A New Splitting Scheme and Existence of Elliptic Region for Gasflood Modeling

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Summary

Analytical models of gasflooding are important for enhanced-oil-recovery (EOR) screening, for interpretation of laboratory data, and for streamline modeling. Introduction of two Lagrangian coordinates linked with one of the components and with the overall two-phase flux results in splitting the compositional model into an auxiliary system and an independent scalar equation. The number of equations in the auxiliary system is less by one if compared with the compositional model, making analytical modeling possible for more practical cases. The auxiliary system contains only thermodynamic functions and is independent of transport properties. Therefore, phase transitions and minimum miscibility pressure for gas injection are also independent of transport properties. The new splitting method is applicable for both self-similar solutions of continuous gas injection and nonself-similar solvent-slug problems. Analytical solution for four-component oil displacement by a nitrogen-based solvent was obtained using the splitting technique. The compositional two-phase model contains an elliptic region if and only if an elliptic region is also present in the auxiliary system. Calculations for several four-component mixtures exhibit existence of an elliptic region in compositional modeling.

Introduction

Miscible displacement is characterized by the injection of fluids that mix totally or partially with reservoir fluid (Lake 1989; Latil 1980). Basically, there are three main distinct miscible-hydrocarbon-solvent processes—miscible-slug injection, enriched-gas injection, and high-pressure lean-gas injection (van Poolen 1980). The miscible-slug process consists of the injection of a slug of liquid hydrocarbon driven by a chase fluid, which may be natural gas or even water. The enriched-gas process is essentially the injection of a slug of enriched natural gas displaced by lean gas or water. In the third process, lean gas is injected at high pressure to achieve retrograde evaporation of oil and the formation of a miscible phase between gas and oil phases flowing in the reservoir. The most important technical problem of miscible-hydrocarbon injection, besides its cost, is related to the high mobility ratio of solvent to crude oil. As oil price increased, carbon dioxide became a natural substitute for hydrocarbons in miscible flooding processes. It is also suitable for continuous injection during production lifetime, depending on the thermodynamic behavior of its mixture with reservoir fluid. Recently, more attention has been given to the injection of inert gas, such as nitrogen (N_2) or flue gas, with promising results.

The derivation of analytical models for two-phase multicomponent displacement was motivated by the planning and result-interpretation of partially miscible corefloods, by performing sensitivity studies and EOR screening and by developing streamline simulators. An $(n-1)\times(n-1)$ hyperbolic system of conservation laws describes 1D two-phase displacement of oil by gas at large scale, assuming mixture-volume conservation, where n is the number of components (Johns et al. 1993; Orr et al. 1995; Johns and Orr 1996; Wang and Orr 1997; Orr 2007). The continuous injection of gas results in a Riemann problem, while the displacement of oil by a gas slug with another gas drive is described by an initial and boundary-value problem with piecewise initial data (Bedrikovetsky 1993).

The elementary hyperbolic waves of the 2×2 system for two-phase three-component displacement can be described both analytically and graphically (Wachmann 1964; Hirasaki 1981; Hirasaki 1982; Zick 1986). Analytical 1D models for continuous gas injection with different types of ternary phase diagrams and boundary conditions related to injection of different fluids were developed by solving the Riemann problem that arises in these cases (Lake 1989; Wachmann 1964; Hirasaki 1981; Hirasaki 1982; Zick 1986). Injection of gaseous solvents with gas drives also allows for analytical modeling by solving the problem of elementary-wave interactions (Bedrikovetsky 1993; Barenblatt et al. 1991).

The semianalytical solutions for n-component gasflooding, obtained by numerical combination of shock and rarefaction waves, allow thermodynamic analysis, minimum-miscibility-pressure calculations, and recovery estimates (Johns et al. 1993; Orr et al. 1995; Johns and Orr 1996; Wang and Orr 1997; Orr 2007; Bedrikovetsky 1993). This technique was developed for any number of components.

A hyperbolic system for partially miscible gas injection is similar to that for polymer flooding. The solution of two-phase multicomponent displacement of oil by a polymer solution can be projected into that for one phase. With the projection, elementary waves of a two-phase system are mapped onto those for a one-phase system. The observation that concentration waves in a two-phase environment can be lifted from one-phase multicomponent flow was used for the development of a semianalytical Riemann-problem solver for two-phase n-component polymer flooding (Johansen and Winther 1989; Johansen et al. 1989; Dahl et al. 1992). The exact solutions for this problem, with adsorption governed by a Langmuir isotherm, were obtained by use of a one-phase solution (Johansen and Winther 1989; Rhee et al. 1970). The projection technique cannot be extended for nonself-similar problems of oil displacement by polymer slugs.

A projection approach analogous to that of one-/two-phase polymer flooding was applied to derive the exact solutions for gasflooding for the case of n-component ideal mixtures (Bedrikovetsky and Chumak 1992a; Bedrikovetsky and Chumak 1992b). The model of ideal mixtures is analogous to a multicomponent Langmuir sorption; it results in full exact integrability of one-phase and two-phase cases. The projection maps the two-phase compositional system onto the reduced system with the number of equations less by one. These solutions, along with the definition of the reduced system, were further used for the injection of different gases into different oils (Entov 1997; Entov and Voskov 2000). The projection technique can be applied for real mixtures; it involves equation-of-state-based flash calculations and the semianalytical combining of shock and rarefaction waves (Johansen et al. 2005; Wang et al. 2005). For two-phase compositional models, the projection technique is valid only for the case of continuous gas injection and cannot be applied for nonself-similar slug-injection problems.

Another approach, which is the introduction of the Lagrangian coordinate linked to aqueous-phase conservation and using it with an Eulerian coordinate instead of the traditional independent variables Eulerian coordinate and time, also results in splitting the polymer-flood system into an auxiliary thermodynamics

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system containing only sorption isotherms and a scalar hydrodynamics equation containing relative permeability and viscosity of both phases (Pires et al. 2006). The number of equations in the auxiliary system is less by one than that of the total system, which results in several analytical models. This splitting technique is valid for any initial-boundary-value problem, including the problems of continuous and slug injections.

The problem of nonstrict hyperbolicity of a multicomponent polymer-flooding system (existence of parameter values where different eigenvelocitites of the system coincide) was investigated with regard to the appearance of both rarefaction and shock waves of the same index in the solution of the displacement problem (Johansen and Winther 1989; Johansen et al. 1989; Dahl et al. 1992). Hyperbolicity of the conservation-law system and the possible existence of elliptic regions have never been discussed for either polymer flooding or for gas injection (i.e., for two-phase compositional flows).

The existence of elliptic regions in three-phase flow is a well-known phenomenon and has a long research history (Marchesin and Plohr 2001; Guzmán and Fayers 1997a; Guzmán and Fayers 1997b). It results in different types of analytical models and in a change of numerical procedures (Kulikovskii et al. 2001). Hyperbolicity of a three-phase-flow system is highly dependent on the relative permeability model (Guzmán and Fayers 1997a; Guzmán and Fayers 1997b; Juanes and Patzek 2004a; Juanes and Patzek 2004b; Juanes and Patzek 2004c). Yet, physics assumptions on three-phase relative permeability result in the disappearance of the elliptic region in the flow model (Guzmán and Fayers 1997b; Juanes and Patzek 2004a; Juanes and Patzek 2004b; Juanes and Patzek 2004c).

The vanishing of the elliptic region when physics assumptions are imposed on the model supports the idea that the existence of the elliptic region is physically unrealistic and is a consequence of inaccurate determination of empirical functions. Even more, it is recommended to reject the empirical functions that give rise to a mixed-type conservation-law model (Juanes and Patzek 2004a; Juanes and Patzek 2004b; Juanes and Patzek 2004c; Fayers and Matthews 1984).

Nevertheless, the conservation-law model for the commingled flow of gas, oil, and water layers between two parallel plates has a mixed hyperbolic/elliptic type (Talon et al. 2004; Shariati et al. 2004). The model does not contain any empirical functions. The elliptic region develops on the displacement tip where the parallel-flow assumption breaks down. Crossing the elliptic region by compositional path corresponds to nonclassical shocks (LeFloch 2002).

For this paper, an elliptic region was observed for two-phase multicomponent flow in porous media for a so-called compositional model. The introduction of two Lagrangian coordinates, one linked with the nth-component motion and the other linked with overall flow, and the use of them instead of the Eulerian coordinate and time in the compositional model, result in the splitting of the two-phase multicomponent system into an auxiliary system and a lifting equation. The splitting reduces the problem of hyperbolicity of the compositional model to that for the auxiliary system. The existence of an elliptic region was observed for the auxiliary system of the four-component gas-flooding model.

The structure of the paper is as follows. First, we briefly derive the compositional model for mixtures, with conservation of the mixture volume. Then, two Lagrangian coordinates are introduced instead of the traditional independent variables (x_D, t_D), which results in the splitting of the compositional system into an auxiliary system and a scalar lifting equation. Phase diagrams for two four-component mixtures that are built by equation-of-state-based flash calculations for different pressures and temperatures are presented in the next section. Further calculations based on phase diagrams show that the auxiliary system contains elliptic regions for some investigated mixtures. Finally, the example of oil displacement by inert gas is presented, with component concentrations and saturation profiles and with the recovery curve.

Mathematical Model for Solvent Miscible Flooding

We consider 1D two-phase multicomponent solvent miscible flooding under the following assumptions:

- Negligible dissipation effects—capillary pressure, gravity, and diffusion
- Instantaneous thermodynamic equilibrium
- Constant pressure and temperature
- Equal and constant individual densities of components in both phases.

The detailed derivation of the compositional model under the those assumptions can be found elsewhere (Orr 2007; Bedrikovetsky 1993; Zick 1986); this section contains only a brief introduction of conservation laws for component masses and of geometric parameters on a phase diagram.

Under thermodynamic equilibrium and constant pressure and temperature, the two-phase n-component system has $n-2$ degrees of freedom. We choose the independent mass fractions of components $i = 2, 3, \ldots, n-1$ in the gas phase as independent variables. The vector of independent phase-mass fractions determines all component concentrations in phases:

$$\vec{g} = \left(c_{2g}, c_{3g}, \ldots, c_{(n-1)g}\right). \quad (1)$$

Under those conditions, the total two-phase flux is conserved, and n mass balances for n components are replaced by n–1 volume-conservation laws for n–1 components.

$$\frac{\partial C_i}{\partial t_D} + \frac{\partial F_i}{\partial x_D} = 0$$
$$x_D = \frac{x}{l}, \quad t_D = \frac{u\,t}{\Phi\, l}, \quad (2)$$

where the overall ith-component volume fraction and flux are

$$C_i = c_{il}\,S + c_{ig}\,(1-S) \quad (3)$$

and

$$F_i = c_{il} f + c_{ig}\,(1-f). \quad (4)$$

Here, f is the fractional flow of liquid:

$$f(S, \vec{g}) = \frac{k_{rl}(S, \vec{g}) \big/ \mu_l(\vec{g})}{k_{rl}(S, \vec{g}) \big/ \mu_l(\vec{g}) + k_{rg}(S, \vec{g}) \big/ \mu_g(\vec{g})}. \quad (5)$$

Initial and boundary conditions for continuous gas injection correspond to given compositions of injected gas and displaced oil:

$$C_i(x_D, 0) = C_i^I$$
$$C_i(0, t_D) = C_i^J. \quad (6)$$

The boundary conditions for the displacement of oil by solvent slug with lean gas drive are

$$C_i(0, t_D) : \begin{cases} C_i^J, & t_D < 1 \\ C_i^D, & t_D > 1 \end{cases}, \quad (7)$$

where C_i^D is the composition of the gas driving the solvent slug.

At this point, we introduce the following variables:

$$\alpha_i(\vec{g}) = \frac{c_{il} - c_{ig}}{c_{nl} - c_{ng}}, \quad i = 2, 3, \ldots n-1 \quad (8)$$

and

$$\beta_i(\vec{g}) = c_{ig} - \alpha_i c_{ng}, \quad i = 2, 3, \ldots, n-1. \quad (9)$$

Fig. 1 shows the geometrical meaning of α_i and β_i. Vertices 1, 2, … , n correspond to pure components in this phase diagram. Tie Line GL connects equilibrium phase compositions, G_iL_i is

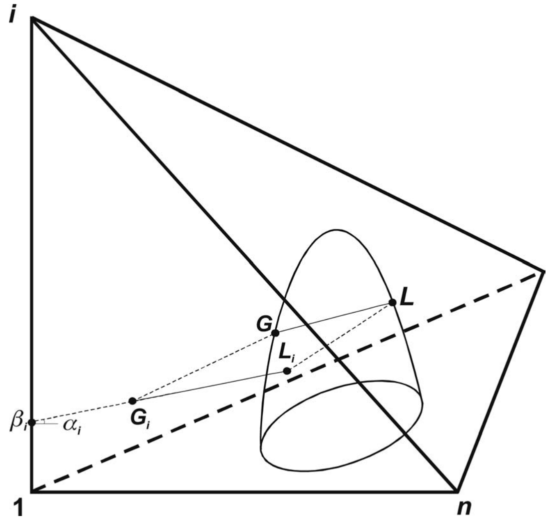

Fig. 1—Phase diagram and geometrical meaning of α and β.

the tie-line projection on the plane (1-i-n). The slope of the straight line G_iL_i is equal to α_i, and the intersection of G_iL_i with the axes C_i is equal to β_i.

Applying the new variables, Eq. 2 takes the form

$$\frac{\partial C}{\partial t_D} + \frac{\partial F\left(C, \vec{\beta}\right)}{\partial x_D} = 0$$

$$\frac{\partial\left(\vec{\alpha}\left(\vec{\beta}\right) C + \vec{\beta}\right)}{\partial t_D} + \frac{\partial\left(\vec{\alpha}\left(\vec{\beta}\right) F + \vec{\beta}\right)}{\partial x_D} = 0. \quad (10)$$

In Eq. 10, C is equal to C_n, the overall volumetric fraction of the nth component, and F is equal to F_n, the overall volumetric fractional flow of the nth component.

The unknowns in Eq. 10, composed of n–1 equations, are C and β_i, i = 2,3, . . . , n–1.

After the introduction of variables (Eqs. 8 and 9), the initial and boundary conditions (Eq. 6) for continuous gas injection become

$$C(x_D, 0) = C_n^I$$

$$\beta_i(x_D, 0) = \beta_i\left(\vec{g}^I\right) \quad (11)$$

and

$$C(0, t_D) = C_n^J)$$

$$\beta_i(0, t_D) = \beta_i\left(\vec{g}^J\right). \quad (12)$$

Usually, the reference length l in Eq. 2 is a core size or distance between injection and production rows. The solution of the continuous-injection problem is self-similar and is independent of l. For displacement of oil by a rich-gas slug with lean-gas drive, l is the length of the slug.

$$l = \frac{\Omega_s}{\phi A}, \quad (13)$$

where Ω_s is the slug volume and A is the reservoir cross section.

The boundary conditions (Eq. 7) for injection of a rich-gas slug with lean-gas drive take the form

$$C(0, t_D) : \begin{cases} C_n^J, & t_D < 1 \\ C_n^D, & t_D > 1 \end{cases}$$

$$\beta_i(0, t_D) : \begin{cases} \beta_i\left(\vec{g}^J\right), & t_D < 1 \\ \beta_i\left(\vec{g}^D\right), & t_D > 1 \end{cases}. \quad (14)$$

Splitting Between Hydrodynamics and Thermodynamics

The conservation-law form of Eq. 10 allows the introduction of the following potential:

$$C = -\frac{\partial \phi}{\partial x_D}, \quad F = \frac{\partial \phi}{\partial t_D}. \quad (15)$$

Eq. 10 is the equality of mixed second derivatives of the potential taken in different orders.

The potential $\phi(x_D, t_D)$ is equal to the nth-component volume flowing through any curve connecting points (0, 0) and (x_D, t_D):

$$\phi(x_D, t_D) = \int_{0,0}^{x_D, t_D} F \mathrm{d}t_D - C \mathrm{d}x_D. \quad (16)$$

From Eq. 10, it follows that the integral (Eq. 16) is independent of the curve (i.e., it is a function of x_D and t_D).

Let us introduce the following variable:

$$\psi = x_D - t_D. \quad (17)$$

From the incompressibility of the total flux, it follows that $\psi(x_D,t_D)$ is equal to the overall mixture volume flowing through a curve connecting points (0, 0) and (x_D, t_D).

After the following transformation of independent variables,

$$\Theta : (x_D, t_D) \rightarrow (\psi, \phi), \quad (18)$$

Eq. 10 becomes

$$\frac{\partial}{\partial \phi}\left(\frac{C}{F - C}\right) - \frac{\partial}{\partial \psi}\left(\frac{1}{F - C}\right) = 0 \quad (19)$$

and

$$\frac{\partial \vec{\beta}}{\partial \phi} + \frac{\partial \vec{\alpha}\left(\vec{\beta}\right)}{\partial \psi} = 0. \quad (20)$$

Derivation of Eqs. 19 and 20 is presented in Appendix A.

Potential ϕ is a Lagrangian coordinate associated with overall flux of the nth component. Phase velocities of the nth component in oil and gas phases are not equal. Nevertheless, overall volume balance of the nth component, described by Eq. 10, allows the introduction of a mean velocity of the nth component:

$$\frac{\mathrm{d}x_D}{\mathrm{d}t_D} = \frac{F(x_D, t_D)}{C(x_D, t_D)}. \quad (21)$$

From Eq. 10, it follows that potential ϕ is constant along mean trajectories $x_D = x_D(t_D)$ of the nth component. The relative flux of the nth component by means of the mean trajectory is zero.

Potential ψ is a Lagrangian coordinate associated with overall flux. It follows from the total volume conservation that mean velocity is unity. The linear coordinate in the reference system linked to overall flux is ψ.

The transformation equation (Eq. 18) changes from Eulerian coordinate x_D and time t_D to Lagrangian coordinates ϕ and ψ. The transformed system of Eqs. 19 and 20 has the form of conservation laws. Eq. 19 is mass balance of the nth component; Eq. 20 consists of conservation laws for components 2, 3, . . . , n – 1. Therefore, in subsequent text, the dependencies $\alpha_i = \alpha_i(\beta_2, \beta_3, \ldots, \beta_{n-1})$ are called flux functions.

Eq. 20 separates from Eq. 19 because n – 2 equations (Eq. 20) are independent of unknown $C(\psi,\phi)$. Eq. 18 splits the (n – 1) × (n – 1) conservation-law system (Eq. 10) into the (n – 2) × (n – 2) system (Eq. 20) and the scalar hyperbolic equation (Eq. 19).

The unknowns in the system (Eq. 20) are β_i, i = 2, 3, . . . , n – 1. The hyperbolic equation (Eq. 19) contains the unknown $C(\psi,\phi)$ and the known vector function $\beta_i(\psi,\phi)$, which is the solution of Eq. 20.

Eq. 20 is called the auxiliary system associated with the large system (Eq. 10). It is important to mention that the system (Eq. 10) contains thermodynamic functions and transport properties, while the auxiliary system (Eq. 20) contains only thermodynamic functions. The so-called "lifting equation" (Eq. 19) contains a hydrodynamic-flux function $F(C,\vec{\beta})$ that depends on both phase relative permeabilities and phase viscosities.

The Cauchy data for the Riemann problem of continuous gasflooding (Eqs. 11 and 12) and the initial-boundary data for the injection of a solvent slug with gas drive (Eq. 14) are transformed by mapping (Eq. 18) to those for the lifting equation (Eq. 19) and for the auxiliary system (Eq. 20). (The detailed derivations are presented in Appendix B.) Finally, the solution for the compositional model (Eq. 10) is split into solving the auxiliary problem (Eq. 20) first and then solving the lifting problem (Eq. 19).

Therefore, the change of variables (Eq. 18) splits problems of two-phase multicomponent displacement into those for the auxiliary thermodynamic system and for the lifting equation.

The calculation of eigenvalues for Eqs. 10 and 20 using standard hyperbolic techniques (Dafermos 1999; Courant and Friedrichs 1985; Kulikovskii and Sveshnikova 1995) shows that the eigenvalues of the large and auxiliary systems for β waves are related by

$$\Lambda_k\left(C,\vec{\beta}\right)=\frac{F+1/\lambda_k\left(\vec{\beta}\right)}{C+1/\lambda_k\left(\vec{\beta}\right)},\ k=2,3,\ldots,n-1, \quad (22)$$

where λ_k are wave speeds of the auxiliary system. The relationship (Eq. 22) is obtained by straightforward calculation.

Moreover, the transformation equation (Eq. 18) maps rarefaction β waves of the compositional model (Eq. 10) into rarefaction waves of the auxiliary system (Eq. 20).

The Hugoniot-Rankine conditions of mass balance for all components on the shock (Dafermos 1999; Courant and Friedrichs 1985; Kulikovskii and Sveshnikova 1995) for the initial compositional model (Eq. 10) are

$$\begin{gathered}[C]D=[F]\\ \left[\vec{\alpha}\left(\vec{\beta}\right)C+\vec{\beta}\right]D=\left[\vec{\alpha}\left(\vec{\beta}\right)F+\vec{\beta}\right],\end{gathered} \quad (23)$$

where $[A]=A^{+}-A^{-}$ is a jump of the value A and D is the shock speed.

The mass-balance conditions on the shock for the auxiliary system (Eq. 20) are

$$\left[\vec{\beta}\right]V=\left[\vec{\alpha}\left(\vec{\beta}\right)\right], \quad (24)$$

where V is the shock-wave speed for the auxiliary system.

Simple algebraic transformations of Eqs. 23 and 24 show that the shock-wave speeds V of the auxiliary system are linked with the shock-wave speeds of the large system D by

$$D=\frac{F+V}{C+V}. \quad (25)$$

Transformation (Eq. 18) maps β-shock-wave loci of the compositional model (Eq. 10) onto those for the auxiliary system (Eq. 20).

The splitting technique (Eqs. 18 through 20) yields the following procedure for the solution of the large compositional system (Eq. 10) subject to initial and boundary conditions (Eqs. 11, 12, and 14):

- Solution of the auxiliary problem (Eq. 20 and Eqs. B-1 through B-7) and determination of $\beta(\psi,\phi)$;
- Solution of the lifting problem (Eqs. 19 and B-8) and calculation of $C(\psi,\phi)$;
- Inverse transformation (Eq. 18) of independent variables.

Now, we present formulas to obtain the analytical compositional model $C(x_D,t_D)$, $\beta(x_D,t_D)$ from the solution in Lagrangian coordinates $C(\psi,\phi)$, $\beta(\psi,\phi)$.

It follows from Eqs. 16 and 17 that the transformation maps point $x_D=t_D=0$ to the origin $\phi=\psi=0$. Integrating Eqs. A-2 and A-3 from Appendix A along any path linking points (0, 0) and (ψ,ϕ), we obtain

$$t_D(\psi,\phi)=\int_{0,0}^{\psi,\phi}\frac{1}{F-C}\mathrm{d}\phi+\frac{C}{F-C}\mathrm{d}\psi \quad (26)$$

and

$$x_D(\psi,\phi)=\int_{0,0}^{\psi,\phi}\frac{1}{F-C}\mathrm{d}\phi+\frac{F}{F-C}\mathrm{d}\psi. \quad (27)$$

In this paper, we solve the auxiliary problem to show the compositional path in the phase diagram, to emphasize splitting between thermodynamics and hydrodynamics for gas-based EOR processes and to prove independence of phase transitions of transport properties in oil/gas/rock systems.

Let us show that phase transitions occurring during any non-self-similar gas-based EOR displacements throughout the 1D reservoir are determined only by the thermodynamics of the oil/gas system and are independent of transport properties.

In Appendix B, it is shown how the initial-boundary-value problems (Eqs. 11, 12, and 14) are projected onto the auxiliary problems (Eqs. B-7 through B-9). It is possible to prove that any Cauchy or initial-boundary-value problem for the compositional model (Eq. 10) can be projected onto the corresponding Cauchy or initial-boundary-value problem for the auxiliary system.

The solution of the large system $\beta_i(x_D,\ t_D)$ realizes the mapping from the plane $(x_D,\ t_D)$ to the set of tie lines in a four-vertices tetrahedron of a four-component phase diagram. The image of the domain of the plane (x_D,t_D), $x_D>0$, $t_D>0$, defines 2D surfaces in the tetrahedron. The auxiliary solution $\beta_i(\psi,\ \phi)$ also maps the domain of the plane $(\psi,\ \phi)$, where the initial-boundary-value problem is defined, onto a 2D surface in the tetrahedron. From the splitting of the compositional model (Eq. 10) into the auxiliary system (Eq. 20) and the lifting problem (Eq. 19), it follows that these surfaces coincide.

The auxiliary solution depends on the thermodynamic functions α_i and β_i and on the composition fractions of the initial and boundary conditions. Therefore, the 2D solution image in the tetrahedron is independent of transport properties (i.e., fractional-flow curves, phase relative permeability, and phase viscosities).

As a particular case, minimum miscibility pressure is also independent of phase relative permeabilities and phase viscosities. The phenomenon of compositional-path independence of hydrodynamic properties of an oil/gas/rock system was observed for continuous-gas-injection processes (Wang and Orr 1997).

Splitting significantly reduces the amount of calculation for sensitivity study with respect to the transport properties: The auxiliary thermodynamic problem may be solved once for given reservoir and injected compositions; variation of relative permeabilities and viscosities should be performed only in the solution of one transport equation.

Phase Behavior

To calculate the flux functions $\alpha=\alpha(\beta)$ of the auxiliary system, it is necessary to model the phase behavior of the fluids analyzed. The Peng-Robinson equation of state (Peng and Robinson 1976) was used to determine the phase diagram of the mixtures. The list of the analyzed four-component systems is presented in **Table 1.**

TABLE 1—SYSTEMS ANALYZED

Components	Pressure (bar)	Temperature (K)
C_1-C_3-C_6-C_{10}	50	350
C_1-C_3-C_6-C_{10}	100	300
N_2-C_3-C_6-C_{10}	100	350
N_2-C_3-C_6-C_{10}	200	350
N_2-C_3-C_6-C_{10}	300	350

TABLE 2—COMPONENTS PARAMETERS

	N_2	C_1	C_3	C_6	C_{10}
T_c (K)	126.2	190.5	369.8	507.4	617.6
P_c (bar)	34.0	46.0	42.4	30.1	20.9
ω	0.0372	0.0126	0.1541	0.2998	0.4904

Table 2 presents critical parameters of each component (Prausnitz 1969).

The phase-equilibrium diagrams were built using in-house software. Mole fractions were converted to volume fractions, assuming that individual densities of each component in both phases are equal.

Flash calculations were performed for the set of basic mixture points that were selected in a way to cover the domain of β_2 and β_3 variation uniformly.

Figs. 2 through 6 present, respectively, two-phase diagrams for the five systems listed in Table 1. For evaluated temperatures and pressures, the two-phase region is relatively small for all systems. Phase diagrams are used to calculate flux functions for the auxiliary system (Eq. 20).

Flux functions $\alpha = \alpha(\beta)$ are also determined from the same basic mixture points as were used for building phase diagrams (Figs 2 through 6). The values of fluxes α_2, α_3 and concentrations β_2, β_3 were calculated in the same points by Eqs. 8 and 9, respectively. Flux functions $\alpha_i(\beta_2, \beta_3)$, i=2, 3 were approximated by second-order polynomials using the least-squares method.

Existence of Elliptic Region

In this section, it is shown that the auxiliary system (Eq. 20) can contain an elliptic region (i.e., it is not always hyperbolic). All eigenvalues for the compositional two-phase model are real if and only if the same is applicable for the auxiliary system (Pires et al. 2006). Therefore, from the fact that the auxiliary system contains an elliptic region, it follows that the compositional system also has an elliptic region.

The auxiliary system of the four-component-flow problem consists of two equations

$$\frac{\partial \beta_2}{\partial \phi} + \frac{\partial \alpha_2(\beta_2, \beta_3)}{\partial \psi} = 0$$

and

$$\frac{\partial \beta_3}{\partial \phi} + \frac{\partial \alpha_3(\beta_2, \beta_3)}{\partial \psi} = 0. \quad \ldots\ldots (28)$$

The 2 × 2 quasilinear system (Eq. 28) is called hyperbolic if two eigenvalues are real. It is called elliptic if two eigenvalues are complex. If eigenvalues are not equal for all variables (β_2, β_3), the system is strictly hyperbolic (Dafermos 1999).

It follows from Eq. 22 that if the n-component auxiliary system (Eq. 20) presents elliptic regions [some pairs of eigenvalues $\lambda_k(\vec{\beta})$ are complex], the general system (Eq. 10) also is a mixed hyperbolic/elliptic type [pairs of complex eigenvalues $\Lambda_k(C, \vec{\beta})$].

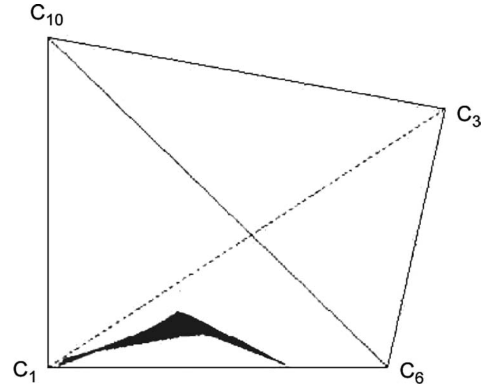

Fig. 3—Phase diagram for C_1-C_3-C_6-C_{10} at 100 bar and 300 K.

Eigenvalues λ_k of Eq. 28 are roots of the determinant of the matrix

$$\det\left\{\begin{bmatrix} \frac{\partial \alpha_2}{\partial \beta_2} & \frac{\partial \alpha_2}{\partial \beta_3} \\ \frac{\partial \alpha_3}{\partial \beta_2} & \frac{\partial \alpha_3}{\partial \beta_3} \end{bmatrix} - \lambda I\right\} = 0, \quad \ldots\ldots (29)$$

where I is the identity matrix. Eq. 28 is hyperbolic (i.e., Eq. 29 has two real roots) if discriminant

$$z = \left(\frac{\partial \alpha_2}{\partial \beta_2} + \frac{\partial \alpha_3}{\partial \beta_3}\right)^2 - 4\left(\frac{\partial \alpha_2}{\partial \beta_2}\frac{\partial \alpha_3}{\partial \beta_3} - \frac{\partial \alpha_2}{\partial \beta_3}\frac{\partial \alpha_3}{\partial \beta_2}\right) \quad \ldots\ldots (30)$$

is nonnegative. If, in some region of the plane (β_2,β_3), function z assumes negative values, the region is called elliptic.

For all studied values of pressure and temperature (Table 1), the C_1-C_3-C_6-C_{10} system of hydrocarbons was strictly hyperbolic (i.e., $z > 0$). Substitution for methane by N_2 in the mixture resulted in the appearance of an elliptic region.

The following flux functions were determined for the N_2-C_3-C_6-C_{10} system at 100 bar and 350 K:

$$\alpha_2(\beta_2, \beta_3) = 14.23(\beta_2)^2 + 1.095\beta_2 - 0.3104\beta_3 + 811.2\beta_2\beta_3$$

and

$$\alpha_3(\beta_2, \beta_3) = 9953(\beta_3)^2 + 0.2\beta_2 + 29.91\beta_3 - 102.1\beta_2\beta_3. \quad \ldots\ldots (31)$$

Fig. 7 shows a 3D plot of $z(\beta_2, \beta_3)$ as calculated by Eq. 30 from compositions of coexisting phases and parameters α and β in basic points. The fact that function z can take negative values is visible from projection of the surface $z = z(\beta_2, \beta_3)$ in **Fig. 8** on the plane (β_3, z). Fig. 8 shows the region where function z is negative.

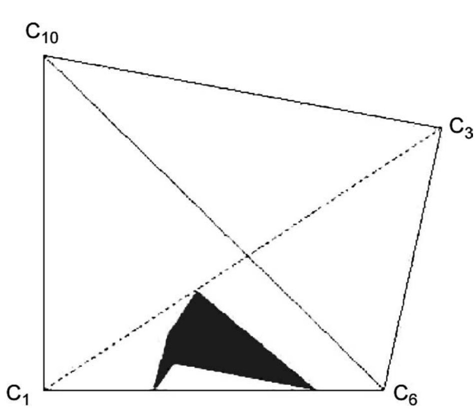

Fig. 2—Phase diagram for C_1-C_3-C_6-C_{10} at 50 bar and 350 K.

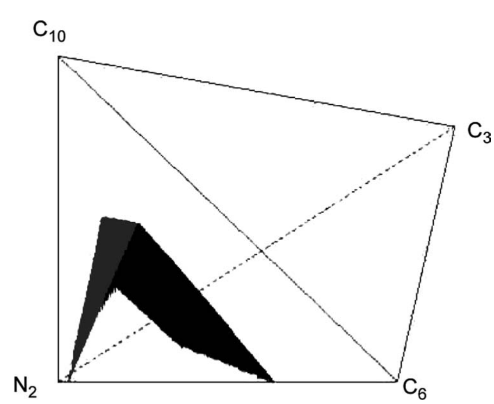

Fig. 4—Phase diagram for N_2-C_3-C_6-C_{10} at 100 bar and 350 K.

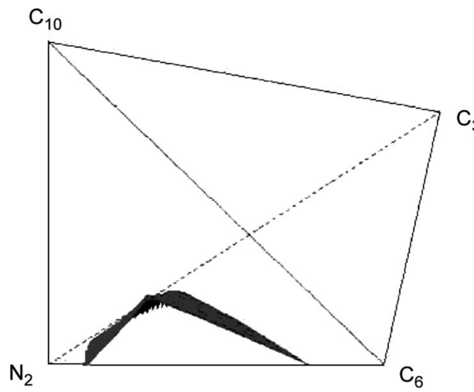

Fig. 5—Phase diagram for N_2-C_3-C_6-C_{10} at 200 bar and 350 K.

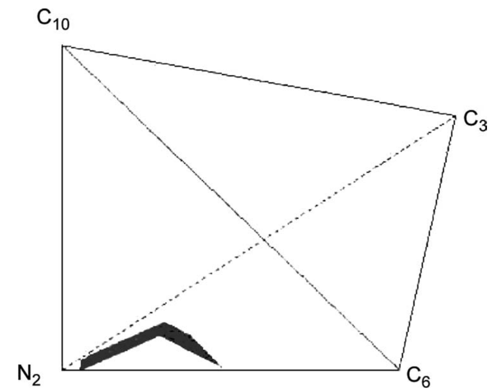

Fig. 6—Phase diagram for N_2-C_3-C_6-C_{10} at 300 bar and 350 K.

For the values (β_2, β_3) where $z(\beta_2, \beta_3) < 0$, auxiliary-system Eq. 28 is elliptic.

Similar results were obtained for this system at the same temperature and greater pressures. The following flux functions were determined at 200 bar and 350 K:

$$\alpha_2(\beta_2,\beta_3) = 11(\beta_2)^2 - 0.8688\beta_2 + 0.09526\beta_3 + 424.5\beta_2\beta_3$$

and

$$\alpha_3(\beta_2,\beta_3) = 5233(\beta_3)^2 - 6.126\beta_2 - 4.262\beta_3 + 212.6\beta_2\beta_3. \quad \text{(32)}$$

And at 300 bar and 350 K,

$$\alpha_2(\beta_2,\beta_3) = -2.856\beta_2 - 0.8497\beta_3 + 382.8\beta_2\beta_3$$

and

$$\alpha_3(\beta_2,\beta_3) = 4130(\beta_3)^2 + 8.725\beta_2 - 47.632\beta_3 - 633.9\beta_2\beta_3. \quad \text{(33)}$$

Figs. 9 and 10 show the elliptic regions for system N_2-C_3-C_6-C_{10} at 200 bar and 350 K and at 300 bar and 350 K, respectively.

To check if the existence of elliptic regions is a property of the compositional model (Eq. 10) and not an artifact of the polynomial adjustment of flux functions (Eqs. 31 through 33), the Jacobian matrix (Eq. 29) was calculated numerically—partial derivatives were obtained from flash calculations for the mixtures in neighborhoods of the basic mixture points. Forms of elliptic regions were close to those shown in Figs. 8 through 10.

It is important to find out for which multicomponent mixtures and for which equations of state the elliptic region could be expected. The formulation is similar to those already posed (Juanes and Patzek 2004a; Juanes and Patzek 2004b; Juanes and Patzek 2004c), where physics assumptions on the behavior of a three-phase system in porous media resulted in hyperbolicity of the flow system. The analogous formulations for the existence of an elliptic region in a compositional model are

- Which thermodynamic property corresponds to the existence of only real eigenvalues of the matrix $\left|\frac{\partial \alpha_i}{\partial \beta_j}\right|$?
- Which thermodynamic systems possess a set of tie lines that have complex eigenvalues?

This research involves differential geometry of tie-line manifolds. A more practical question would be, "Which combinations of adjustment parameters in different equations of state correspond to the mixed hyperbolic/elliptic type of auxiliary system?" In this paper, we present only examples of the existence of the elliptic region, leaving the other research for a future paper.

Probably, splitting of the compositional model into thermodynamic and hydrodynamic models is more fundamental fact than that presented in this paper. The model (Eq. 10) assumes that pressure is constant during the displacement, which is relevant for displacement processes. For compositional flows toward a well, where pressure drawdown is important, Darcy's law is coupled with mass-balance equations (Eq. 10). Splitting for this system is performed asymptotically, where a small parameter is dimensionless well rate (Oladyshkin and Panfilov 2006).

Oil Displacement by N_2-Based Solvent

Consider displacement of oil by N_2 with low-concentration hydrocarbon additives at pressure P = 300 bar and temperature T = 350 K. The phase diagram for the mixture is shown in Fig. 6. At some mixture points, the discriminant is negative (Fig. 10) (i.e., the system contains an elliptic region). Compositions of injected and displaced fluids are presented in **Table 3.** Points J and I on the phase diagram **(Fig. 11)** correspond to compositions of injected gas and reservoir oil, respectively.

The auxiliary system (Eq. 28) is a conservation-law system with quadratic-polynomial flux functions (Eq. 33), and the auxiliary problem (Eqs. 11 and 12) is a Riemann problem. The solution

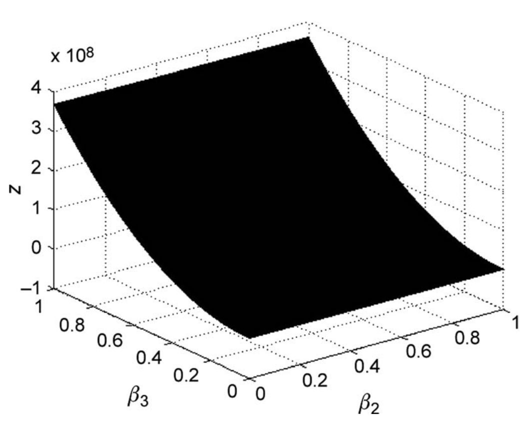

Fig. 7—Function z for system N_2-C_3-C_6-C_{10} at 100 bar and 350 K.

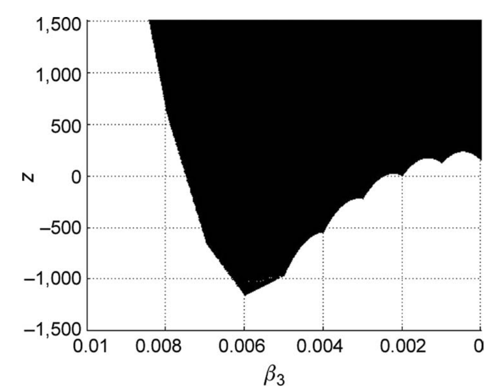

Fig. 8—Eigenvalue analysis for system N_2-C_3-C_6-C_{10} at 100 bar and 350 K.

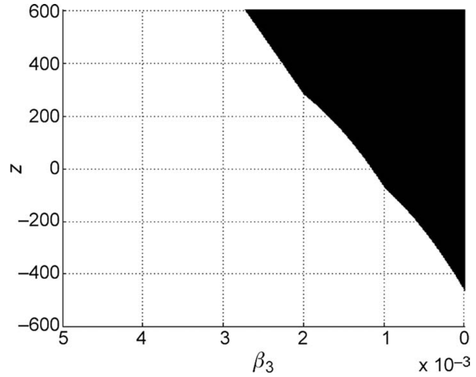

Fig. 9—Eigenvalue analysis for system N_2-C_3-C_6-C_{10} at 200 bar and 350 K.

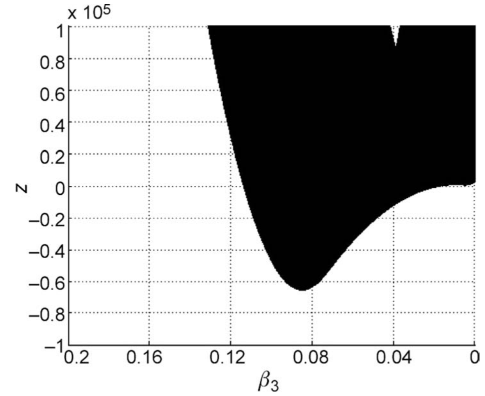

Fig. 10—Eigenvalue analysis for system N_2-C_3-C_6-C_{10} at 300 bar and 350 K.

was obtained using Pakman software, which is based on Riemann-problem solutions for 2 × 2 conservation-law systems with second-order-polynomial fluxes (Marchesin and Plohr 2001).

The solution of the auxiliary problem under consideration consists of two shocks that can be expressed by the structural formula $J \to M \to I$. The values of shock speed and β values for points that are present in the solution are given in **Table 4.**

Fig. 12 presents trajectories of shock waves corresponding to solution $J \to M \to I$ of the auxiliary system on the plane (ψ,ϕ). Initial condition I and boundary condition J are set along straight lines. Two shocks in the solution correspond to the appearance of Zone M between Zones I and J.

The structure of displacement zone with trajectories of shock fronts is presented in **Fig. 13** on the plane (x_D,t_D). The solution consists of a jump from J to 1 with speed D_{J1}, of rarefaction 1–2, of shock 2–M, constant state M, shock $M \to 3$, constant state 3 followed by another jump from 3 to I.

The compositional path corresponding to the described solution is shown in Fig. 11.

Table 5 presents shock speed, saturation, and β values for points that are present in the solution.

Profiles of liquid saturation and four concentrations are shown in **Fig. 14.** Evaporation shock moves with low speed $D_{J1} = 0.041$. Phase transitions during condensed drive occur along fronts that move with speeds $D_{2M} = 0.986$ and $D_{M3} = 0.987$. Phase transition does not occur on the shock $D_{3I} = 1.010$.

Welge's method for calculating an average value in a self-similar solution (Welge 1952) can be generalized for the case of average component concentration in the compositional model (Bedrikovetsky 1993). This method allows calculating the recovery factor for each component. The recovery factor for the fourth (the heaviest) component vs. time in pore volumes injected (PVI) is shown in **Fig. 15.**

Since phase transitions occur at a sequence of shocks with speeds near unity, solvent breakthrough takes place at approximately 1 PVI; production of initial reservoir oil takes place until that moment. The speed of jump $J \to 1$ is very low. Therefore, complete evaporation of oil into injected gas occurs after several PVI. Recovery factor increases slowly until the moment of complete evaporation.

Summary and Conclusions

The $(n-1)\times(n-1)$ system of conservation laws for two-phase n-component flow in porous media with interphase mass transfer can be split into an $(n-2)\times(n-2)$ auxiliary system and one independent lifting equation. The splitting is obtained from the change of independent variables (x_D,t_D) to flow potentials (ψ, ϕ). The flow potentials are Lagrangian coordinates associated with mean flux of the nth component and with the overall flux, respectively. This change of coordinates changes the conservation law for the nth component into the lifting equation. It transforms conservation laws of components 2,3, . . . , n–1 into the auxiliary system. The auxiliary system contains only equilibrium thermodynamic variables (equilibrium fractions of each phase), while the large system contains both hydrodynamic (phase relative permeabilities and viscosities) functions and equilibrium thermodynamic variables. Therefore, phase transitions occurring during displacement are determined by the auxiliary system (i.e., they are independent of hydrodynamic properties of fluids and rock).

Approximation of auxiliary flux functions by quadratic polynomials allows for exact solution of the 2×2 auxiliary problem by use of the Pakman software. The solution could contain any sequence of rarefaction and shock waves. Therefore, the splitting technique provides an analytical model of four-component displacement of real reservoir mixtures.

The auxiliary system for four-component two-phase flow in porous media was analyzed, elliptic regions were found in systems containing N_2 at pressures ranging from 100 to 300 bar and at a temperature of 350 K. Systems of hydrocarbons are hyperbolic for this range of pressures and temperatures.

TABLE 3—INITIAL AND INJECTION CONDITIONS						
	N_2	C_3	C_6	C_{10}	β_2	β_3
Injection	90%	5%	5%	0%	0.05	0.05
Initial	11%	4%	45%	40%	0.012	0.023

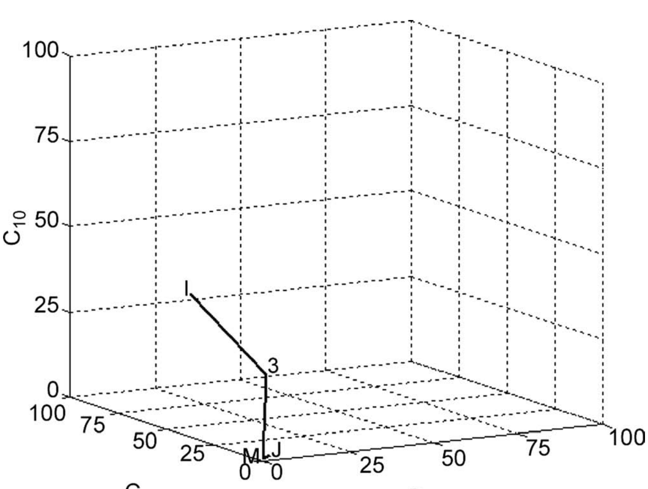

Fig. 11—Initial, injection, and intermediate mixture points along with the compositional path on phase diagram.

TABLE 4—SOLUTION OF AUXILIARY PROBLEM

	β_2	β_3	V
I	0.012	0.023	0.005
M	0.013	0.047	
J	0.05	0.05	0.060

Nomenclature

c = volumetric fraction
C = overall volumetric fraction of nth component
C_i = overall volumetric fraction of ith component
D = shock speed for the large system
f = liquid fractional flow
F = overall volumetric fractional flow of nth component
F_i = overall volumetric fractional flow of ith component
$\vec{g}$ = vector of independent mass fractions of gas phase
G = gas-phase composition
k_r = relative permeability
l = reservoir size
L = liquid phase composition
n = number of components
P = pressure
S = liquid volumetric fraction
t = time
T = temperature
t_D = dimensionless time
u = total flux
V = shock speed for the auxiliary system
x = distance
x_D = dimensionless distance
α = geometric parameter of thermodynamic equilibrium
β = geometric parameter of thermodynamic equilibrium
Θ = transformation of independent variables
λ = eigenvalue of auxiliary system
Λ = eigenvalue of large system
μ = viscosity
ϕ = potential for nth-component flow
Φ = porosity
ψ = flow potential of overall flux
ω = acentric factor
Ω = closed domain

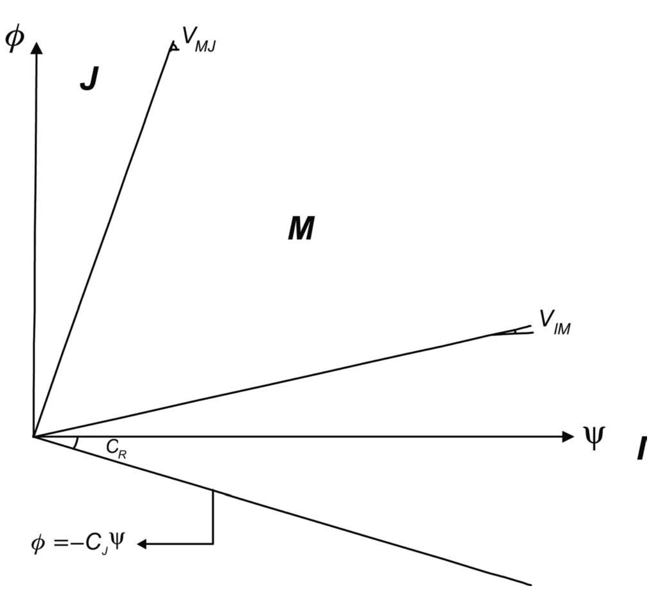

Fig. 12—Solution of auxiliary problem in the plane of Lagrangian coordinates (Ψ, ϕ).

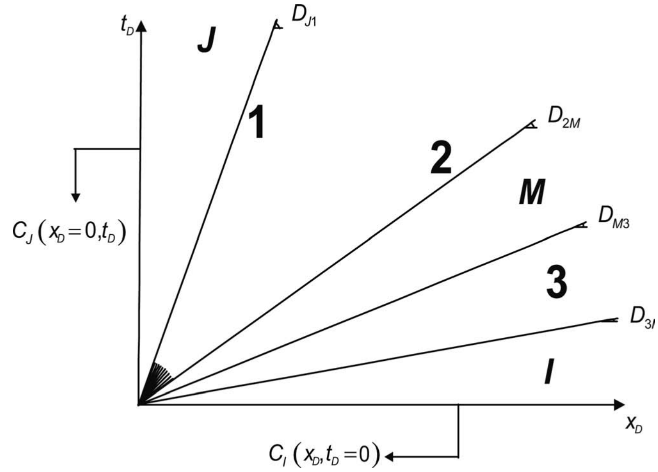

Fig. 13—Solution of displacement problem on plane (x_D,t_D).

Subscripts

c = critical
g = gas phase
i = component index
k = wave index
l = liquid phase

Superscripts

D = drive gas
I = initial oil
J = injected solvent

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TABLE 5—SOLUTION OF DISPLACEMENT PROBLEM

	β_2	β_3	S	D
J	0.05	0.05	0	0.041
1	0.05	0.05	0.559	
2	0.05	0.05	0.680	0.986
M	0.013	0.047	0.750	
3	0.012	0.023	0.751	0.987
I	0.012	0.023	0.800	1.010

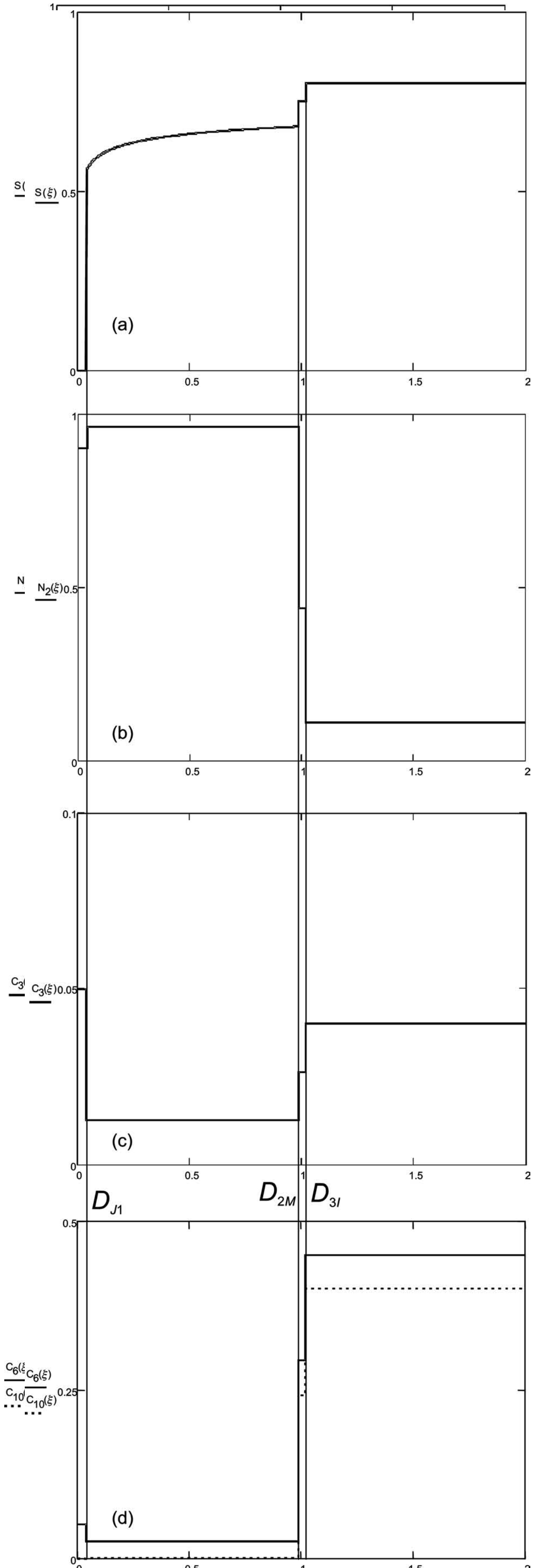

Fig. 14—Liquid-saturation (a) and component-concentration [(b) through (d)] profiles.

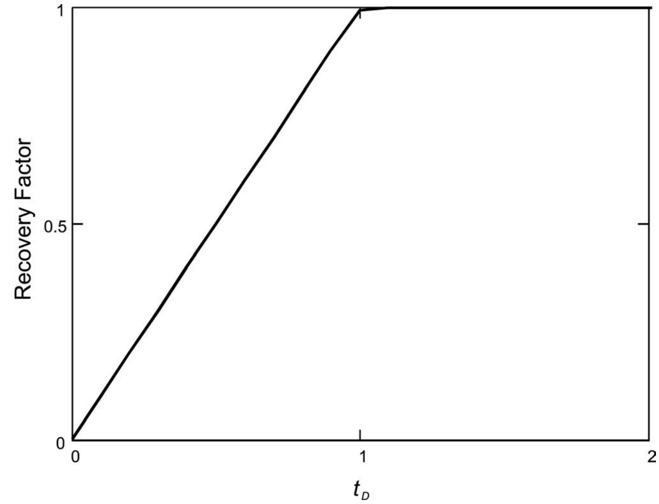

Fig. 15—Recovery factor of heaviest component.

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Appendix A—Derivation of Conservation-Law System in Lagrangian Coordinates

Let us derive Eq. 19. The differentials of two potentials (Eqs. 16 and 17) are:

$$\mathrm{d}\phi = F\mathrm{d}t_D - C\mathrm{d}x_D$$

and

$$\mathrm{d}\psi = -\mathrm{d}t_D + \mathrm{d}x_D. \quad \text{(A-1)}$$

The differentials $\mathrm{d}t_D$ and $\mathrm{d}x_D$ can be calculated from Eq. A-1, which is a 2×2 linear system with respect to unknowns $\mathrm{d}x_D$ and $\mathrm{d}t_D$.

$$\mathrm{d}t_D = \frac{1}{F-C}\mathrm{d}\phi + \frac{C}{F-C}\mathrm{d}\psi \quad \text{(A-2)}$$

and

$$\mathrm{d}x_D = \frac{1}{F-C}\mathrm{d}\phi + \frac{F}{F-C}\mathrm{d}\psi. \quad \text{(A-3)}$$

Taking full differential of both sides of Eq. A-2 yields

$$\mathrm{d}^2 t_D = 0 = \left[\frac{\partial}{\partial\phi}\left(\frac{C}{F-C}\right) - \frac{\partial}{\partial\psi}\left(\frac{1}{F-C}\right)\right]\mathrm{d}\phi\mathrm{d}\psi. \quad \text{(A-4)}$$

Eq. 19 follows from Eq. A-4.

Let us show that if $C(x_D,t_D)$, $\beta_i(x_D,t_D)$, $i = 2, 3, \ldots, n-1$ is a solution of Eq. 10, and $\phi\,(x_D,t_D)$ and $\psi(x_D,t_D)$ are the potential functions (Eqs. 16 and 17), then the function $\beta_i(\psi,\phi)$ obeys the following conservation law:

$$\oint_{\partial\Omega} \alpha_i \mathrm{d}\phi - \beta_i \mathrm{d}\psi = 0, \quad \text{(A-5)}$$

where Ω is a closed domain $\Omega \subset R^2$.

Eq. 10 was derived from the conservation law of ith component volume balance in the integral form:

$$\oint_{\partial\Omega} (\alpha_i F + \beta_i)\,\mathrm{d}t_D - (\alpha_i C + \beta_i)\,\mathrm{d}x_D = 0. \quad \text{(A-6)}$$

Here, Ω is an arbitrary domain in plane (x_D,t_D) with smooth boundary $\partial\Omega$.

Applying the Green-Gauss theorem to Eq. A-6 yields

$$\int\int\left[\frac{\partial(\alpha_i C + \beta_i)}{\partial t_D} + \frac{\partial(\alpha_i F + \beta_i)}{\partial x_D}\right]\mathrm{d}x_D\mathrm{d}t_D = 0. \quad \text{(A-7)}$$

From Eq. A-6, and using the definition of potentials in Eqs. 16 and 17, we obtain:

$$\begin{aligned}
&\oint_{\partial\Omega} (\alpha_i F + \beta_i)\,\mathrm{d}t_D - (\alpha_i C + \beta_i)\,\mathrm{d}x_D \\
&\quad = \oint_{\partial\Omega} \alpha_i\,(F\,\mathrm{d}t_D - C\,\mathrm{d}x_D) - \beta_i\,(\mathrm{d}x_D - \mathrm{d}t_D) \\
&\quad = \oint_{\partial\Omega} \alpha_i \mathrm{d}\phi - \beta_i \mathrm{d}\psi = 0. \quad \text{(A-8)}
\end{aligned}$$

In domains Ω where the solution is a smooth function, from the integral conservation-law (Eq. A-8) the system of partial-differential equations (Eq. 20) follows. In narrow domains around shock trajectories, from Eq. A-8 the Hugoniot-Rankine conditions follow.

Appendix B—Initial-Boundary-Value Problem for Auxiliary System and Lifting Equation

Now we formulate initial-boundary-value problems for continuous gas injection and for injection of a rich-gas slug with gas drive for independent variables (ψ,ϕ).

The initial and boundary conditions (Eqs. 11, 12, and 14) allow calculation of both potentials along the axes x_D and t_D where the conditions are set.

Performing the integration (Eq. 16) in x_D accounting for Eq. 11, we obtain the potential ϕ along the axis x_D:

$$\begin{aligned} t_D = 0 : \phi &= -C_n^I \psi \\ \psi &= x_D. \end{aligned} \quad \text{(B-1)}$$

Therefore, the initial conditions (Eq. 11) in coordinates (ψ, ϕ) become

$$\phi = -C^I \psi : C = C^I \quad \text{(B-2)}$$

and

$$\phi = -C^I \psi : \vec{\beta} = \vec{\beta}^I. \quad \text{(B-3)}$$

Integration (Eq. 16) in t_D, accounting for boundary conditions (Eq. 12), allows calculation of the potential ϕ along the axis t_D.

$$\begin{aligned} x_D = 0 : \phi &= -F_n^J \psi \\ \psi &= -t_D \end{aligned}. \quad \text{(B-4)}$$

The boundary conditions (Eq. 12) take the form

$$\phi = -F^J \psi : C = C^J \quad \text{(B-5)}$$

and

$$\phi = -F^J \psi : \vec{\beta} = \vec{\beta}^J. \quad \text{(B-6)}$$

The boundary condition (Eq. 14) for slug injection gives the following value of potential ϕ:

$$x_D = 0 : \phi = \begin{cases} -F_n^J \psi, -1 < \psi < 0 \\ F_n^J - F_n^D(\psi + 1), -\infty < \psi < -1 \end{cases}. \quad \text{(B-7)}$$

Therefore, the boundary conditions (Eq. 14) become

$$C = \begin{cases} C^J, \phi = -F^J \psi, -1 < \psi < 0 \\ C^D, \phi = -F^J - F^D(\psi - 1), -\infty < \psi < -1 \end{cases} \quad \text{(B-8)}$$

and

$$\vec{\beta} = \begin{cases} \vec{\beta}^J, \phi = -F^J \psi, -1 < \psi < 0 \\ \vec{\beta}^D, \phi = -F^J - F^D(\psi - 1), -\infty < \psi < -1 \end{cases}. \quad \text{(B-9)}$$

Therefore, the transformation (Eq. 18) separates the initial and boundary conditions for the large system (Eq. 10) into the initial-boundary-value problem for the auxiliary system (Eq. 20) and the initial-boundary-value problem for the lifting equation (Eq. 19).

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