &= \int pl^2 f(l, p, \tau|r_s, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\tau \\
T_{2s}(r_s, x, t) &= \int p\tau^2 f(l, p, \tau|r_s, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\tau \\
L_{Ts}(r_s, x, t) &= \int pl\tau f(l, p, \tau|r_s, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\tau \\
P_s(r_s, x, t) &= \int pf(l, p, \tau|r_s, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\tau .
\end{aligned} \tag{9}$$

The last value, $P_s(r_s, x, t)$, is the overall probability for a particle to pass a pore without being captured. The values $L_s(r_s, x, t)$ and $T_s(r_s, x, t)$ are an average particle's travel distance and flight time, correspondingly. These values take into account only the particles that are not captured/deposited, which is reflected by the multiplier p. The remaining values in Eq. (9) are the different second moments of the particle flight.

We will apply a more detailed characteristic than $P_s(r_s, x, t)$: a probability for a particular particle to pass through a pore of a specific size:

$$P(r_s, l, r_p, x, t) = \int pf(p, \tau|r_s, l, r_p, x, t)\mathrm{d}p\mathrm{d}\tau . \tag{10}$$

As follows from Eqs. (4), (8)

$$P_s(r_s, x, t) = \int P(r_s, l, r_p, x, t)f(l, r_p, x, t)\mathrm{d}r_p\mathrm{d}l. \tag{11}$$

The particle flight distribution $f(l, p, \tau|r_s, x, t)$, introduced in the previous subsection, is associated with operator **S** describing one "step" of a particle: transformation of a distribution of the particles when all of them advance on one pore. For an arbitrary distribution $g(r_s, x, t)$, by definition,

$$\mathbf{S}g = \int pf(l, p, \tau|r_s, x, t)g(r_s, x - l, t - \tau)\mathrm{d}l\mathrm{d}\tau\mathrm{d}p. \tag{12}$$

Such a defined operator **S** is linear and depends on r_s, x and t.

We will consider motion of the particles in the limit of infinitely many infinitely small steps, each step corresponding to passage of one pore (when the macroscopic scale much exceeds the pore length). Similarly to Refs. [26,27] (see also the derivation below), it may be concluded that in this limit motion of the particles is described by the operator

$$\mathbf{D} = \lim \frac{\mathbf{1} - \mathbf{S}}{T_s(r_s, x, t)}. \tag{13}$$

In order to find this operator, it is necessary to expand function $g(r_s, x - l, t - \tau)$ in Eq. (12) up to the second-order terms. Cutting the expansion exactly on the second, but not higher or lower order terms is substantiated in Refs. [26,27], by application of the theory of continuous stochastic semigroups. As proven in Ref. [28], such semigroups describe random processes with continuous trajectories only when the infinitesimal semigroup operator **D** is a second order differential operator.

Substituting expansion

$$g(r_s, x - l, t - \tau) \approx (1 - p)g(r_s, x, t) + pg(r_s, x, t) - l\frac{\partial g}{\partial x} - \tau\frac{\partial g}{\partial t} + \frac{l^2}{2}\frac{\partial^2 g}{\partial x^2} + l\tau\frac{\partial^2 g}{\partial x \partial t} + \frac{\tau^2}{2}\frac{\partial^2 g}{\partial t^2}$$

into Eq. (12), and integrating, we obtain

$$\mathbf{D}(r_s, x, t)g = \frac{\partial g}{\partial t} + \upsilon_s\frac{\partial g}{\partial x} - D_{xs}\frac{\partial^2 g}{\partial x^2} - D_{xts}\frac{\partial^2 g}{\partial x \partial t} - D_{ts}\frac{\partial^2 g}{\partial t^2} + \Lambda_s g(r_s, x, t) \tag{14}$$

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with

$$\begin{aligned} &\upsilon_s = \lim \frac{L_s(r_s, x, t)}{T_s(r_s, x, t)}; \qquad D_{xs} = \lim \frac{L_{2s}(r_s, x, t)}{T_s(r_s, x, t)}; \qquad D_{xts} = \lim \frac{L_{Ts}(r_s, x, t)}{T_s(r_s, x, t)}; \\ &D_{ts} = \lim \frac{T_{2s}(r_s, x, t)}{T_s(r_s, x, t)}; \qquad \Lambda_s = \lim \frac{1 - P_s(r_s, x, t)}{T_s(r_s, x, t)}. \end{aligned} \tag{15}$$

Here υ_s is an average rate of particle transfer; D_{is} $(i = x; xt; t)$ are the different dispersion coefficients; Λ_s is the capture coefficient showing how many particles are captured per unit time. These coefficients are similar to those introduced in Ref. [27] for monodisperse suspensions, and discussion of their properties, as well as expressions for them in some model cases may be applied without changes. Transfer to the limit in Eq. (15) will be omitted, assuming that the values of l, τ or L_s, T_s, etc. are "small enough".

As will be shown below, operator **D** defines a governing differential equation for distribution of the particles $C(r_s, x, t)$. As follows from Eqs. (8), (9), the coefficients in this operator depend implicitly on the distribution $H(l, r_p, x, t)$. An equation for the last distribution, also derived below, requires an additional limiting relation for the frequency $\Lambda(r_s, l, r_p, x, t)$ of capturing specific particles by specific pores. We define this coefficient in terms of the probability $P(r_s, l, r_p, x, t)$ (see Eq. (10)):

$$\Lambda(r_s, l, r_p, x, t) = \lim \frac{1 - P(r_s, l, r_p, x, t)}{T_s(r_s, x, t)}. \tag{16}$$

3. Derivation of the governing system of equations

3.1. Assumptions

The goal of this section is to derive a system of the two equations determining the particle and the pore size distributions. The derivation is based on the theory of continuous time random walks (CTRW) [25,28]. It follows the pattern developed in Refs. [26,27] and is to some extent similar to derivations presented in Ref. [18]. However, there is an important advance, in that we take into account both the distribution of the particles and the varying distribution of the pore sizes. That is why we present the derivation in detail. The reader who is not interested in details of the derivation can proceed from Section 3.5.

In order to simplify the derivation, we assume that a particle moves by jumps, staying a certain time τ at the entrance of a pore and then jumping to the entrance of the next pore. The particle may be captured during the stay. A more realistic representation of the particle motion would only introduce corrections of the order of a pore.

We consider the case of capture by size exclusion: each captured particle totally closes the pore where it is captured, and this pore is excluded from the flow. The capture is irreversible. A more general case, where a captured particle does not always totally close the pore, but just decreases its size, was considered in Ref. [23]. Similar considerations may be applied in the theory developed here, by introducing a probability $p(r_s \to r_s', r_p)$ that a pore of a size r_s is reduced to r_s' after capturing a particle of the size r_p. We do not consider incomplete capturing here for the reasons discussed in the Introduction.

3.2. Integral equation for the particle size distribution

The goal of this and of the next subsection is to demonstrate that the particle size distribution $C(r_s, x, t)$ obeys the equation $\mathbf{D}C = 0$, where the differential operator **D** is the operator determined by Eqs. (12), (14). The derivation, similar to Refs. [26,27], consists in expressing $C(r_s, x, t)$ in terms of the particle step operator, and further application of the operator $(\mathbf{1} - \mathbf{S})/T_s$, which in the limit of infinitely small steps gives operator **D**.

Denote the initial distribution of the particles at $t = 0$ by $C_0(r_s, x, t) = C_0(r_s, x)\delta(t)$, where $\delta(t)$ is the Dirac delta-function. The distribution of the particles after n steps is obtained by n-wise application of operator **S** to the initial distribution:

$$C_n(r_s, x, t) = [\mathbf{S}^n C_0](r_s, x, t).$$

The particles can reach a fixed point x by different numbers of steps n at a moment $t' < t$ and stay there until the moment t. More precisely,

$$\begin{aligned} C(r_s, x, t) = \sum_n \int_{t'} &\{\text{The particle came to point } x \text{ at time } t' < t \text{ after } n \text{ steps}\} \\ &\times \{\text{the particle is still there and is not captured}\} \end{aligned}$$

or

$$C(r_s, x, t) = \sum_n \int_0^\infty [\mathbf{S}^n C_0](r_s, x, t') B(t - t' | r_s, x, t') \mathrm{d}t'. \tag{17}$$

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Here, we have introduced the designation $B(t-t'|r_s,x,t')$ for the probability that the particle is still there and is not captured, which is expressed in the following way. By definition, $\int pf(l,\tau,p|r_s,x,t')\mathrm{d}l\mathrm{d}p$ is a probability density that a particle stays at point x during the time interval τ after time t' and is not captured. Correspondingly,

$$B(t-t'|r_s,x,t')=\int_{t-t'}^{\infty}\int pf(l,p,\tau|r_s,x,t')\mathrm{d}l\mathrm{d}p\mathrm{d}\tau.$$

We represent B as a difference

$$B(t-t'|r_s,x,t')=\int_0^{\infty}\int pf(l,p,\tau|r_s,x,t')\mathrm{d}l\mathrm{d}p\mathrm{d}\tau-\int_0^{t-t'}\int pf(l,p,\tau|r_s,x,t')\mathrm{d}l\mathrm{d}p\mathrm{d}\tau.$$

According to Eq. (9), the first addendum is equal to $P_s(r_s,x,t')$. Similarly to Ref. [27], we further introduce

$$S_T(T|r_s,x,t)=\frac{1}{P_s(r_s,x,t)}\int_0^T\int pf(l,p,\tau|r_s,x,t)\mathrm{d}l\mathrm{d}p\mathrm{d}\tau. \tag{18}$$

The value of $S_T(T|r_s,x,t)$ is a cumulative distribution function of the time interval spent by a particle in a pore, provided that the particle is not captured. Applying definition (9) of $P_s(r_s,x,t)$, it is easy to show that

$$S_T(0|r_s,x,t)=0; \qquad S_T(\infty|r_s,x,t)=1.$$

In fact, unity is reached very fast, during a time of the order of particle flight.

Finally,

$$B(t-t'|r_s,x,t')=P_s(r_s,x,t')(1-S_T(t-t'|r_s,x,t')).$$

Substituting the last expression into Eq. (17),

$$C(r_s,x,t)=\sum_n\int_0^{\infty}[\mathbf{S}^nC_0](r_s,x,t')P_s(r_s,x,t')(1-S_T(t-t'|r_s,x,t'))\mathrm{d}t'. \tag{19}$$

This is the integral equation for the particle distribution C. Following Refs. [18,25], we introduce the distribution of the first achievement of point x:

$$W(r_s,x,t')=\sum_n[\mathbf{S}^nC_0](r_s,x,t'). \tag{20}$$

Then the equation for C is simplified:

$$C(r_s,x,t)=\int_0^{\infty}W(r_s,x,t')P_s(r_s,x,t')(1-S_T(t-t'|r_s,x,t'))\mathrm{d}t'. \tag{21}$$

3.3. *Differential equation for the particle size distribution*

In order to transfer from the integral equation (19) to a differential equation, we apply to both sides of this equation operator $(\mathbf{1}-\mathbf{S})/T_s(r_s,x,t)$. Operator $\mathbf{1}-\mathbf{S}$ is inverse to the sum $\sum_{n=1}^{\infty}\mathbf{S}^n$ and cancels this sum. As a result, we obtain:

$$\frac{1-\mathbf{S}}{T_s(r_s,x,t)}C(r_s,x,t)=\int_0^t\mathrm{d}t'C_0(r_s,x,t')\times P_s(r_s,x,t')\frac{1-S_T(t-t'|r_s,x,t')}{T_s(r_s,x,t)}.$$

Recalling that the initial distribution $C_0(r_s,x,t')$ is equal to $C_0(r_s,x)\delta(t')$, we simplify the right-hand side in the last equation:

$$\frac{1-\mathbf{S}}{T_s}C(r_s,x,t)=C_0(r_s,x)P_s(r_s,x,0)\frac{1-S_T(t|r_s,x,0)}{T_s}. \tag{22}$$

Now, let us consider the limit of infinitely many small steps. As discussed in Section 2.3 (see Eq. (13)), in this limit operator $(\mathbf{1}-\mathbf{S})/T_s$ tends to the differential operator $\mathbf{D}$ determined by Eq. (14). As was demonstrated in Ref. [27], expression $(1-S_T(t|r_s,x,0))/T_s$ on the right-hand side of Eq. (22) tends to $\delta(t)$.[1] Thus, Eq. (22) is reduced to

$$\mathbf{D}(r_s,x,t)C(r_s,x,t)=C_0(r_s,x)P_s(r_s,x,0)\delta(t). \tag{23}$$

In particular, for $t>0$ the right-hand side of the last equation is equal to zero, and it is reduced to (cf. Eq. (14))

$$\frac{\partial C}{\partial t}+v_s\frac{\partial C}{\partial x}=D_{xs}\frac{\partial^2 C}{\partial x^2}+D_{xts}\frac{\partial^2 C}{\partial x\partial t}+D_{ts}\frac{\partial^2 C}{\partial t^2}-\Lambda_sC(r_s,x,t). \tag{24}$$

Thus, we recover the equation particle transfer, the simplified version of which was discussed in Refs. [26,27].

The right-hand side of Eq. (23) should also be discussed. It represents the initial condition for the equation for concentration. While $C_0(r_s,x)$ is the initial particle distribution, $P_s(r_s,x,0)$ is the "entrance probability": a probability that the particles start entering the porous medium, but are not captured at the zeroth step.

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[1] If a sequence of random variables X_ε with the distribution functions $F_\varepsilon(x)$ tends to a distribution concentrated at zero at $\varepsilon\to 0$, then, in rather wide assumptions, $(1-F_\varepsilon(x))/E(X_\varepsilon)\to\delta(x)$ [24].

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3.4. Equation for the number of pores

The coefficients of Eq. (24) depend on the distribution $H(l, r_p, x, t)$ of free pores. The goal of this section is to derive a governing equation for $H(l, r_p, x, t)$, depending only on $C(r_s, x, t)$ and, thus, to obtain a closed system of equations for these two functions.

In the assumption that each deposited particle "kills" a pore, and this process is irreversible, the following balance equation holds:

$$H(l, r_p, x, t) = H(l, r_p, x, 0) - \sum_n \int_{t'} \{\text{a particle has come to point } x \text{ at some moment } t' < t \text{ after } n \text{ steps}\} \\ \times \{\text{the particle came exactly into the pore of a radius } r_p\} \times \{\text{the particle was captured there}\}.$$

The first multiplier under the integral sign is the same as in the integral equation for the particle motion (see Section 3.2). It is equal to $C_n(r_s, x, t') = [\mathbf{S}^n C_0](r_s, x, t')$. The second multiplier is equal to $f(l, r_p, x, t)$, by definition of this function. The third multiplier is $1 - P(r_s, l, r_p, x, t')$, since $P(r_s, l, r_p, x, t')$ is a probability that the particle of a size r_s passes a pore of (l, r_p). Finally, the expression for $H(l, r_p, x, t)$ assumes the form of

$$H(l, r_p, x, t) = H(l, r_p, x, 0) - \sum_n \int_0^t \mathrm{d}t' \int \mathrm{d}r_s [\mathbf{S}^n C_0](r_s, x, t') \times f(l, r_p, x, t')(1 - P(r_s, l, r_p, x, t')).$$

Recalling definition (20) of the function $W(r_s, x, t)$,

$$H(l, r_p, x, t) = H(l, r_p, x, 0) - \int_0^t \mathrm{d}t' \int \mathrm{d}r_s W(r_s, x, t') \times f(l, r_p, x, t')(1 - P(r_s, l, r_p, x, t')).$$

Differentiating by t,

$$\frac{\partial H(l, r_p, x, t)}{\partial t} = -f(l, r_p, x, t) \int \mathrm{d}r_s W(r_s, x, t)(1 - P(r_s, l, r_p, x, t)). \tag{25}$$

This equation, still, does not express $H(l, r_p, x, t)$ in terms of H and C. It is necessary to exclude function $W(r_s, x, t)$. This task is resolved by application of Eq. (21), at the limit of the infinitely many small steps. Divide Eq. (25) by $f(l, r_p, x, t)$:

$$\frac{1}{f(l, r_p, x, t)} \frac{\partial H(l, r_p, x, t)}{\partial t} = -\int \mathrm{d}r_s W(r_s, x, t)(1 - P(r_s, l, r_p, x, t)) \tag{26}$$

and apply to both its parts the integral operator $\int_0^t P_s(r_s, x, t') \frac{1 - S_T(t - t'|r_s, x, t')}{T_s(r_s, x, t')}(\circ)\mathrm{d}t'$. It should be noted that this operator tends to the unity operator in the limit of small steps. Indeed, probability $P_s(r_s, x, t')$ tends to unity, otherwise coefficient Λ_s in Eq. (15) would be infinite. Meanwhile $\frac{1 - S_T(t - t'|r_s, x, t')}{T_s(r_s, x, t')} \to \delta(t)$, as discussed above. Thus, the left-hand side of Eq. (26) in the limit of infinitely small steps remains unchanged, while the right-hand side is transformed to

$$-\int_0^t \int \left\{W(r_s, x, t')(1 - S_T(t - t'|r_s, x, t'))P_s(r_s, x, t')\right\} \frac{1 - P(r_s, l, r_p, x, t')}{T_s(r_s, x, t')} \mathrm{d}r_s \mathrm{d}t'. \tag{27}$$

In the last expression, the fraction $(1 - P(r_s, l, r_p, x, t'))/T_s(r_s, x, t')$ tends to $\Lambda(r_s, l, r_p, x, t')$ (see definition (16)). It is possible to substitute in it t' by t, since, due to multiplier $(1 - S_T(t - t'|r_s, x, t'))$, integration over $\mathrm{d}t'$ is carried out over the last time step. The remaining expression under the integral (27) (in braces) coincides with a similar expression in Eq. (21). Thus, in the limit of small steps we may rewrite Eq. (26) in the form of

$$\frac{\partial H(l, r_p, x, t)}{\partial t} = -f(l, r_p, x, t) \int \Lambda(r_s, l, r_p, x, t) C(r_s, x, t) \mathrm{d}r_s. \tag{28}$$

This is the desired equation for the pore size distribution.

3.5. Mass conservation law for the carrying fluid

Eqs. (24), (28) form a system of two equations for the two distributions $C(r_s, x, t)$, $H(l, r_p, x, t)$. Since the number of equations is equal to the number of unknowns, the system is formally closed. However, may implicitly depend on macroscopic characteristics of the flow, like the flow rate U and porosity ϕ (cf. Eq. (6)). This is not important for very dilute suspensions, where these values may be considered as invariable. However, if the deposition changes the parameters of the porous medium, the porosity and the flow velocity vary correspondingly. Let us consider these changes in more detail.

A natural assumption is incompressibility of the solid and of the liquid and additivity of their volumes. The incompressibility equation for liquid has the form of

$$\frac{\partial \phi(1 - C_v)}{\partial t} + \frac{\partial (1 - C_v)U}{\partial x} = -I_l. \tag{29}$$

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Here C_v is the overall volumetric concentration of the particles in the flow, I_l describes transfer of the liquid into an inactive state. Inactive liquid is contained in the pores, which are excluded from the flow by the captured particles, and forms a "dead porosity" ϕ_i [6]. We have assumed that an event of capture of a particle excludes a pore from the flow. The particle volume, $v_s(r_s)$, may be different from the volume of the pore, $v_p(l, r_p)$. For example, for a cylindrical pore and a spherical particle, $v_p(l, r_p) = \pi l r_p^2$ and $v_s(r_s) = 4\pi r_s^3/3$. The remaining volume (in the simplest assumptions, $v_p(l, r_p) - v_s(r_s)$) is occupied by liquid, which becomes separated from the flow. The part of the space occupied by this liquid is exactly the "dead volume", or inactive porosity, ϕ_i:

$$\frac{\partial \phi_i}{\partial t} = I_l. \tag{30}$$

The part of space $1 - \phi - \phi_i$ belongs to the immobile solid, consisting of the rock and the precipitated particles. Its evolution is described by the equation

$$\frac{\partial (1 - \phi - \phi_i)}{\partial t} = I_s. \tag{31}$$

Finally,

$$\frac{\partial \phi}{\partial t} = -I_\phi = -(I_l + I_s). \tag{32}$$

Let us determine the different values entering these equations in terms of the microscopic distributions introduced above. The volumetric concentration C_v is, obviously,

$$C_v = \int v_s(r_s) C(r_s, x, t) \mathrm{d}r_s. \tag{33}$$

In order to determine the different porosities from the microstructure of the porous medium, it is necessary to introduce a more detailed distribution $\Sigma(r_s, l, r_p, x, t)$: the number of particles of size r_s captured in pores of sizes (l, r_p). Obviously,

$$H(l, r_p, x, t) = H_0 - \int \Sigma(r_s, l, r_p, x, t) \mathrm{d}r_s.$$

An equation for $\Sigma(r_s, l, r_p, x, t)$ may be derived by application of the procedure from the previous subsection, and the result is similar to Eq. (28):

$$\frac{\partial \Sigma(r_s, l, r_p, x, t)}{\partial t} = f(l, r_p, x, t) \Lambda(r_s, l, r_p, x, t) C(r_s, x, t). \tag{34}$$

The initial condition for this equation is $\Sigma = 0$ at $t = 0$. Thus, $\Sigma(r_s, l, r_p, x, t)$ may, in principle, be determined, provided that $C(r_s, x, t)$ and $H(l, r_p, x, t)$ (and, therefore, $f(l, r_p, x, t)$) are already found. On the basis of this function, the different porosities may be found:

$$\begin{aligned} &1 - \phi - \phi_i = 1 - \phi_0 + \int v_s(r_s) \Sigma(r_s, l, r_p, x, t) \mathrm{d}r_s \mathrm{d}r_p \mathrm{d}l; \\ &\phi_i = \int (v_p(l, r_p) - v_s(r_s)) \Sigma(r_s, l, r_p, x, t) \mathrm{d}r_s \mathrm{d}r_p \mathrm{d}l; \\ &\phi = \phi_0 - \int v_p(l, r_p) \Sigma(r_s, l, r_p, x, t) \mathrm{d}r_s \mathrm{d}r_p \mathrm{d}l, \end{aligned} \tag{35}$$

or

$$\phi = \int v_p(l, r_p) H(l, r_p, x, t) \mathrm{d}r_p \mathrm{d}l. \tag{36}$$

Here ϕ_0 is an initial porosity of the porous medium. The source terms in Eqs. (30) to (32) may also be found by proper integration of the right-hand side of Eq. (34). For example,

$$I_l = \int f(l, r_p, x, t) \Lambda(r_s, l, r_p, x, t) C(r_s, x, t) (v_p(l, r_p) - v_s(r_s)) \mathrm{d}l \mathrm{d}r_p \mathrm{d}r_s. \tag{37}$$

If the kernels and coefficients are determined in terms of U, ϕ, as in the model (6), then the system of Eqs. (24), (28) should be completed by the porosity equation (36) and the continuity equation (29), in which the volumetric concentration and the source term are determined from Eqs. (33), (37). The four equations, (24), (28), (29), (36) form a closed system of four equations for the four unknowns $C(r_s, x, t)$, $H(l, r_p, x, t)$, $\phi(x, t)$, and $U(x, t)$. Then porosity ϕ_i may be found by solving Eq. (34) and integrating according to Eq. (35). For more sophisticated pictures of flow and capture on the microlevel Eq. (34) may become a part of the total system or be considered instead of Eq. (28).

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For examples presented below we will use a simplified version of Eq. (29). We will assume that the suspension is dilute, $C_v \ll 1$, although it still may affect porosity, and that the inactive porosity may be neglected: $I_l \approx 0$. With these assumptions, Eq. (29) is reduced to

$$\frac{\partial \phi}{\partial t} + \frac{\partial U}{\partial x} = 0. \tag{38}$$

The equation for porosity (36) will be applied without further simplifications.

3.6. Discussion

We have derived the basic system for evolution of the particle and pore size distribution. For convenience, we reproduce this system again. It consists of the equation for evolution of the particle size distribution (24):

$$\frac{\partial C}{\partial t} + v_s \frac{\partial C}{\partial x} = D_{xs} \frac{\partial^2 C}{\partial x^2} + D_{xts} \frac{\partial^2 C}{\partial x \partial t} + D_{ts} \frac{\partial^2 C}{\partial t^2} - \Lambda_s C(r_s, x, t), \tag{39}$$

and of equation for evolution of the pore size distribution (28) (where $f(l, r_p, x, t)$ is expressed in terms of $H(l, r_p, x, t)$ by Eq. (2)):

$$\frac{\partial H(l, r_p, x, t)}{\partial t} = -f(l, r_p, x, t) \int \Lambda(r_s, l, r_p, x, t) C(r_s, x, t) \mathrm{d}r_s. \tag{40}$$

These equations are connected. The coefficients in Eq. (24) depend on $f(l, r_p, x, t)$, as follows from Eqs. (15), (9) and the equations of Section 2.2. This dependence reflects the fact that deposition changes the structure of the porous medium, which results in a change of its transport properties. Coefficient $\Lambda(r_s, l, r_p, x, t)$ in Eq. (40) also depends on $f(l, r_p, x, t)$ (cf. Eqs. (16), (10)) and on $C(r_s, x, t)$. Additionally, if the distributions for the particle flights are related to the global flow characteristics, this system should be completed with the equation for porosity evolution and the continuity equation for the carrying liquid, as discussed in the previous subsection.

Eq. (39) resembles a standard "advection-dispersion-reaction" equation, which is commonly used for description of the deep bed filtration [6]. The fact that the particles are distributed by sizes does not affect the structure of this equation. It may be considered as a series of the modified advection-dispersion equations for the particles of each size r_s. Interaction between the particles of different sizes is only reflected in the coefficients of this equation. The different particles "compete" to be deposited in the same pores, and in this indirect way the coefficients in an equation for a particular size r_s are dependent on the particles of other sizes. For the dilute suspensions, the equations for the different radii become independent. Generally, Eqs. (39), (40) are linear with regard to concentration $C(r_s, x, t)$, but they are highly nonlinear with regard to the pore size distribution $H(l, r_p, x, t)$.

The main distinctive feature of Eq. (39) is that along with the standard terms it contains new terms $D_{xts}\partial^2 C/\partial x \partial t + D_{ts}\partial^2 C/\partial t^2$ on the right-hand side. These terms are responsible for the dispersion of the time step and mixed dispersion, like the common diffusion term $D_{xs}\partial^2 C/\partial x^2$ is responsible for the dispersion of the spatial step. The new terms and physical meanings of the new transport coefficients are discussed in detail in Refs. [26,27]. Below we will show in the numerical examples that the term with the temporal derivative is responsible for the hyperexponential deposition profiles.

Eq. (40) is a natural generalization of the capture equation from the classical filtration theory onto the case of distributed particles and pores. Nontriviality of this equation is hidden in the dependence of the capture coefficient $\Lambda(r_s, l, r_p, x, t)$ on the distribution $f(l, r_p, x, t)$, as formulated in Section 2. With the help of Eq. (15), Eq. (16) for the capture coefficient may be represented as (omitting the limit)

$$\Lambda(r_s, l, r_p, x, t) = \frac{1 - P(r_s, l, r_p, x, t)}{L_s(r_s, x, t)} v_s(r_s, x, t)$$

and, similarly,

$$\Lambda_s(r_s, x, t) = \frac{1 - P_s(r_s, x, t)}{L_s(r_s, x, t)} v_s(r_s, x, t).$$

Hence, as in the standard filtration theory [6], the capture frequencies may be treated as amounts proportional to the particle velocities, with the proportionality coefficients being the capturing probabilities per unit length.

For the particular case described by Eqs. (6), (7), it is shown in Appendix A that

$$\begin{aligned} v_s &= \frac{U}{\phi} V(r_s, x, t); \qquad D_{xs} = \frac{U}{\phi} R_D(r_s, x, t); \qquad D_{xts} = D_{xts}(r_s, x, t); \\ D_{ts} &= \frac{\phi}{U} R_t(r_s, x, t); \qquad \Lambda_s = \frac{U}{\phi} \lambda_s(r_s, x, t); \qquad \Lambda(r_s, l, r_p, x, t) = \frac{U}{\phi} \lambda(r_s, l, r_p, x, t). \end{aligned} \tag{41}$$

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With application of the dependences (41), Eqs. (39), (40) may be represented in the form of

$$\phi\frac{\partial C}{\partial t}+UV\frac{\partial C}{\partial x}=UR_D\frac{\partial^2 C}{\partial x^2}+\phi D_{xts}\frac{\partial^2 C}{\partial x\partial t}+\frac{\phi^2 R_t}{U}\frac{\partial^2 C}{\partial t^2}-U\lambda_s C \tag{42}$$

$$\frac{\partial H(l,r_p,x,t)}{\partial t}=-f(l,r_p,x,t)\frac{U}{\phi}\int\lambda(r_s,l,r_p,x,t)C(r_s,x,t)\mathrm{d}r_s. \tag{43}$$

Here coefficients R_D, R_t, D_{xts} have the dimension of a distance; the coefficient of velocity reduction V is dimensionless; and the capture coefficients λ_s, λ possess the dimension of inverse distance. All these coefficients depend on the distribution $f(l|r_p,x,t)$. If the effect of the particle deposition on $f(l|r_p,x,t)$ is negligible, these coefficients are geometric properties of the porous medium. This is the case of diluted suspensions considered in Ref. [27].

Coefficients R_D, R_t, D_{xts} have the meaning of different correlation lengths of the porous medium, and may vary in different ranges (from micrometers to centimeters) depending on the microscale heterogeneity. In many cases they may be treated as constants, although expressions for them may be rather nontrivial (see discussion in Appendices and especially Eq. (81)).

It is important to mark dependence of the transport coefficients on the flow velocity. While the convective dispersion coefficient D_{xs} increases with velocity, as usual for convective diffusion, the temporal dispersion coefficient D_{ts} is inversely proportional to velocity. Thus, while convective diffusion is important for fast flows, the temporal dispersion prevails for slow flows.

Correction to velocity V in Eq. (42) was discussed in detail in Refs. [23,24,27]. It is caused by the two effects. First, the pore space is micro-tortuous, and the particles do not always select a straight path between the two points. This may lead to a certain slowing down of the particles compared to the flow. On the other hand, the particles, due to their finite sizes, move close to the centers of the capillaries, that is, where the flow velocity is larger than the average. This may lead to an increase of the particle velocity: $V>1$. Other reasons for the different values of V are also possible (stagnant zones, vortices [3], or, inverse, fast conducting channels, which most of the particles "select"). The fact that the porosity ϕ considered above is not the complete porosity of the system, but only its part accessible for the particles, also contributes to V.

The filtration theory expressed by Eqs. (42), (43) is reduced to the standard filtration theory for (effectively) monodisperse supensions; when the temporal and mixed dispersions may be neglected (e.g. at high filtration rates); and when the correction V to the particle flow velocity is equal to unity.

4. Averaging of the system of equations

The system of equations obtained in the previous section combines integral and partial differential operators. Its full solution would require development of special numerical methods. Direct modeling of the propagation of a particle population in a stochastic porous medium is possible [10], but it will probably require a very large particle ensemble and very long simulation times in order to avoid dispersion and to approach real solutions of the systems (42) and (43). Averaging of the system should overcome these difficulties, reducing it to a system of partial differential equations for a number of practically important characteristics.

It was shown in Ref. [24] for a similar system of equations that it allows for averaging/upscaling for a monodisperse suspension. In this paper we transfer the procedure developed in Ref. [24] onto system Eqs. (42), (43). Then we extend these methods onto polydisperse systems containing the particles of n distinct sizes.

The derivations below are only applicable to the system Eqs. (42), (43), but not to the more general system Eqs. (39), (40). In other words, they are valid only if the particle deposition is determined by geometric or force factors, but is not affected by kinetics of the flow. As established in Appendix A, in this case kernel $\lambda(r_s,l,r_p,x,t)$ may be factorized and represented in the form of

$$\lambda(r_s,l,r_p,x,t)=\frac{p(r_s,l,r_p)}{\rho(r_s,x,t)}. \tag{44}$$

The probability of capturing $p(r_s,l,r_p)$ is depends only on the particle and pore sizes, while the characteristic distance $\rho(r_s,x,t)$ may depend on the current size distribution of the pores available for the flow.

4.1. Averaging for a monodisperse suspension

If all the sizes r_s of the particles are the same and equal to a fixed value R_s, the particle distribution has the form of

$$C(r_s,x,t)=c(x,t)\delta(r_s-R_s).$$

Correspondingly, Eq. (43) accounting for Eqs. (2), (44), is reduced to

$$\frac{\partial H(l,r_p,x,t)}{\partial t}=-\frac{H(l,r_p,x,t)}{h(x,t)}c(x,t)\frac{U(x,t)p(l,r_p)}{\phi(x,t)\rho(x,t)}. \tag{45}$$

Remark that for this case a more detailed distribution $\Sigma(r_s,l,r_p,x,t)$ for this case coincides with $H(l,r_p,x,t)$ and, therefore, Eq. (34) is also reduced to Eq. (45).

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Integrating the last equation over l, r_p, we obtain

$$\frac{\partial h(x,t)}{\partial t} = -\frac{c(x,t)}{h(x,t)}\frac{U(x,t)}{\phi(x,t)\rho(x,t)}\int p(l,r_p)H(l,r_p,x,t)\mathrm{d}l\mathrm{d}r_p. \tag{46}$$

An idea proposed in Ref. [24] is to substitute independent variable t by h. Since variables l, r_p, x in Eqs. (45), (46) do not participate in differentiation, we may divide one equation by another, to obtain

$$\frac{\partial H(l,r_p,x,h)}{\partial h} = -\frac{p(l,r_p)H(l,r_p,x,h)}{\int p(l,r_p)H(l,r_p,x,h)\mathrm{d}l\mathrm{d}r_p}.$$

An advantage of the last equation is that it is a closed equation with regard to H, being independent of c. Moreover, this equation may be solved. The denominator in its right-hand side is a function of x, h:

$$n(x,h) = \int p(l,r_p)H(l,r_p,x,h)\mathrm{d}l\mathrm{d}r_p. \tag{47}$$

Therefore, the equation may be reduced to the form

$$\frac{\mathrm{d}H}{\mathrm{d}N} = -p(l,r_p)H \tag{48}$$

where

$$N(x,h) = \int_{h_0(x)}^{h}\frac{\mathrm{d}h'}{n(x,h')}. \tag{49}$$

A solution of Eq. (48) is

$$H = H_0(l,r_p,x)\exp[-p(l,r_p)N(x,h)]. \tag{50}$$

If $h_0(x)$ is the initial pore concentration, then $H_0(l,r_p,x)$ is the initial pore distribution at a macroscopic point x. Eq. (50) determines dependence $H(l,r_p,x,h)$ within an unknown function $N(x,h)$. The last function may be found in the following way. First Eq. (49) is represented as an equation for $h(x,N)$: at constant x

$$\frac{\mathrm{d}h}{\mathrm{d}N} = n(x,h).$$

Expressing now $n(x,h)$ from Eq. (47), we obtain:

$$\frac{\mathrm{d}h}{\mathrm{d}N} = \int p(l,r_p)H_0(l,r_p,x)\exp[-p(l,r_p)N]\mathrm{d}l\mathrm{d}r_p. \tag{51}$$

That is,

$$h(x,N) = h_0(x) + \int_0^N \mathrm{d}N'\left\{\int p(l,r_p)H_0(l,r_p,x)\exp[-p(l,r_p)N']\mathrm{d}l\mathrm{d}r_p\right\}. \tag{52}$$

The limits of integration and the initial value are selected in agreement with Eq. (49), since $N = 0$ at $h = h_0(x)$. It should be noted that for practical purposes numerical solution of the differential equation (51) may be more advantageous than a solution determination of the dependence $N(h,x)$ by inversion of the algebraic equation (52).

With a known dependence $H(l,r_p,x,h)$, porosity ϕ also becomes a known function of x and h, since, by definition, H is the concentration of pores per unit volume, as follows from Eq. (36):

$$\phi(x,h) = \int v(l,r_p)H(l,r_p,x,h)\mathrm{d}l\mathrm{d}r_p. \tag{53}$$

The value of $\rho(x,t)$ also becomes $\rho(x,h)$, due to Eq. (82) of Appendix A. Now Eq. (46) becomes:

$$\begin{aligned}\frac{\partial h(x,t)}{\partial t} &= -c(x,t)U(x,t)\Phi(h,x),\\ \Phi &= \frac{1}{h\phi(x,h)\rho(x,h)}\int p(l,r_p)H(l,r_p,x,h)\mathrm{d}l\mathrm{d}r_p.\end{aligned} \tag{54}$$

Let us now consider the equation for concentration (42), which, for a monodisperse suspension, is reduced to an equation for concentration $c(x,t)$. As follows from Appendix A, the coefficients in this equation depend on distribution $f(l|r_p,x,t)$, which, according to Eqs. (2), (3), is expressed in terms of $H(r_p,l,x,t)$. The procedure above shows existence of a universal dependence $H(r_p,l,x,h)$. Thus, the coefficients in the modified Eq. (42) depend on x and h:

$$\phi\frac{\partial c}{\partial t} + UV(x,h)\frac{\partial c}{\partial x} = UR_D(x,h)\frac{\partial^2 c}{\partial x^2} + \phi D_{xts}(x,h)\frac{\partial^2 c}{\partial x\partial t} + \frac{\phi^2 R_t(x,h)}{U}\frac{\partial^2 c}{\partial t^2} - U\lambda_s(x,h)c. \tag{55}$$

Eqs. (54), (55), (53) coupled with the continuity equations (38) form a closed system of equations for the four macroscopic functions: $c(x,t)$, $U(x,t)$, $\phi(x,t)$ and $h(x,t)$. The closing relations for these equations are presented in this section and in Appendix A.

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4.2. Averaging for a polydisperse suspension

Averaging of the system (42), (43) for a polydisperse system is generally similar to averaging of a monodisperse system, although some steps are nontrivial. We consider flow of a suspension where the particles possess n distinct sizes $R_1, \ldots, R_n$. The particle size distribution $C(r_s, x, t)$ is expressed in terms of the concentrations $c_i(x, t)$ of the particles of individual sizes:

$$C(r_s, x, t) = \sum_{i=1}^{n} c_i(x, t)\delta(r_s - R_i).$$

Correspondingly, Eq. (43) assumes the form of

$$\frac{\partial H(l, r_p, x, t)}{\partial t} = -\frac{H(l, r_p, x, t)}{h(x, t)} \frac{U(x, t)}{\phi(x, t)} \sum_{i=1}^{n} \frac{p_i(l, r_p)}{\rho_i(x, t)} c_i(x, t) \tag{56}$$

with

$$p_i(l, r_p) = p(R_i, l, r_p); \qquad \rho_i(x, t) = \rho(R_i, x, t). \tag{57}$$

Similarly to the case of a monodisperse system, we will look for the solution of Eq. (56) in the form of $H(l, r_p, x, t) = H(l, r_p, x, h_1, \ldots, h_n)$, where unknown functions $h_i(x, t)$ are determined from the equations

$$\frac{\partial h_i(x, t)}{\partial t} = -\frac{U(x, t)c_i(x, t)}{h(x, t)\phi(x, t)\rho_i(x, t)} = -\frac{U(x, t)c_i(x, t)}{h(x, h_1, \ldots, h_n)\phi(x, h_1, \ldots, h_n)\rho_i(x, h_1, \ldots, h_n)} \quad (i = 1, \ldots, n). \tag{58}$$

The last equality is valid, since h, ϕ and ρ_i become functions of $h_1, \ldots, h_n$ due to Eqs. (2), (36), (82), as far as dependence $H(l, r_p, x, h_1, \ldots, h_n)$ is determined (which we do later). Unlike the value of h for a monodisperse system, functions h_i do not have a meaning of the total number of pores, but, rather, of the amounts on which this number decreases due to the deposition of particles of size R_i. Therefore, it is reasonable to set initial conditions

$$h_i(x, 0) = 0. \tag{59}$$

By the chain rule,

$$\frac{\partial H(l, r_p, x, h_1, \ldots, h_n)}{\partial t} = \sum_{i=1}^{n} \frac{\partial H}{\partial h_i} \frac{\partial h_i}{\partial t}.$$

Comparison of this equation to Eqs. (56), (58) shows that it is possible to define

$$\frac{\partial H}{\partial h_i} = p_i(l, r_p)H.$$

This equation is consistent, since it follows from it that $\partial^2 H/\partial h_i \partial h_j = \partial^2 H/\partial h_j \partial h_i$. Its solution, taking in account the initial condition (59), is

$$H(l, r_p, x, h_1, \ldots, h_n) = H_0(l, r_p, x) \exp\left[\sum_{i=1}^{n} p_i(l, r_p)h_i\right]. \tag{60}$$

The last formula generalizes solution (50) for monodisperse suspensions. Further upscaling is similar to the monodisperse case. In particular, equations for the concentrations $c_i(x, t)$ of the particles of the different radii assume the form of

$$\phi\frac{\partial c_i}{\partial t} + UV_i(x, \{h_j\})\frac{\partial c_i}{\partial x} = UR_{Di}(x, \{h_j\})\frac{\partial^2 c_i}{\partial x^2} + \phi D_{xtsi}(x, \{h_j\})\frac{\partial^2 c_i}{\partial x \partial t} + \frac{\phi^2 R_{ti}(x, \{h_j\})}{U}\frac{\partial^2 c_i}{\partial t^2} - U\lambda_{si}(x, \{h_j\})c_i. \tag{61}$$

Eqs. (58), (61) coupled with the continuity equation (38) form a closed system of $(2n+1)$ equations for $(2n+1)$ variables h_i, c_i, U, in which porosity is determined from Eqs. (36), (60).

It is seen that the complexity of the averaged system increases with the polydispersity of the suspension. The number of equations is proportional to the number of the distinct radii. Numerical methods are required for solution of the system, as demonstrated in the example below.

5. Example: a bidisperse suspension in a monodisperse porous medium

The goal of the present section is to illustrate the developed formalism on a representative example. This example is designed to be, on one hand, as simple as possible, but on the other hand, to demonstrate specific behavior of the model and characteristic difficulties arising at the numerical solution.

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5.1. Simplified system of equations

Let us consider a suspension where all the particles may have only two possible radii: R_{s1} and R_{s2} (the case of a monodisperse suspension was considered in Refs. [24,27]). All the pores of the porous medium, where filtration occurs, have the same radius R_p. We assume additionally that, in spite of the fact that the pore lengths are somehow distributed, the capturing probabilities p_i are independent of them. A similar formalism is obtained under the assumption that the pore radii are also distributed, but this does not influence the capture probabilities. For example, this assumption may be reasonable if the pore sizes are much larger than the particle sizes, and deposition is caused by the surface forces. The last statement formally contradicts the above assumption that capture occurs by size exclusion. However, as mentioned above, this assumption has only been introduced for simplicity of the derivation, and the basic results are valid for other capture mechanisms.

The porous medium is assumed to be initially homogeneous, so that initial distributions are independent of x. However, these distributions may vary with time.

Under the assumptions above solution (60) is reduced to

$$H(l,x,h_1,h_2) = H_0(l)\exp(p_1h_1+p_2h_2). \tag{62}$$

Correspondingly, the porosity and the number of particles are expressed as

$$h(x,h_1,h_2) = \int H(l,h_1,h_2)\mathrm{d}l = h_0\exp(p_1h_1+p_2h_2), \tag{63}$$

$$\phi(x,h_1,h_2) = \int v_p(l)H(l,h_1,h_2)\mathrm{d}l = \phi_0\exp(p_1h_1+p_2h_2). \tag{64}$$

Additionally, as shown in Appendix B, the values of ρ_i are constants, as well as the coefficients V_i, R_{Di}, D_{xtsi}, R_{ti}, λ_{si} in Eq. (61). Moreover, as shown in this Appendix, Eq. (84), $\lambda_{si} = p_i/\rho_i$.

A complete system of equations for this case consists of the continuity equations and the simplified Eqs. (58), (61):

$$\frac{\partial\phi(h_1,h_2)}{\partial t} + \frac{\partial U}{\partial x} = 0; \tag{65}$$

$$\frac{\partial h_i(x,t)}{\partial t} = -\frac{U(x,t)c_i(x,t)}{h_0\phi_0\rho_i\exp(2p_1h_1+2p_2h_2)} \quad (i=1,2); \tag{66}$$

$$\phi\frac{\partial c_i}{\partial t} + UV_i\frac{\partial c_i}{\partial x} = UR_{Di}\frac{\partial^2 c_i}{\partial x^2} + \phi D_{xtsi}\frac{\partial^2 c_i}{\partial x\partial t} + \frac{\phi^2 R_{ti}}{U}\frac{\partial^2 c_i}{\partial t^2} - U\frac{p_i}{\rho_i}c_i \quad (i=1,2). \tag{67}$$

Further simplification of this system is possible. The two equations for h_1, h_2 may be substituted by a single equation for porosity ϕ. Differentiating Eq. (64) with regard to t and substituting $\mathrm{d}h_i/\mathrm{d}t$ from Eq. (66), after obvious transformations we obtain

$$\frac{\partial\phi}{\partial t} = -\frac{\phi_0 U}{h_0\phi}\left(p_1\frac{c_1}{\rho_1} + p_2\frac{c_2}{\rho_2}\right). \tag{68}$$

This equation may also be combined with the continuity equation (38), to obtain

$$\frac{\partial U}{\partial x} = \frac{\phi_0 U}{h_0\phi}\left(p_1\frac{c_1}{\rho_1} + p_2\frac{c_2}{\rho_2}\right). \tag{69}$$

Eqs. (68), (69), coupled with equations for the concentrations (67), form the closed system of equations for variables ϕ, U, c_1, c_2. All the coefficients in this system are constants.

In a similar way, we obtain a general system of equations for transfer of polydisperse suspensions in a monodisperse porous medium (or in a porous medium where the rules of deposition are independent of the pore radii). Under the assumptions listed above this system is reduced to $(n+2)$ equations:

$$\phi\frac{\partial c_i}{\partial t} + UV_i\frac{\partial c_i}{\partial x} = UR_{Di}\frac{\partial^2 c_i}{\partial x^2} + \phi D_{xtsi}\frac{\partial^2 c_i}{\partial x\partial t} + \frac{\phi^2 R_{ti}}{U}\frac{\partial^2 c_i}{\partial t^2} - U\frac{p_i}{\rho_i}c_i \quad (i=1,\ldots,n)$$

$$\frac{\partial\phi}{\partial t} = -\frac{\phi_0 U}{h_0\phi}\sum_{i=1}^{n}\frac{p_i}{\rho_i}c_i$$

$$\frac{\partial U}{\partial x} = \frac{\phi_0 U}{h_0\phi}\sum_{i=1}^{n}\frac{p_i}{\rho_i}c_i.$$

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5.2. Initial and boundary conditions

Let us specify initial and boundary conditions for system (67) to (69). Consider a problem of injection of a finite portion of suspension pushed by pure liquid, which corresponds to a typical deep bed filtration experiment [13,14]. First, the initial porosity and the injection velocity should be specified:

$$\phi(x, t = 0) = \phi_0; \tag{70}$$

$$U(x = 0, t) = U^0. \tag{71}$$

The boundary conditions at the injection and at the production ends may be posed as usual conditions for an advection-dispersion equation. If a finite portion of the suspension is injected during time T_0 and is pushed by a pure liquid afterwards, then

$$\begin{aligned} &c_i(x = 0, t) = c_i^0 \quad (0 \le t \le T_0); \\ &c_i(x = 0, t) = 0 \quad (t \ge T_0); \\ &\frac{\partial c_i}{\partial x}(x = L, t) = 0. \end{aligned} \tag{72}$$

In the following calculations we assume that the injection time T_0 is comparable to the characteristic convective times UV_i/L and, thus, represents the characteristic time scale of the problem. Generalizations onto incomparable times may easily be obtained (although may lead to numerical difficulties).

Eqs. (67) contain the second time derivative of the concentrations. As was discussed in Refs. [26,27], this requires, apart from a usual initial condition, an additional initial condition for c_i, or a condition at another time. Specification of the two initial conditions at the same time would lead to an instable elliptic problem. It is physically evident, on the other hand, that specifying the only initial concentration would be sufficient:

$$c_i(x, t = 0) = c_{0,i}. \tag{73}$$

It was proven in Refs. [26,27] that, instead of the second condition, a condition of limitedness of the solution at infinity may be posed. However, this is inconvenient for numerical computations.

For the considered problem, the following solution may be suggested. Let us consider a large time interval, $T_1 = \alpha T_0$, after which (almost) all the free particles are either captured, or removed from the porous medium by the pure injected liquid. It is possible to set the following "final" condition:

$$c_i(x, t = T_1) = 0. \tag{74}$$

Ideally, T_1 should tend to infinity. However, numerical tests show that the concentration decreases rather rapidly after pure water is injected. In our calculations we used $\alpha = 5$. This has shown to be large enough: further increase of time T_1 did not lead to significant modification of the solution in the practically interesting period $0 \le t \le 2.5T_0$, after which both forward and backward concentration fronts have been passed. Of course, other situations are possible, where α should be increased, and for any particular case this question needs further numerical study.

5.3. Dimensionless form

It is convenient to solve the system of equations above in the dimensionless variables, since this makes it possible to exclude the parameters of the order of unity and to study dependence of the system on the key dimensionless complexes. Let us introduce the following dimensionalization:

$$\begin{aligned} &\phi = \phi_0 \Phi, \qquad c_i = c^0 C_i, \qquad U = U_0 u, \\ &x = LX, \qquad t = (L\phi_0/U_0)T, \\ &R_{Di} = aLr_{Di}, \qquad D_{xtsi} = aLr_{xti}, \qquad R_{ti} = aLr_{ti}, \\ &\rho_i = L\gamma_0\gamma_i, \qquad p_i = p_0 P_i. \end{aligned}$$

Here $\phi_0,\ c^0,\ U_0,\ L,\ \gamma_0,\ p_0$ are characteristic values possessing the dimensions of the corresponding dimensional values (ϕ_0 and γ_0 are dimensionless); the variables $\Phi,\ C_\alpha,\ u,\ X,\ T,\ r_{D\alpha},\ r_{xt\alpha},\ r_{t\alpha},\ \gamma_\alpha,\ P_\alpha$ are dimensionless and are supposed to be of the order of unity — otherwise the corresponding terms in the transport equations may be neglected. The value of a is the inverse Peclet number, expressing the characteristic ratio of the dispersion and advection terms. There are two other characteristic dimensionless numbers:

$$b = \frac{p_0}{\gamma_0}; \qquad A = \frac{c^0}{h_0}.$$

The value of b is responsible for the characteristic frequency of particle entrapment related to the linear scale of the problem. The value of A compares the amount of injected particles with the initial amount of the available pores.

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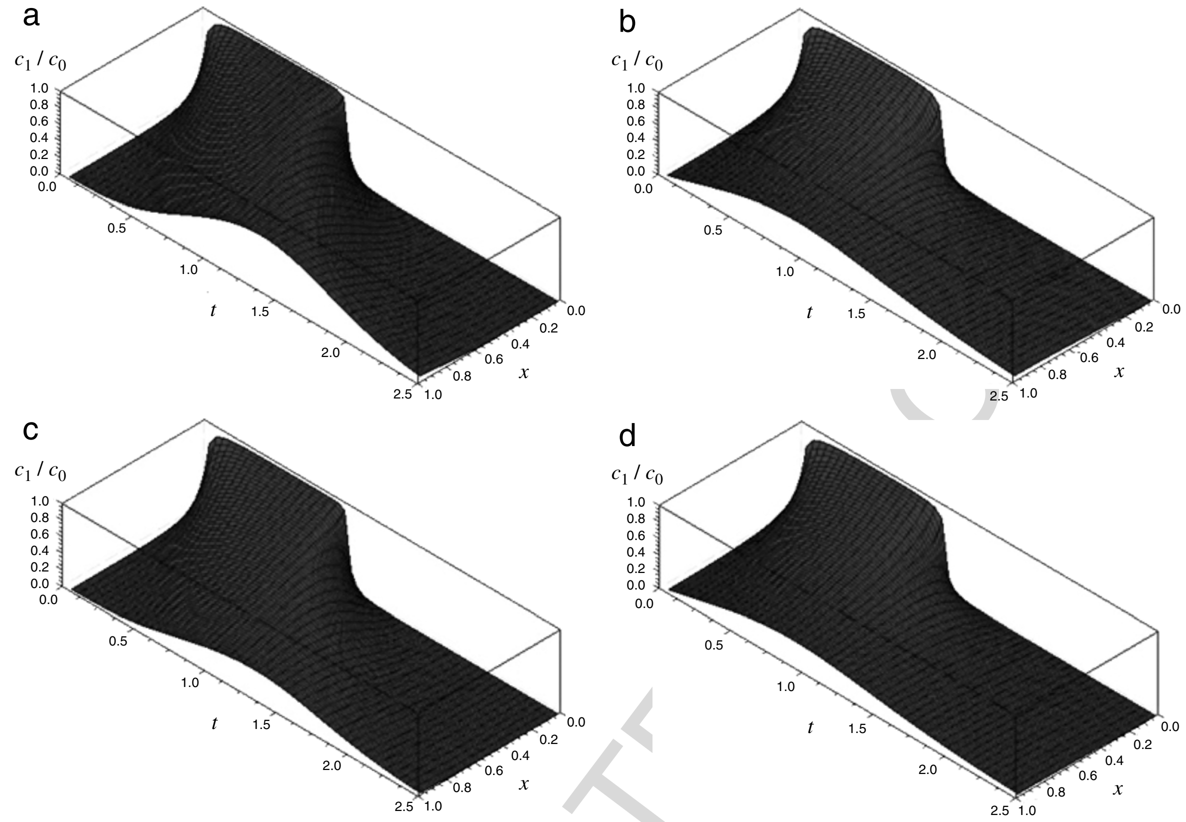

Fig. 1. Particle concentration dependences on space and time for the four cases: (a) $a = 0.05$, $b = 0.1$; (b) $a = 0.5$, $b = 0.1$; (c) $a = 0.05$, $b = 1$; (d) $a = 0.5$, $b = 1$. Here a is the inverse Peclet number (dispersion to advection ratio); b is the particle capture frequency ratio. Parameter A (initial concentration of particles to pores) is equal to 0.02 in all the calculations.

In dimensionless variables, the system of transport equations is

$$\Phi \frac{\partial C_i}{\partial T} + uV_i \frac{\partial C_i}{\partial X} = a\left(ur_{Di} \frac{\partial^2 C_i}{\partial X^2} + \Phi r_{xti} \frac{\partial^2 C_i}{\partial X \partial T} + \frac{\Phi^2 r_{ti}}{u} \frac{\partial^2 C_i}{\partial T^2} \right) - bu \frac{P_i}{\gamma_i} C_i \tag{75}$$

$$\begin{aligned} \frac{\partial u}{\partial X} &= Ab \frac{u}{\Phi} \sum_{i=1}^{n} \frac{P_i}{\gamma_i} C_i \\ \frac{\partial \phi}{\partial T} &= -Ab \frac{u}{\Phi} \sum_{i=1}^{n} \frac{P_i}{\gamma_i} C_i. \end{aligned} \tag{76}$$

It is also important to consider the equation for accumulation of the entrapped particles. Denoting by σ_i the accumulated amounts of the particles of the type i, from equations of Section 3.5 we obtain

$$\frac{\partial \sigma_i}{\partial t} = U \frac{p_i}{\rho_i} c_i \quad (i = 1, \ldots, n).$$

The dimensionless accumulated amounts S_i are introduced by $\sigma_i = c_0 S_i$ $(i = 1, \ldots, n)$. A dimensionless equation for them is

$$\frac{\partial S_i}{\partial T} = \phi_0 bu \frac{P_i}{\gamma_i} C_i \quad (i = 1, \ldots, n). \tag{77}$$

We also calculate the total accumulation σ as a sum of σ_i and, correspondingly, $S = \sum S_i$.

5.4. A method for numerical solution

Direct discretization of the system of equations (75), (76) leads to a system of nonlinear algebraic equations for each point on a rectangular grid, which requires iterative methods for its solution. A simpler solution procedure may be obtained

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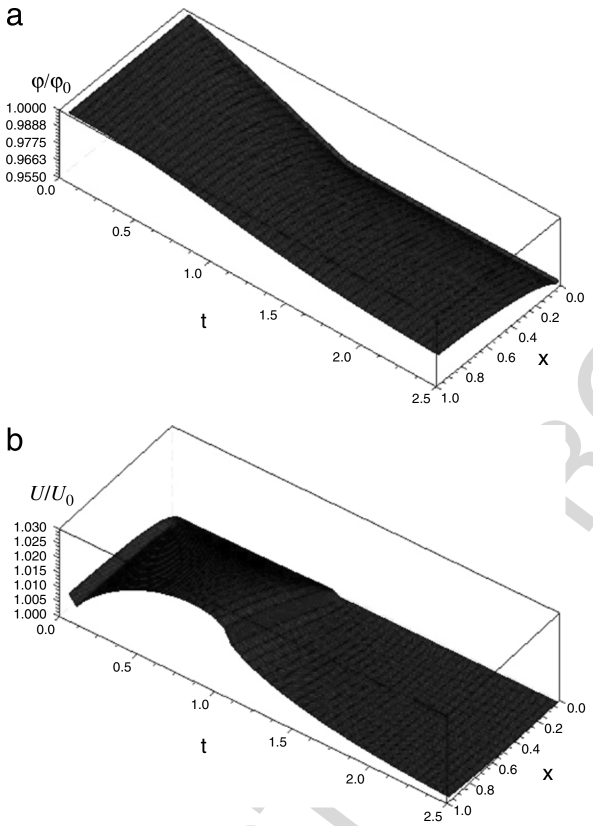

Fig. 2. Dependences of (a) porosity, and (b) velocity on x and t. Parameters for calculation: $a = 0.5$, $b = 0.1$, $A = 0.2$.

if Eqs. (76) for (Φ, u) are solved separately from the Eqs. (75) for C_i. Each time, solving the equations for (Φ, u), we assume the concentrations to be known. Then we solve the equations for concentrations, taking porosity and flow rate from the previous iteration. The iterative solution procedure may be schematized as

$$\Phi^{(0)}, u^{(0)} \to C_i^{(0)} \to \Phi^{(1)}, u^{(1)} \to \cdots \to C_i^{(n)} \to \cdots \tag{78}$$

until convergence.

Under known $C_i^{(\alpha-1)}$, Eqs. (76) for $\Phi^{(\alpha)}$, $u^{(\alpha)}$ are solved by direct integration. Afterwards Eqs. (75) for $C_i^{(\alpha)}$ are solved, with the coefficients taken at $\Phi^{(\alpha)}$, $u^{(\alpha)}$, by usual discretization methods. In this paper, we apply numerical differences, taking the central differences when possible. Iterations (78) are repeated until convergence.

In our calculations, reproduction of the velocity and porosity fields with an accuracy of 0.1% was usually achieved after 3–4 iterations for low dispersion (low parameter a) and after 5–7 iterations for high dispersion. This indicates a reasonable robustness and rapidness of the proposed numerical procedure. Care should be taken for the cases of high particle concentrations, where complete plugging of a porous medium may be achieved and the porosity becomes zero at some step. Special problems may arise, for example, in the cases where the porosity $\Phi^{(\alpha)}$ becomes zero at an intermediate iteration, but this is not the right final answer. In the computations below, we select the initial conditions avoiding complete plugging.

5.5. Sample calculations

As mentioned in the Introduction, one of the observed effects, which cannot be explained by the classical filtration theory, is hyperexponential distribution of the entrapped particles along a porous medium. The classical “homogeneous” theory predicts that under boundary conditions (72) the final dependence of $\ln S$ on X will be linear. However, many experiments contradict this prediction [3,13]. It was suggested [13,14,20] to explain the hyperexponentiality by the different sizes of the particles and (or) to the different ways of their entrapment. We have pointed out [27] that the time dispersion term may also result in a non-standard particle distribution. Temporal dispersion means that more particles than expected stay near the entrance and, simultaneously, more particles move further away. The model developed above makes it possible to compare the relative roles of the two mechanisms.

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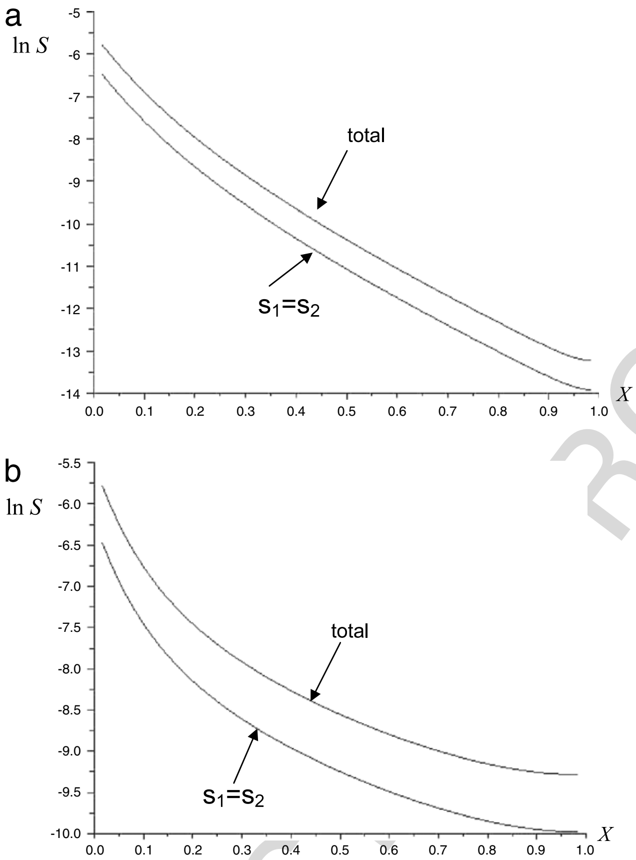

Fig. 3. Logarithms of the retention profiles for (a) low dispersion-advection ratio ($a = 0.05$); (b) high dispersion-advection ratio ($a = 0.5$). Other parameters: $b = 0.1, A = 0.02$. The retention profiles are (almost) exponential for case (a) and hyperexponential for case (b).

We have obtained a series of numerical solutions of the system of equations (75), (76), for injection of a suspension of particles of two different sizes ($n = 2$). A SciLab-based numerical implementation of the procedure outlined in the previous section was created.

The basic parameters for calculations were selected in such a way that they are representative for realistic conditions, where the dispersion effects may be observed. The inverse Peclet number a for convective dispersion is equal to the ratio of the correlation length of the medium to the characteristic scale of the problem. In the computations the basic value was 0.05, and we used a high value of 0.5 for comparison. Even at $a = 0.05$ (that is, for the correlation length of 5 cm with the filter size of 1 m) contribution of the dispersive mechanisms is significant. Smaller values of a were also explored, but in some cases they caused numerical instabilities of the selected discretization. The particle-to-pore ratio A varied from 0.02 to 0.2. It affected the rate of plugging, but not the fact of hyperexponentiality of the deposition profiles. The same is true for the dimensionless entrapment rate b varying from 0.1 to 1. Other coefficients were set to unity, apart from $r_{xti} = 0$.

Typical concentration dependences on time and space are presented in Fig. 1, at the two different dispersion-advection ratios a and the two different entrapment ratios b. Within the range of the parameters under study, the particle–pore ratio A does not affect the shapes of the dependences. The behavior of concentrations in Fig. 1 is as expected. The particles are injected with the dimensionless concentrations $C_i^0 = 1$ at times between 0 and $T_0 = 1$. Afterwards pure liquid is injected, pushing the suspension forward. At this stage the particle concentration decreases due to capturing and dispersion. Higher values of a result in higher washout of the concentration profile, while higher b leads to a faster decrease of the concentration.

A characteristic evolution of porosity and velocity is shown in Fig. 2. Porosity decreases with time, since more particles are deposited. The decrease of the porosity slows down when pure liquid is injected. Porosity increases with the spatial coordinate, since fewer particles reach the "far end" of the porous medium and deposit there. Velocity behaves non-monotonously. This behavior is explained on the basis of Eq. (38). While the decrease of the porosity slows down, the velocity becomes more uniformly distributed along the porous medium.

Our goal was to analyze behavior of the distribution of the retained particles. The deposition profiles are plotted in semi-logarithmic coordinates, so that the "classical" exponential behavior is represented by a straight line. In all our calculations, the main influential parameter determining the type of this dependence has turned to be the dispersion parameter a. Hyperexponential behavior was observed for cases of high dispersions. Comparative behavior of the deposition profiles for the cases of low and high dispersions is presented in Fig. 3. The deposition profiles for the case of low dispersion are

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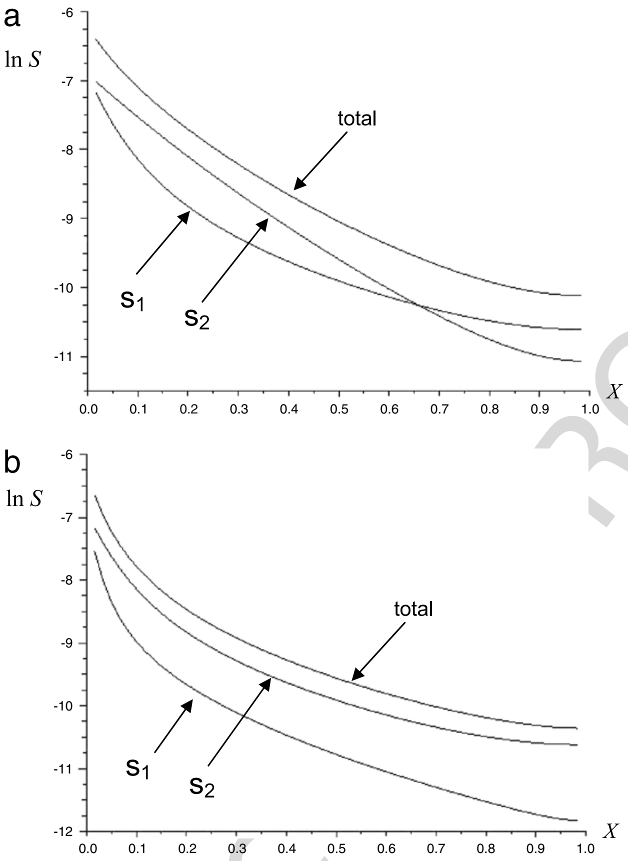

Fig. 4. Comparison of the impacts of the spatial and temporal dispersion on the hyperexponentiality of the retention profiles. Dimensionless ratios: $a = 0.5$, $b = 0.1$, $A = 0.2$. (a) Low temporal dispersion for the second component: $r_{t,2} = 0.1$, while $r_{t,1} = r_{x,1} = r_{x,2} = 1$. (b) Low spatial dispersion for the second component: $r_{x,2} = 0.1$, while $r_{x,1} = r_{t,1} = r_{t,2} = 1$.

(almost) exponential, with the deviations at $X = 1$ caused by the specific boundary condition. The profiles corresponding to high dispersion show the clearly pronounced hyperexponential behavior. Similar effects were observed in our previous paper [27], as well as in other papers based on the CTRW approach (e.g. Ref. [18]).

Parameter a rescales both spatial and temporal dispersion coefficients. In order to distinguish the contributions of the two dispersions, we have run the two numerical simulations. In the first simulation the coefficient $r_{t,2}$ was set to 0.1, while all the other coefficients $r_{t,i}$ and $r_{x,i}$ were set to unity. Thus, in this case the temporal dispersion was high for the first component and low for the second component, while spatial dispersions were equal. In the second simulation, the temporal dispersions were equal (and high) for both components, while the spatial dispersion for the second component was set to $r_{x,2} = 0.1$. We also set different initial concentrations, in order to make the components distinguishable on the plots.

The sets of deposition profiles for the two cases are presented in Fig. 4. It is seen from Fig. 4a that the deposition profile for component 1 (with high temporal dispersion) is strongly hyperexponential, while the profile for component 2 is almost exponential, apart from the least points. Meanwhile, as follows from Fig. 4b, variation of the spatial dispersion does not affect hyperexponentiality of the deposition profiles.

It has been suggested in the literature [13,20] that the main mechanism for hyperexponentiality is that particles of different types are captured with different probabilities. The goal of the next series of simulations was to compare the effects of the temporal dispersion and of the unequal capture probabilities onto the deposition profiles. The results are shown in Fig. 5. The basic case (Fig. 5a) corresponds to the low temporal dispersion and to the equal capture probabilities. In this case the deposition profiles are almost exponential. Selection of high temporal dispersion (Fig. 5b) results in the clearly hyperexponential distribution. The final simulation, shown on Fig. 5c, corresponds to low temporal dispersion, as for Fig. 5a, but to the different capture probabilities for the two kinds of particles: $p_2 = 0.2p_1$. For this case the distributions of the concentration logarithms become less linear, but the difference from the basic case on Fig. 5a is only slight. Thus, at least for this case, the contribution of the temporal dispersion is more significant than the contribution of the difference in the capture probabilities.

It should be noticed that both the temporal dispersion and the dispersion of the capture probability lead to hyperexponential profiles, while "hypoexponential" deposition profiles have not been observed in our simulations. The

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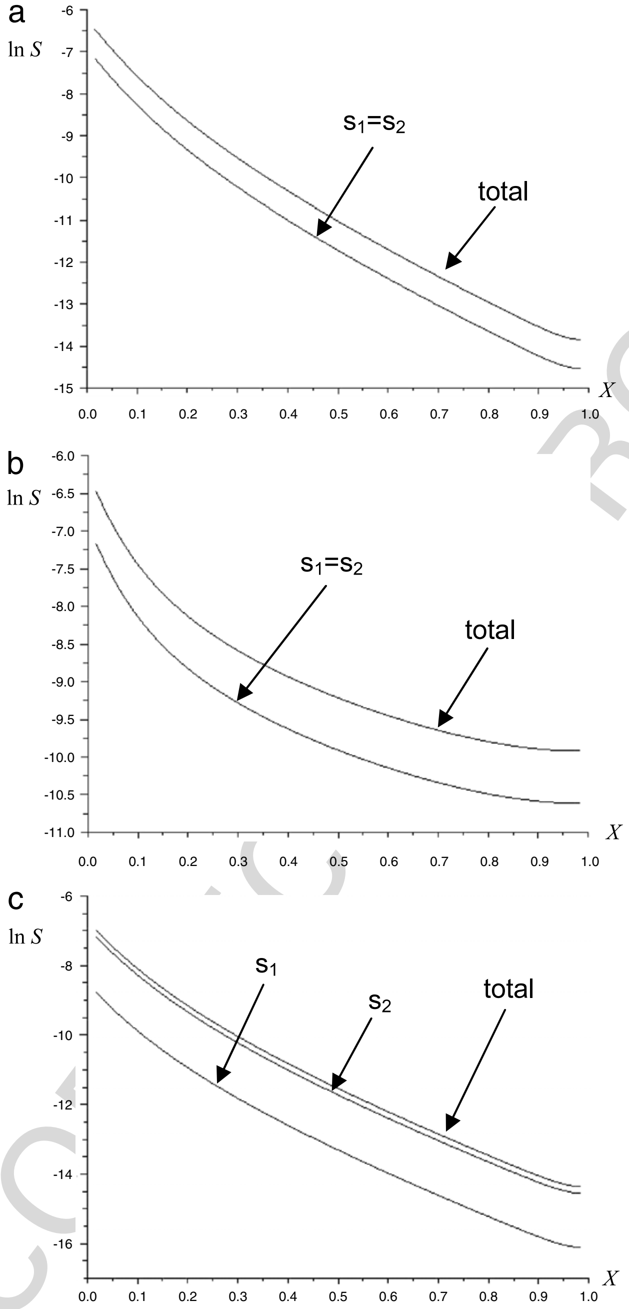

Fig. 5. Impact of different temporal dispersions compared to the impact of the different capture probabilities on the hyperexponentiality of the retention profiles. For all cases, $b = 0.1$, $A = 0.2$. For cases (a), (b) the capture probabilities are equal: $p_1 = p_2 = 1$, but the advection-dispersion ratios are different: (a) $a = 0.05$, (b) $a = 0.5$. Case (c): Low $a = 0.05$, and the different p_i: $p_1 = 1,\ p_2 = 0.2$.

effects of particle size distribution and of the temporal dispersion on the deposition profiles are competing, and cannot be directly distinguished on the plots.

Conclusions

We have developed a complete semi-microscopic stochastic theory of filtration of polydisperse suspensions in a porous medium characterized by a pore size distribution. The theory covers the gap between the microscopic mechanisms of filtration of polydisperse suspensions and macroscopic equations describing their flow.

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The components of the developed scheme are: A complete stochastic system of transport equations for the particle concentration distributed by the particle sizes, and for the pore distribution by the pore sizes and lengths; a method for their averaging and reduction to a system of hydrodynamic equations for particles of different sizes; and an approach to numerical solution of the resulting system of equations.

Sample calculations demonstrate that the theory is able to predict the effect of "hyperexponential" behavior of the deposition profiles, which has previously been discussed in the literature. It is demonstrated that the main effect responsible for such a behavior of the profiles is a new effect of the temporal dispersion. This effect plays the same role for the temporal coordinate as diffusion plays for the spatial coordinate. Numerical simulations demonstrate that the temporal dispersion leads to a stronger nonlinearity of the logarithms of the deposition profiles than the commonly considered effect of the different retentions of particles of different sizes.

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Appendix A

The goal of this appendix is to study in detail the transport coefficients in Eqs. (39), (40) for the specific case where the distribution of one step is given by Eq. (7), that is, when the flows in the separate pores are rescaled together with the overall flow, and when the kinetics does not affect the particle entrapment. We start with the evaluation of the moments (9). Substituting distribution (7) into the expression for L_s, we obtain:

$$L_s(r_s, x, t) = \int pl f_U(\phi l/U\tau | r_s, l, r_p) \frac{\phi l}{U\tau^2} f(p|r_s, l., r_p) f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\tau.$$

After introduction of the new integration variable $u = l/U\tau$ this expression is reduced to

$$L_s(r_s, x, t) = \int pl f_U(\eta | r_s, l, r_p) f(p|r_s, l., r_p) f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\eta.$$

That is, L_s is independent of U, as might be expected. In a similar way we obtain for other moments (9):

$$T_s(r_s, x, t) = \frac{\phi}{U} \int \frac{pl}{u} f_U(\eta | r_s, l, r_p) f(p|r_s, l., r_p) f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\eta. \tag{79}$$

- inversely proportional to U/ϕ;

$$L_{2s}(r_s, x, t) = \int pl^2 f_U(\eta | r_s, l, r_p) f(p|r_s, l., r_p) f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\eta$$

- independent of U/ϕ;

$$L_{Ts}(r_s, x, t) = \frac{\phi}{U} \int p\frac{l^2}{u} f_U(\eta | r_s, l, r_p) f(p|r_s, l., r_p) f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\eta$$

- inversely proportional to U/ϕ;

$$T_{2s}(r_s, x, t) = \frac{\phi^2}{U^2} \int p\frac{l^2}{u^2} f_U(\eta | r_s, l, r_p) f(p|r_s, l., r_p) f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\eta$$

- inversely proportional to $(U/\phi)^2$;

$$P_s(r_s, x, t) = \int p f_U(\eta | r_s, l, r_p) f(p|r_s, l., r_p) f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}\eta$$

- independent of U/ϕ, and finally,

$$P(r_s, l, r_p, x, t) = \int p f_U(\eta | r_s, l, r_p) f(p|r_s, l, r_p)\mathrm{d}p\mathrm{d}\eta \tag{80}$$

- independent of U/ϕ and, moreover, of x and t.

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Referring now to Eqs. (15) for the transport coefficients, we recover relations (41). In these relations,

$$\begin{aligned}
V(r_s, x, t) &= \frac{\int plf_U(u|r_s, l, r_p)f(p|r_s, l., r_p)f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}u}{\int p\frac{l}{u}f_U(u|r_s, l, r_p)f(p|r_s, l., r_p)f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}u}\\
R_D(r_s, x, t) &= \frac{\int pl^2f_U(u|r_s, l, r_p)f(p|r_s, l., r_p)f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}u}{\int plf_U(u|r_s, l, r_p)f(p|r_s, l., r_p)f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}u}\\
D_{xts}(r_s, x, t) &= \frac{\int p\frac{l^2}{u}f_U(u|r_s, l, r_p)f(p|r_s, l., r_p)f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}u}{\int \frac{pl}{u}f_U(l/U\tau|r_s, l, r_p)f(p|r_s, l., r_p)f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}u}\\
R_t(r_s, x, t) &= \frac{\int p\frac{l^2}{u^2}f_U(u|r_s, l, r_p)f(p|r_s, l., r_p)f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}u}{\int \frac{pl}{u}f_U(l/U\tau|r_s, l, r_p)f(p|r_s, l., r_p)f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}u}.
\end{aligned} \qquad (81)$$

Additionally, since $P(r_s, l, r_p, x, t)$ is independent of x and t, it follows from Eqs. (lambda), (79) and (80) that (omitting the limit) Q1

$$\Lambda(r_s, l, r_p, x, t) = \frac{U}{\phi}\frac{p(r_s, l, r_p)}{\rho(r_s, x, t)}.$$

Here

$$\begin{aligned}
&p(r_s, l, r_p) = 1 - P(r_s, l, r_p);\\
&\rho(r_s, x, t) = \int \frac{pl}{u}f_U(u|r_s, l, r_p)f(p|r_s, l., r_p)f(l|r_p, x, t)\mathrm{d}p\mathrm{d}l\mathrm{d}u.
\end{aligned} \qquad (82)$$

Appendix B. Coefficients in the transport equations for special cases

The goal of this subsection is to determine coefficients in the transport equations derived in Section 4.1 for filtration of a bidisperse suspension in a monodisperse porous medium. In view of monodispersity, all the distributions are independent of r_p. Further, it follows from Eq. (62), (63), that the distribution of the pore lengths l remains invariable:

$$f(l, x, t) = f_0(l) = H_0(l)/h_0.$$

Distribution $f_U(u|r_s, l, r_p)$ is equal to $f_{U1}(u|l)$ if $r_s = R_{s1}$ and $f_{U2}(u|l)$ if $r_s = R_{s2}$. Correspondingly, $\rho(r_s, x, t)$ is either $\rho(R_{s1}, x, t)$ or $\rho(R_{s2}, x, t)$. We denote these values by $\rho_i(x, t)$ $(i = 1, 2)$. According to Eq. (82) and the previous analysis,

$$\rho_i(x, t) = \int \frac{pl}{u}f_{Ui}(u|l)\delta(p - p_i)f_0(l)\mathrm{d}p\mathrm{d}l\mathrm{d}u = (const)_i. \qquad (83)$$

Thus, the values of ρ_i are independent of x, t. Similar analysis is possible for other coefficients. Consider, for example, $V(r_s, x, t)$. Similarly to coefficient ρ, it is reduced to $V_i(x, t)$ $(i = 1, 2)$. According to the transformed equation (81),

$$V_i = \frac{\int plf_{Ui}(u|l)\delta(p - p_i)f_0(l)\mathrm{d}p\mathrm{d}l\mathrm{d}u}{\int p\frac{l}{u}f_{Ui}(u|l)\delta(p - p_i)f_0(l)\mathrm{d}p\mathrm{d}l\mathrm{d}u} = (const)_i.$$

Hence, these coefficients are also independent of x, t. In other words, the coefficients in Eq. (61) become constants, since these coefficients depend on x, t via $f(l|r_p, x, t)$, and this function remains invariable in our case.

Further, if $P_i = P(R_i, l, r_p)$ are constants, then it follows from Eq. (11) that $P_{si} = P_s(R_i, x, t)$ are also constants equal to P_i. Therefore, $p_{si} = 1 - P_{si} = p_i$, and, as follows from the last Eq. (15), Eq. (16) (and taking into account re-norming transferring T_s into the values of ρ_i),

$$\lambda_{si} = \frac{p_i}{\rho_i}. \qquad (84)$$

References

[1] M. Elimelech, J. Gregory, X. Jia, R.A. Williams, Particle Deposition and Aggregation, Butterworth-Heinemann, USA, 1995.
[2] F. Civan, Reservoir Formation Damage, Gulf Publishing Company, USA, 2000.
[3] S.A. Bradford, S. Torkzaban, Vadose Zone J. 7 (2) (2008) 667–681.
[4] S.A Bradford, S. Torkzaban, S.L. Walker, Water Res. 41 (2007) 3012–3024.
[5] T. Iwasaki, J. Am. Water Works Ass. (1937) 1591–1602.
[6] J.P. Herzig, D.M. Leclerc, P. le Goff, Ind. Eng. Chem. 62 (8) (1970) 8–35.
[7] K. Khilar, S. Fogler, Migration of Fines in Porous Media, Kluwer Academic Publishers, Dordrecht, London, Boston, 1998.
[8] A. Santos, P. Bedrikovetsky, S. Fontoura, J. Membr. Sci. 308 (2008) 115–127.
[9] J.E. Altoé, P. Bedrikovetsky, A.L.S. de Souza, F.S. Shecaira, J. Pet. Sci. Eng. 51 (2006) 68–84.

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[10] M.M. Sharma, Y.C. Yortsos, AIChE J. 33 (10) (1987) 1636–1643; 1644–1653; 1654–1662.
[11] R. Ciegis, O. Iliev, Z. Lakdawala, Comput. Methods Appl. Math. 7 (2) (2007) 118–134.
[12] S. Rief, A. Latz, A. Wiegmann, Filtration 6 (2) (2006) 169–172.
[13] N. Tufenkji, J.A. Redman, M. Elimelech, Environ. Sci. Technol. 37 (2003) 616–623.
[14] S.A. Bradford, N. Toride, J. Environ. Qual. 36 (2007) 1346–1356.
[15] A. Bouddor, J.-L. Auriault, M. Mhamdi-Alaoui, Transp. Porous Media 25 (1996) 121–146.
[16] J.S. Chang, S. Vigneswaran, J.K. Kandasamy, L.J. Tsai, Separation Sci. Technol. 43 (2008) 1771–1784.
[17] T. Ilina, M. Panfilov, M. Buès, I. Panfilova, Tranp. Porous Med. 71 (2008) 311–329.
[18] B. Berkowitz, A. Cortis, M. Dentz, H. Scher, Rev. Geophys. 44 (2006) 1–49, RG2003.
[19] G. Kosakowski, B. Berkowitz, H. Scher, J. Contaminant Hydrol. 47 (2001) 29–51.
[20] N. Tufenkji, M. Elimelech, Langmuir 21 (2005) 841–852.
[21] A. Cortis, Phys. Rev. E 76 (2007) 030102(R).
[22] A. Cortis, Y. Chen, H. Scher, B. Berkowitz, Phys. Rev. E 70 (2004) 041108.
[23] A.A. Shapiro, P.G. Bedrikovetsky, A. Santos, O.O. Medvedev, Transp. Porous Med. 67 (2007) 135–164.
[24] P.G. Bedrikovetsky, Transp. Porous Med. 75 (2008) 335–369.
[25] E.W. Montroll, B.J. West, in: E.W. Montrol, J.L. Leibowitz (Eds.), Fluctuation Phenomena, North-Holland Publishing Company, 1979, pp. 62–175 (Chapter 2).
[26] A.A. Shapiro, Physica A 375 (2007) 81–96.
[27] A.A. Shapiro, P.G. Bedrikovetsky, Physica A 387 (2008) 5963–5978.
[28] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. II, Wiley, New York, 1979.
[29] G. Kampel, G.H. Goldsztein, J.C. Santamarina, Appl. Phys. Lett. 92 (2008) 084101.

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