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New Classes of Exact Solutions to Nonlinear Sets of Equations in the Theory of Filtration and Convective Mass Transfer

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Abstract—New classes of exact solutions to nonlinear sets of equations encountered in the theory of filtration and convective mass transfer of reacting media are described. Focus is placed on general first-order sets in which the chemical reaction rates depend on arbitrary functions. General solutions to some first-order systems with power-law nonlinearities are found. A set of new exact solutions with a functional separation of variables involving arbitrary functions is constructed. The results obtained are used for solving the problems of the theory of filtration of one-component and multicomponent suspensions with an arbitrary kinetics of particle accumulation.

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REDUCTION OF NONLINEAR EQUATIONS OF CONVECTIVE MASS TRANSFER AND FILTRATION TO CANONICAL FORM

Some exact solutions to nonlinear sets of equations of the first and second orders encountered in the theories of filtration and mass transfer of reactive media are described in the literature [1–17].

Consider the simplest nonlinear model of convective mass transfer in a two-component system with a bulk chemical reaction, which is described by a nonlinear sets of partial differential equations of the first order:

$$\frac{\partial u}{\partial \tau} + a_1\frac{\partial u}{\partial \xi} = F_1(u, w), \quad \frac{\partial w}{\partial \tau} + a_2\frac{\partial w}{\partial \xi} = F_2(u, w), \quad (1)$$

where τ is the time; ξ, the spatial coordinate; a_1 and a_2, the rates of convective mass transfer; F_1 and F_2, the rates of chemical reactions. It is assumed in writing system (1) that the diffusion of both components can be ignored. If the first (second) medium is quiescent, then $a_1 = 0$ ($a_2 = 0$).

System (1) with $a_2 = 0$ for the kinetic power-law functions

$$F_1(u, w) = \alpha u^n w^m, \quad F_2(u, w) = \beta u^n w^m$$

is used in the mathematical modeling of a two-phase bubble reactor [10, 11]. A similar system with $n = 1$ is encountered in the problems of the theory of filtration dealing with the desalting of soils by groundwaters [1, 3].

The set of equations (1) with

$$F_1(u, w) = F_2(u, w) = u + f(w),$$

is one of the main objects of study in the mathematical theory of the dynamics of sorption and chromatography [18–20].

It should be noted that set (1) is used for describing the stability of a plug-flow chemical tubular reactor [21] (as the diffusion proceeds at a low rate, the presence of the terms involving a second derivative can not cause a noticeable instability) and a continuous stirred reactor. Similar systems are also encountered in the simplest models of nonisothermal chemical reactors, where one of the quantities to be found is the concentration and the other is the temperature [21, 22]. The transition to characteristic variables

$$x = \frac{\xi - a_2\tau}{a_1 - a_2}, \quad t = \frac{\xi - a_1\tau}{a_2 - a_1}, \quad (a_1 \neq a_2)$$

allows us to reduce set (1) to canonical form:

$$\frac{\partial u}{\partial x} = F_1(u, w), \quad \frac{\partial w}{\partial t} = F_2(u, w). \quad (2)$$

Exact solutions to some systems of the form (2) will be given below.

The processes in which the deep filtration of a suspension of particles in a porous medium occurs are the pumping of the seawater accompanying recovered oil into reservoirs, the penetration of drilling waters into the reservoirs of a productive zone, the filtration of slurries from sand particles in gravel-bed filters, industrial filtration, the transportation of fine particles in oil fields, the carryover of impurities by groundwaters, the traveling of bacteria, viruses, and the like. The main feature of the process consists in the capture of particles by the porous medium, which causes a decrease in its permeability. The particles are captured due to size exclusion (particles are larger than pore sizes), surface adsorption, sedimentation, diffusion, the action of electric forces, and so on.

For a suspension with a single suspended component, the system describing the process consists of a material balance equation for the accumulated particles and suspension and an equation accounting for the accumulation kinetics [2, 7, 23]:

$$\frac{\partial(u+w)}{\partial t}+\frac{\partial u}{\partial x} = 0, \quad \frac{\partial w}{\partial t} = f(w)u, \tag{3}$$

where one of the components, u, is the suspension and the second, w, is the accumulated substance (deposit); $f(w)$ is the filtration coefficient.

Substituting the right-hand side of the second equation in system (3) for $\frac{\partial w}{\partial t}$ in the first equation and passing from the variables x and t to new characteristic variables $z = -x$ and $\eta = x - t$, we obtain set (2) of the special form:

$$\frac{\partial u}{\partial z} = uf(w), \quad \frac{\partial w}{\partial \eta} = -uf(w). \tag{4}$$

The solution of the boundary-value problem for the pumping of a suspension into the particle-free reservoir described by the set of equations (3) will be considered below.

TRANSFORMATIONS OF THE SETS EQUATIONS OF THE SPECIAL FORM

In solving chemical engineering problem, we usually consider systems (2) in which the kinetic functions are proportional to:

$$\frac{\partial u}{\partial x} = aF(u, w), \quad \frac{\partial w}{\partial t} = bF(u, w), \quad (ab < 0). \tag{5}$$

Introduction of an analog of the stream function $\varphi = \varphi(x, t)$ using the formulas

$$u = a\frac{\partial \varphi}{\partial t}, \quad w = b\frac{\partial \varphi}{\partial x}$$

reduces system (5) to a nonlinear hyperbolic equation of the second order:

$$\frac{\partial^2 \varphi}{\partial x \partial t} = F\left(a\frac{\partial \varphi}{\partial t}, b\frac{\partial \varphi}{\partial x}\right).$$

Some equations of this kind are considered by Polyanin et al. [24].

Any the set of equations of equations of the special form

$$\frac{\partial u}{\partial x} = f_1(u)g_1(w), \quad \frac{\partial w}{\partial t} = f_2(u)g_2(w) \tag{6}$$

can be reduced using the transform

$$\bar{u} = a\int\frac{f_2(u)}{f_1(u)}du, \quad \bar{w} = b\int\frac{g_1(w)}{g_2(w)}dw \tag{7}$$

to a particular case of the set of equations (5):

$$\frac{\partial \bar{u}}{\partial x} = af(\bar{u})g(\bar{w}), \quad \frac{\partial \bar{w}}{\partial t} = bf(\bar{u})g(\bar{w}).$$

Here, we used the notation $f(\bar{u}) \equiv f_2[u(\bar{u})]$, and $g(\bar{w}) \equiv g_1[w(\bar{w})]$; the functions $u(\bar{u})$ and $w(\bar{w})$ are found by the inversion of functions (7).

Using the transform

$$\tilde{u} = \int\frac{du}{f_1(u)}, \quad \tilde{w} = \int\frac{dw}{g_2(w)}, \tag{8}$$

the set of equations (6) is reduced to canonical form:

$$\frac{\partial \tilde{u}}{\partial x} = \Phi(\tilde{w}), \quad \frac{\partial \tilde{w}}{\partial t} = \Psi(\tilde{u}). \tag{9}$$

Here, we used the notation $\Phi(\tilde{w}) \equiv g_1[w(\tilde{w})]$ and $\Psi(\tilde{u}) \equiv f_2[u(\tilde{u})]$; the functions $u(\tilde{u})$ and $w(\tilde{w})$ are found by the inversion of functions (8).

Canceling out the variable u in Eq. (9), we come to a nonlinear hyperbolic equation:

$$\frac{\partial^2 \tilde{w}}{\partial x \partial t} = \Phi(\tilde{w})\Theta\left(\frac{\partial \tilde{w}}{\partial t}\right), \tag{10}$$

where $\Theta(z) = \Psi'_u(u)|_{z - \Psi(u) = 0}$.

If $\Psi(\tilde{u}) = a\tilde{u} + b$, then $\Theta(z) =$ constant and Eq. (10) can be completely integrated in four cases: $\Phi(w) = k_1 w + k_2$ (linear equation), $\Phi(w) = ke^{\lambda w}$ (Liouville equation), $\Phi(w) = k\sin(\lambda w + \sigma)$ (sine-Gordon equation), and $\Phi(w) = k\sinh(\lambda w)$ (hyperbolic sine-Gordon equation) [24, 25].

EXACT SOLUTIONS TO NONLINEAR SYSTEMS OF EQUATIONS OF CONVECTIVE MASS TRANSFER

In this section, we will give exact solutions to some classes of nonlinear sets of first-order equations of the form (2) to which the equations of convective mass transfer in two-component systems with a bulk chemical reaction without diffusion are reduced.

It is obvious that the set of equations (2) generally admits of exact traveling-wave solutions:

$$u = u(z), \quad w = w(z), \quad z = kx - \lambda t,$$

where k and λ are arbitrary constants and the functions $u(z)$ and $w(z)$ are described by an autonomous set of ordinary differential solutions:

$$ku'_z - F_1(u, w) = 0, \quad \lambda w'_z + F_2(u, w) = 0.$$

System 1. Consider a special case of the set of equations (2) in which the rates of chemical reactions involve a power-law function of the concentration of one of the reacting components:

$$\frac{\partial u}{\partial x} = uf(w), \quad \frac{\partial w}{\partial t} = u^k g(w). \tag{11}$$

When $k = 1$ and $g(w) = -f(w)$, obvious changes of variables in the set of equations (11) can transform it into the set of equations (4), which is encountered in the theory of filtration. When $k = 1$ and $g(w) = \text{const} \times f(w)$, the set of equations (11) is the special form of system (5).

In particular cases where $k = 0$ and $f(w) = \text{const}$, one of the equations can be solved independently of the other and the set of equations (11) can be easily integrated. In the consideration that follows, $k \neq 0$ and $f(w)$ $\neq$ constant.

The transformation of the dependent variables using

$$U = u^k, \quad W = \int \frac{dw}{g(w)} \tag{12}$$

allows us to come to a simpler set of equations:

$$\frac{\partial U}{\partial x} = \Phi(W)U, \quad \frac{\partial W}{\partial t} = U, \tag{13}$$

where the function $\Phi(W)$ is defined in parametric form (w is the parameter):

$$\Phi = kf(w), \quad W = \int \frac{dw}{g(w)}. \tag{14}$$

Substituting the left hand side of the second equation in the set of equations (13) for U in its first equation yields a second-order equation for the function W:

$$\frac{\partial^2 W}{\partial x \partial t} = \Phi(W)\frac{\partial W}{\partial t}.$$

Integrating the latter equation with respect to t, we obtain

$$\frac{\partial W}{\partial x} = \int \Phi(W)dW + \theta(x). \tag{15}$$

Equation (15) can be considered as an ordinary differential equation of the first order with respect to the variable x. After its general solution is obtained, it is necessary to replace the integration constant C in it with an arbitrary time function $\psi(t)$ because w depends on x and t.

Using formulas (12) and (14) to transfer to the initial variable w, we obtain

$$\frac{\partial w}{\partial x} = kg(w)\int \frac{f(w)}{g(w)}dw + \theta(x)g(w). \tag{16}$$

For the particular case of $\theta(x) = \text{constant}$ in Eq. (16), the set of equations (11) has special solutions:

$$w = w(z), \quad u = [\psi'(t)]^{1/k} v(z), \quad z = x + \psi(t),$$

where the prime stands for the derivative and the functions $w(z)$ and $v(z)$ are described by an autonomous set of ordinary differential equations:

$$v'_z = f(w)v, \quad w'_z = g(w)v^k. \tag{17}$$

The general solution to the above system can be written in the implicit form:

$$\int \frac{dw}{g(w)[kF(w) + C_1]} = z + C_2,$$
$$v = [kF(w) + C_1]^{1/k}, \quad F(w) = \int \frac{f(w)}{g(w)}dw. \tag{18}$$

Examples of constructing general solutions to some nonlinear systems of equations of the form (11) using Eq. (16) are given below.

Example 1. Consider sets of equations with power-law nonlinearities:

$$\text{(a)} \quad \frac{\partial u}{\partial x} = auw^n, \quad \frac{\partial w}{\partial t} = bu^k w \tag{19}$$

is a particular case of the set of equations (11) with $f(w) = aw^n$, $g(w) = bw$. Using Eq. (16), this case can be reduced to the Bernoulli equation:

$$w'_x = \frac{ak}{n}w^{n+1} + bw\theta(x).$$

Its general solution is described by the formulas:

$$u = \left(\frac{\xi}{\psi(t) - ak\int \xi dx}\right)^{1/n}, \quad \xi = \exp(bn\int \theta(x)dx).$$

As the arbitrary function $\theta(x)$ is involved in the solution, it is convenient to introduce a new variable $\varphi(x) = \int \xi dx$. As a result, the general solution to the set of equations (19) is written as

$$u = \left(\frac{-\psi'_t(t)}{bn\psi(t) - ak\varphi(x)}\right)^{1/k},$$
$$w = \left(\frac{\varphi'_x(x)}{bn\psi(t) - ak\varphi(x)}\right)^{1/n}.$$

Here, the change of variable $\psi \longrightarrow bn\psi$ was made.

The common case of a chemical reaction of the second order corresponds to the values of $n = k = 1$. Solutions to some initial- and boundary-value problems in the theory of filtration and in the theory of chemical reactors based on the set of equations (19) with $n = k = 1$ are already obtained [1, 3, 10, 11].

$$\text{(b)} \quad \frac{\partial u}{\partial x} = auw^n, \quad \frac{\partial w}{\partial t} = bu^k w^{1-n} \tag{20}$$

is a particular case of the set of equations (11) with $f(w) = aw^n$, $g(w) = bw^{1-n}$. Substituting these functions into (16), we obtain

$$w'_x = \frac{ak}{2n}w^{n+1} + b\theta(x)w^{1-n}.$$

The change $U = w^n$ reduces this equation to the Riccati equation:

$$U'_x = \frac{1}{2}akU^2 + bn\theta(x). \tag{21}$$

Using the notation $bn\theta = \varphi'_x - \frac{1}{2}ak\varphi^2$, we have the particular solution $U = \varphi(x)$ to Eq. (21). The general solution to the Riccati equation can be expressed in terms of the particular solution. As a result, the general solution to the set of equations (20) is written as

$$w = \left\{\varphi(x) + E(x)\left[\psi(t) - \frac{1}{2}ak\int E(x)dx\right]^{-1}\right\}^{1/n},$$

$$u = \left(\frac{1}{b}w^{n-1}\frac{\partial w}{\partial t}\right)^{1/k}, \quad E(x) = \exp[ak\int\varphi(x)dx],$$

where $\varphi(x)$ and $\psi(t)$ are arbitrary functions.

Solutions to some initial- and boundary-value problems in the theory of filtration based on Eqs. (20) with $k = 1$ and $n = 1/2$ are already obtained [1, 11].

Example 2. Similarly, we can show that the general solution to the system

$$\frac{\partial u}{\partial x} = \frac{cw^n u}{a + bw^n}, \quad \frac{\partial w}{\partial t} = (aw + bw^{n+1})u^k$$

with $b \neq 0$ can be written as

$$w = [\psi(t)e^{F(x)} - be^{F(x)}\int e^{-F(x)}\varphi(x)dx]^{-1/n},$$

$$u = \left(\frac{w'_t}{aw + bw^{n+1}}\right)^{1/k}, \quad F(x) = \frac{ck}{b}x - a\int\varphi(x)dx.$$

It should be noted that in the first equation of the set of equations (11) the function f can additionally depend on the variable x. In this case, $f(w)$ should be substituted for $f(x, w)$ in integral (16).

It should be noted that the system of equations

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + f(w)u, \quad \frac{\partial w}{\partial t} = a\frac{\partial w}{\partial x} + g(w)u^k \tag{22}$$

can be reduced to the set of equations (17) with $z = x/a$ by using the change of variables from x and t to x and $\xi = x + at$. Consequently, the general solution to the set of equations (22) can be obtained using formulas (18) with $z = x/a$, $C_1 = \varphi(x + at)$, and $C_2 = \psi(x + at)$.

For a more complicated set of equations

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + f(w)u, \quad \frac{\partial w}{\partial t} = g(w)u,$$

in which the first equation additionally involves diffusional and convective terms, canceling u from the first equation with the help of the second followed by the integration of the resulting equation with respect to t yields the following second-order equation for the function w:

$$\frac{\partial w}{\partial t} = ag(w)\frac{\partial}{\partial x}\left[\frac{1}{g(w)}\frac{\partial w}{\partial x}\right] + b\frac{\partial w}{\partial x}$$
$$+ g(w)\int\frac{f(w)}{g(w)}dw + \theta(x)g(w).$$

System 2. The set of equations

$$\frac{\partial u}{\partial x} = u^m f(w), \quad \frac{\partial w}{\partial t} = u^k g(w)$$

with $k \neq 0$ and $m \neq 1$ admits of the self-similar solution

$$u = t^{-\frac{1}{k}}V(z), \quad w = W(z), \quad z = xt^{\frac{1-m}{k}},$$

where the functions $V(z)$ and $W(z)$ are described by the following set of ordinary differential equations

$$V'_z = V^m f(W), \quad (1-m)zW'_z = kV^k g(W).$$

It should be noted that when $k = m$ and $g(w) = \text{constant} \times f(w)$, the system under consideration is the special form of the set of equations (5).

System 3. Consider a set of equations that involves two arbitrary functions depending on linear combinations of the variables to be found:

$$\frac{\partial u}{\partial x} = f(a_1u + b_1w), \quad \frac{\partial w}{\partial t} = g(a_2u + b_2w).$$

Its solution obtained using the additive separation of variables when $\Delta = a_1b_2 - a_2b_1 \neq 0$ can be written as

$$u = \frac{1}{\Delta}[b_2\phi(x) - b_1y(t)],$$

$$w = \frac{1}{\Delta}[a_1y(t) - a_2\phi(x)],$$

where the functions $\phi(x)$ and $y(t)$ are described by the ordinary differential equations

$$\frac{b_2}{\Delta}\phi'_x = f(\phi), \quad \frac{a_1}{\Delta}y'_t = g(y).$$

Their integration yields

$$\frac{b_2}{\Delta}\int\frac{d\phi}{f(\phi)} = x + C_1, \quad \frac{a_1}{\Delta}\int\frac{dy}{g(y)} = t + C_2.$$

When $a_1 = a_2 = a$ and $b_1 = b_2 = b$, the solution takes the form:

$$u = b(k_1x - \lambda_1 t) + y(\xi),$$
$$w = -a(k_1x - \lambda_1 t) + z(\xi), \quad \xi = k_2x - \lambda_2 t,$$

where $k_1, k_2, \lambda_1, \lambda_2$ are arbitrary constants, the functions $y(\xi)$ and $z(\xi)$ are described by an autonomous set of ordinary differential equations:

$$k_2y'_\xi + bk_1 = f(ay + bz),$$
$$-\lambda_2 z'_\xi + a\lambda_1 = g(ay + bz).$$

When $a_1 = a_2$, $b_1 = b_2$, and $g(w) = \text{constant} \times f(w)$, the system under consideration is the special form of the set of equations (5).

System 4. The set of equations

$$\frac{\partial u}{\partial x} = e^{\lambda u}f(au + bw), \quad \frac{\partial w}{\partial t} = e^{\beta u}g(au + bw),$$

has an exact solution

$$u = U(z) - \frac{1}{\lambda}\ln(x + C_1),$$
$$w = W(z) + \frac{a}{b\lambda}\ln(x + C_1), \quad z = \frac{t + C_2}{(x + C_1)^{\beta/\lambda}},$$

where the functions $U(z)$ and $W(z)$ are described by the set of ordinary differential equations

$$\beta z U'_z + 1 = -\lambda e^{\lambda U}f(aU + bW),$$
$$W'_z = e^{\beta U}g(aU + bW).$$

When $\beta = \lambda$ and $g(z) = \text{constant} \times f(z)$, the system under consideration is the special form of system (5).

System 5. Consider the set of equations

$$\frac{\partial u}{\partial x} = uf(au^k + bw), \quad \frac{\partial w}{\partial t} = cu^k, \tag{23}$$

which involves an arbitrary function $f(z)$. Using the second equation in (23), we obtain

$$u = \left(\frac{1}{c}\frac{\partial w}{\partial t}\right)^{1/k}. \tag{24}$$

Canceling u from the first equation in (23) with the help of (24) yields the following second-order equation for the function w:

$$\frac{\partial^2 w}{\partial x \partial t} = k\frac{\partial w}{\partial t}f\left(\frac{a}{c}\frac{\partial w}{\partial t} + bw\right).$$

Its equation can be solved using the generalized separation of variables:

$$w = \varphi(x) + C\exp[-\lambda t + k\int f(b\varphi(x))dx],$$
$$u = \left(\frac{1}{c}\frac{\partial w}{\partial t}\right)^{1/k}, \quad \lambda = \frac{bc}{a}.$$

The function u is found from formula (24).

System 6. The set of equations

$$\frac{\partial u}{\partial x} = uf(au^n + bw), \quad \frac{\partial w}{\partial t} = u^k g(au^n + bw)$$

with $n \neq k$ and $ab \neq 0$ can be solved using the generalized separation of variables:

$$u = (C_1t + C_2)^{\frac{1}{n-k}}y(x),$$
$$w = \phi(x) - \frac{a}{b}(C_1t + C_2)^{\frac{n}{n-k}}[y(x)]^n,$$

where the functions $y = y(x)$ and $\phi = \phi(x)$ are described by the set of differential and algebraic equations

$$y'_x = yf(b\phi), \quad y^{n-k} = \frac{b(k-n)}{aC_1n}g(b\phi).$$

When $k = 1$ and $g(z) = \text{constant} \times f(z)$, the set under consideration is the special form of the set of equations (5).

System 7. Consider a set of equations involving two arbitrary functions depending on a complex argument:

$$\frac{\partial u}{\partial x} = u^k f(u^n w^m), \quad \frac{\partial w}{\partial t} = w^s g(u^n w^m).$$

Its self-similar solution for the case of $s \neq 1$ and $n \neq 0$ can be written as

$$u = t^{\frac{m}{n(s-1)}}y(\xi), \quad w = t^{-\frac{1}{s-1}}z(\xi), \quad \xi = xt^{\frac{m(k-1)}{n(s-1)}},$$

where the functions $y(\xi)$ and $z(\xi)$ are described by the set of ordinary differential equations

$$y'_\xi = y^k f(y^n z^m),$$
$$m(k-1)\xi z'_\xi - nz = n(s-1)z^s g(y^n z^m).$$

It should be noted that when $k = s$ and $g(z) = \text{constant} \times f(z)$, the set of equations under consideration is the special form of system (5).

The limiting self-similar solution with $s = 1$ takes the form:

$$u = e^{mt}y(\xi), \quad w = e^{-nt}z(\xi), \quad \xi = e^{m(k-1)t}x,$$

where the functions $y(\xi)$ and $z(\xi)$ are described by the set of ordinary differential equations

$$y'_\xi = y^k f(y^n z^m), \quad m(k-1)\xi z'_\xi - nz = zg(y^n z^m).$$

The exact solution for the case of $k = 1$ and $s = 1$ is written as

$$u = e^{m(px-\lambda t)}y(\xi), \quad w = e^{-n(px-\lambda t)}z(\xi),$$
$$\xi = \alpha x - \beta t,$$

where p, α, β, and λ are arbitrary constants, the functions $y(\xi)$ and $z(\xi)$ are described by an autonomous set of ordinary differential equations:

$$\alpha y'_\xi + mpy = yf(y^n z^m), \quad -\beta z'_\xi + n\lambda z = zg(y^n z^m).$$

System 8. The set of equations with power-law nonlinearity

$$\frac{\partial u}{\partial x} = f_1(x,t)u + g_1(x,t)u^{1-n}w^m,$$

$$\frac{\partial w}{\partial t} = f_2(x,t)w + g_2(x,t)u^n w^{1-m},$$

can be reduced using the transform $U = u^n$, $W = w^m$ to the linear set of equations

$$\frac{\partial U}{\partial x} = nf_1(x,t)U + ng_1(x,t)W,$$

$$\frac{\partial W}{\partial t} = mf_2(x,t)W + mg_2(x,t)U.$$

When $f_{1,2}$ = constant and $g_{1,2}$ = constant, the general solution to this set of equations can be obtained by reducing it to a linear second-order equation with constant coefficients. The solutions to some initial- and boundary-value problems in the theory of chemical reactors based on the initial nonlinear set of equations with $f_{1,2} = 0$, $g_{1,2}$ = constant, and $n = m = 1/2$ are already obtained [10, 11].

SOME NONLINEAR INITIAL- AND BOUNDARY-VALUE PROBLEMS FOR THE TRANSPORT OF SUSPENSIONS IN A POROUS MEDIUM

The problem for the pumping of a suspension into a particle-free reservoir is described by the set of equations (3) with the following initial and boundary conditions:

$$u = w = 0 \text{ when } t = 0, \quad u = 1 \text{ when } x = 0. \qquad (25)$$

As indicated above, the set of equations (3) can be reduced to the set of equations (4), which is a particular case of the set of equations (11), and its solution can be reduced to the integration of an ordinary differential equation. Below, problem (3) with conditions (25) will be solved using another method.

Following the method [7], we introduce a potential

$$\Phi(w) = \int_0^w \frac{dz}{f(z)}, \qquad (26)$$

which makes it possible to reduce the second equation in the set of equations (3) to the form:

$$u = \frac{\partial \Phi(w)}{\partial t}. \qquad (27)$$

The substitution of (27) into the first equation in system (3) and integration from 0 to t according to initial conditions (25) leads to the following quasi-linear partial differential equation of the first order:

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = -f(w)w. \qquad (28)$$

Assuming that $u = 1$ in the second equation of (3) at the inlet ($x = 0$), we provide the satisfaction of the boundary condition for Eq. (28):

$$\Phi(w) = t \quad \text{when } x = 0. \qquad (29)$$

Problem (28), (29) can be solved by the method of characteristics and written in implicit form:

$$\int_w^{\Phi^{-1}(t-x)} \frac{dz}{zf(z)} = x, \qquad (30)$$

where $\Phi^{-1}(w)$ is the function inverse to (26).

To obtain an expression for the suspension concentration $u(x, t)$, it is necessary to differentiate both sides of expression (30) with respect to time:

$$\left(\frac{1}{wf(w)}\frac{\partial w}{\partial t}\right)_{x,t} - \left(\frac{1}{wf(w)}\frac{\partial w}{\partial t}\right)_{0,t-x} = 0.$$

Replacing the partial derivative with the left-hand side of the second equation in the set of equations (3), we obtain

$$\frac{u(x,t)}{w(x,t)} = \frac{u(0,t-x)}{w(0,t-x)}. \qquad (31)$$

This implies that the ratio u/w remains unchanged along the characteristic lines. Using boundary conditions (25) and (29) together with (31), we obtain an expression for the suspension concentration:

$$u(x,t) = \frac{w(x,t)}{\Phi^{-1}(t-x)}. \qquad (32)$$

Consequently, formulas (30), (32), and (26) describe the exact solution in implicit form to problem (3), (25) for $x < t$. For $x > t$, the functions u and w are equal to zero.

Example 1. Consider the set of equations (3) with a constant filtration coefficient $f(w) = \lambda$. In this case, its solution can be written as

$$u = e^{-\lambda x}, \quad w = \lambda(t-x)e^{-\lambda x}.$$

Example 2. Consider the nonlinear set of equations (3) with a linear filtration coefficient:

$$f(w) = 1 - w.$$

Using (26), we can find the potential:

$$\Phi(w) = -\ln(1-w).$$

Using (29), we can determine the boundary value:

$$w(0, t) = \Phi^{-1}(t) = 1-e^{-t},$$

The left-hand side of (30) can be evaluated in implicit form:

$$\ln\frac{w}{1-w} - \ln(1-e^{-t+x}) - t + x = -x,$$

The latter can be used to obtain the solution:

$$u(x, t) = \frac{e^{t-x}}{e^x + e^{t-x} - 1},$$

$$w(x, t) = \frac{e^{t-x} - 1}{e^x + e^{t-x} - 1}.$$

It should be noted that a list of functions $f(w)$ with which integrals (26), (30) or transcendental equation (29) are evaluated in explicit form is given by Herzig et al. (2).

Consider a set of equations more general than (3):

$$\frac{\partial}{\partial t}g(u, w) + a\frac{\partial u}{\partial x} = f_1(w)u, \quad \frac{\partial w}{\partial t} = f_2(w)u.$$

Writing the expression for u from the second equation, substituting it into the first equation and integrating the resulting equation with respect to t, we obtain a partial differential equation of the first order for the function $w(x, t)$:

$$g\left(\frac{1}{f_2(w)}\frac{\partial w}{\partial t}, w\right) + \frac{a}{f_2(w)}\frac{\partial w}{\partial x} = \int\frac{f_1(w)}{f_2(w)}dw + \theta(x). \quad (33)$$

If θ = constant, then the complete integral of Eq. (33) has the form $w = w(C_1x + C_2t + C_3)$. In this case, we can obtain a general integral (containing an arbitrary function) of Eq. (33) in parametric form.

It should be noted that when the right-hand side of the material balance equation contains an additional diffusion term; that is, for the set of equations

$$\frac{\partial}{\partial t}g(u, w) + a\frac{\partial u}{\partial x} = b\frac{\partial^2 u}{\partial x^2} + f_1(w)u,$$

$$\frac{\partial w}{\partial t} = f_2(w)u,$$

the elimination of u from the first equation with the help of the second followed by the integration of the resulting equation with respect to the time t gives the following equation for the function w:

$$g\left(\frac{1}{f_2(w)}\frac{\partial w}{\partial l}, w\right) + \frac{a}{f_2(w)}\frac{\partial w}{\partial x}$$
$$= b\frac{\partial}{\partial x}\left[\frac{1}{f_2(w)}\frac{\partial w}{\partial x}\right] + \int\frac{f_1(w)}{f_2(w)}dw + \theta(x).$$

Consider the generalization of Eq. (3) for the case of a multicomponent system:

$$\frac{\partial}{\partial t}[u + G(w_1, \ldots, w_n)] + \frac{\partial u}{\partial x} = 0, \quad (34)$$

$$\frac{\partial w_k}{\partial t} = F_k(w_1, \ldots, w_n)u, \quad k = 1, \ldots, n. \quad (35)$$

This case is characterized by the accumulation of particles of different components with corresponding concentrations $w_1, \ldots, w_n$.

It follows from the equations of the set of equations (35) that

$$\frac{1}{F_1(w_1, \ldots, w_n)}\frac{\partial w_1}{\partial t} = \ldots$$
$$= \frac{1}{F_n(w_1, \ldots, w_n)}\frac{\partial w_n}{\partial t} = u. \quad (36)$$

To find the exact solution to the set of equations (34)–(35), we assume that it can be written in the special form:

$$w_1 = w_1(w_n), \ldots, w_{n-1} = w_{n-1}(w_n), \quad (37)$$

which implies that the functions $w_1, \ldots, w_{n-1}$ can be expressed in terms of w_n. Using (37) and expressions (36), we obtain a set of $(n-1)$ ordinary differential equations

$$\frac{dw_k}{dw_n} = \frac{F_k(w_1, \ldots, w_n)}{F_n(w_1, \ldots, w_n)}, \quad k = 1, \ldots, n-1. \quad (38)$$

Then, assuming that the solution to the set of equations (38) is found and functions (37) are known, their substitution into (34) and (35) gives a set of two equations

$$\frac{\partial}{\partial t}[u + g(w_n)] + \frac{\partial u}{\partial x} = 0, \quad \frac{\partial w_n}{\partial t} = f_n(w_n)u, \quad (39)$$

where $g(w_n) = G[w_1(w_n), \ldots, w_{n-1}(w_n), w_n]$ and $f_n(w_n) = F_n[w_1(w_n), \ldots, w_{n-1}(w_n), w_n]$.

We introduce a new dependent variable into (39):

$$w = g(w_n) \equiv G[w_1(w_n), \ldots, w_{n-1}(w_n), w_n]. \quad (40)$$

This results in the set of equations (3) in which the function $f = f(w)$ is defined parametrically:

$$f = g'(w_n)f_n(w_n), \quad w = g(w_n), \quad (41)$$

where w_n is the parameter.

This solution of the above initial- and boundary-value problem can be applied to problems with initial and boundary conditions of the form:

$$w_1 = \ldots = w_n = u = 0 \quad \text{when } t = 0,$$
$$u = 1 \quad \text{when } x = 0.$$

Here, the initial conditions correspond to the absence of particles in the reservoir at the initial moment of time, and the boundary condition corresponds to the suspension concentration in the injected liquid. In this case, set (38) of ordinary differential equations is solved with the following initial conditions:

$$w_1 = \ldots = w_{n-1} = 0 \quad \text{when} \quad w_n = 0.$$

It should be noted that in the problem for the flow of an n-component liquid through a porous medium, the material balance equation for accumulated and suspended particles involves a function G defined as the sum of individual suspension components:

$$G(w_1, \ldots, w_n) = \sum_{k=1}^{n} w_k.$$

If all filtration coefficients in (38) are constant quantities, the total filtration coefficient, which is equal to the sum of filtration coefficients for every component, will be constant.

Example 3. Let the first filtration coefficient be a linear function of the concentration of the accumulated component and the second filtration coefficient be constant:

$$f_1 = \lambda(1 - w_1), \quad f_2 = \text{const}.$$

In this case, Eq. (38) has the first integral:

$$\frac{w_2}{f_2} = \int \frac{dw_1}{\lambda(1 - w_1)} = \frac{\ln(1 - w_1)}{\lambda},$$

This allows us to calculate the total concentration of the accumulated substance:

$$w(w_1) = w_1 + w_2 = w_1 + \ln(1 - w_1)^{f_2/\lambda}.$$

Consequently, the expression for the total filtration coefficient takes the form:

$$f(w) = \lambda\left[2 - \exp\left(\frac{\lambda}{f_2}w\right)\right] + f_2.$$

Example 4. The first filtration coefficient is a quadratic function of the concentration of the accumulated substance and the second coefficient is constant:

$$f_1 = a(w_1 + b)^2, \quad f_2 = \text{constant}.$$

In this case, the set of equations (38) likewise has the first integral:

$$\frac{w_2}{f_2} = \int \frac{dw_1}{a(w_1 + b)^2} = \frac{w_1}{ab(w_1 + b)}.$$

The total concentration of the accumulated substance w can be written in terms of the concentration of the first component w_1:

$$w(w_1) = w_1 + w_2 = w_1 + \frac{w_1 f_2}{ab(w_1 + b)}.$$

The inverse function $w_1(w)$ makes it possible to write the concentration of the first component of the accumulated substance in terms of the total concentration:

$$w_1 = \frac{1}{2}\left(w - \frac{f_2}{ab} - b \pm \sqrt{\left(w - \frac{f_2}{ab} - b\right)^2 + 4wb}\right).$$

Consequently, the total filtration coefficient can be written as

$$f(w) = \frac{a}{4}\left(w - \frac{f_2}{ab} - b \pm \sqrt{\left(w - \frac{f_2}{ab} - b\right)^2 + 4wb}\right).$$

The reduction of the set of equations (34)–(35) describing the filtration of a suspension with n mechanisms of particle collection to the set of equations (3) with one particle-collection mechanism by applying transforms (39)–(41) makes it possible to use exact solution (30)–(32) for interpreting laboratory data and determining the sizes of particles in the system with n collection mechanisms. This also gives a chance to use the three-dimensional model of the filtration of a suspension with one mechanism of particle collection for simulating the three-dimensional filtration of a suspension with concurrent diffusion, gravitation, sorption, and electrical particle-collection mechanisms.

The explicit solutions can be used in engineering calculations for estimating the process of solution penetration into the soil during drilling, determining the profile of collected particles, decreasing the intake capacity of boreholes during the pumping of waste water into oil fields, and the like. The explicit solution can also be used to derive formulas for estimating the distribution of concentration fields of viruses, bacteria, and Coli particles in underground waters.

NOTATION

C, C_1, C_2—arbitrary constants;

f, f_1, f_2, g, g_1, g_2—some nonlinear functions of their arguments;

$u, w, w_1, \ldots, w_n$—concentrations of components;

φ, ψ, θ—arbitrary functions of their arguments.

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