Here, as in previous examples, a decisive part is played by the mixed space-time parameter $Nt\frac{r_0^2}{r^2}$.

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STABILITY AND ADMISSIBILITY OF DISCONTINUITIES IN THE SYSTEMS OF EQUATIONS OF TWO-PHASE FILTRATION*

P.G. BEDRIKOVETSKII and M.V. LUR'E

To obtain the additional conditions at a discontinuity in the solution of the non-convex hyperbolic systems of equations of two-phase filtration with an active admixture /1–3/ (**) an approach is proposed that differs from the method of vanishing viscosity. The discontinuous solution is considered as the limit of solutions of the non-equilibrium system, when the characteristic time for thermodynamic equilibrium to become established approaches zero. The admissibility conditions obtained (of the existence of a structure) are the same as the equilibrium conditions in Oleinik's form /5,6/, and ensure the existence and uniqueness of the selfsimiliar solution of the problem of discontinuity disintegration.

The processes of petroleum displacement by hydrodynamically active fluids is defined by systems of non-linear differential equation of hyperbolic type, as in gas dynamics, for which discontinuous solutions are characteristic /7/. The stability of the discontinuity with respect to small perturbations is a generally acceptable requirement in the linearized problem /8,9/. However, for some non-convex sytems of the equations of gas dynamics and elasticity theory, the solution of the problem of discontinuity disintegration, containing stable discontinuities is not unique /6,10/. Supplementary conditions at the discontinuity ensuring the uniqueness of the solution were obtained either by generalizing the concept of stability, or as the limit of the solutions of the corresponding problem in a more comprehensive physical theory of "vanishing viscosity" /8–11/.

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1. Analysis of solutions of the hyperbolic system. The process of two-phase filtration of immiscible liquids with an active admixture under conditions of thermodynamic equilibrium in the first phase and in sorption state is defined by the system of equation of phase-mass balance and of admixture-mass balance

$$\frac{\partial s}{\partial t}+\frac{\partial F}{\partial x}=0, \quad F=F(s,c) \tag{1.1}$$

$$\frac{\partial}{\partial t}(cs+a(c))+\frac{\partial}{\partial x}(cF)=0 \tag{1.2}$$

where x and t are the dimensionless coordinate and time, $s(x, t)$ is the saturation of the first-phase threshold volume, $c(x, t)$ is the admixture solution concentration in the first phase, $a(c)$ is the concentration of adsorbed admixture $a(0)=0$, $a'(c)>0$, $F(s, c)$ is the fraction of the first phase in the stream, and $F_c'<0$.

Let us write the hyperbolic system of quasilinear equations (1.1), (1.2) in Riemann invariants. By virtue of the dependence $F=F(s,c)$ we will consider as the unknowns both (s, c), and (s, F). In the transformation of the hodograph $(x, t)\to[s(x,t), F(x,t)]$ the characteristics of system

$$x^{\cdot}=F\cdot(s+a')^{-1} \tag{1.3}$$

$$x^{\cdot}=F_s' \tag{1.4}$$

become simple waves

$$\xi=F_s'=\frac{dF}{ds} \tag{1.5}$$

$$\xi=\frac{F}{s+a'}=\frac{dF}{ds} \tag{1.6}$$

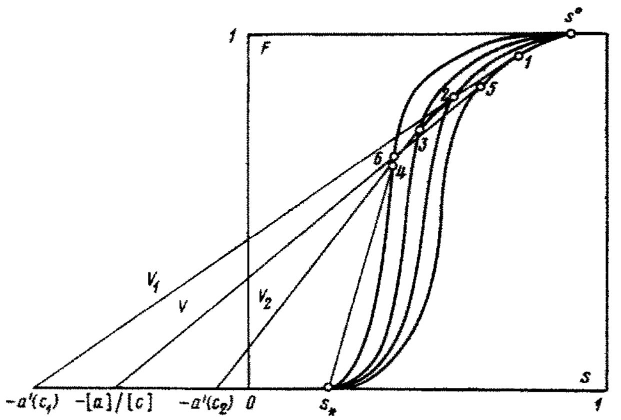

Fig.1

respectively. The characteristics of the families (1.3) and (1.4) will be called the c- and s-characteristics, respectively. The simple waves of families (1.5) and (1.6) will be called s- and c-waves. In the plane (s, F) the simple s-waves are represented by lines $c=$ const; these waves are trajectories of the vector field (1.6). (In Fig.1 the section s^0-s_1 represents the s-wave when $c=c^0$, and the section s_2-s_3 represents the c-wave). Hence the Riemann invariant that is constant along the c-characteristic is the concentration $c=c(s, F)$, and the invariant constant along the characteristic is any function which is constant along the trajectories of the vector field (1.6) and varies monotonically from the trajectory to trajectory.

System (1.1), (1.2) has the following types of discontinuity /1/:

$$V=\frac{[F]}{[s]}, \quad [c]=0 \tag{1.7}$$

$$V=\frac{F^{\pm}}{s^{\pm}+[a]\cdot[c]^{-1}}, \quad [c]\neq 0 \tag{1.8}$$

where $[A]$ is the jump of the parameter A, and V is the velocity of the discontinuity. Discontinuities of types (1.7) and (1.8) are called s- and c-jumps respectively. The following definition of the discontinuity stability of system (1.1), (1.2) is proposed: the discontinuity is stable, when the total number of characteristics in the zone ahead of the jump with a velocity not higher than V, and in the zone behind the jump with velocity not less than V, is equal to three. This is one of the possible generalizations of the definition of a wave adiabatic compared with the Lax form of the stability criterion /6/.

2. Two solutions of the problem of discontinuity decay. Consider the initial system of equations with function $a(c)$, whose graph is shown in Fig.2.

We will solve the problem of discontinuity decay

$$s(x,0)=\begin{cases} s^{\circ}, x<0 \\ s_*, x>0\end{cases}; \quad c(x,0)=\begin{cases} c^{\circ}, x<0 \\ c_*, x>0\end{cases} \tag{2.1}$$

The solution of this problem describes the process of petroleum displacement by a solution of active impurities. The problem has the selfsimilar solution (*) $s=s(\xi)$, $c=c(\xi)$,

*)See the footnote on p.484.

$\xi = x/t$.

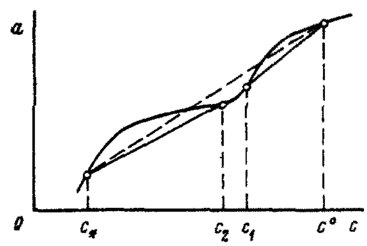

Fig.2

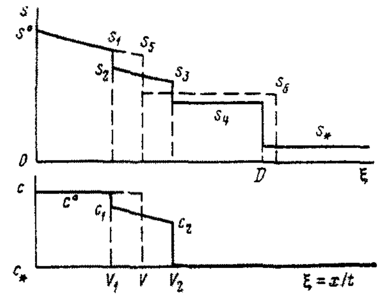

Fig.3

We will rewrite the initial conditions (2.1) in the form

$$s(-\infty) = s^\circ,\ c(-\infty) = c^\circ,\quad s(\infty) = s_*,\quad c(\infty) = c_* \tag{2.2}$$

To construct the solution we shall find c_1 and c_2 such that (Fig.2)

$$a'(c_1) = [a(c^\circ) - a(c_1)](c^\circ - c_1)^{-1},$$
$$a'(c_2) = [a(c_2) - a(c_*)]\ (c_2 - c_*)^{-1}$$

We draw a tangent to the curve $c = c^\circ$ from the point $[-a'(c_1), 0]$ in the plane (s, F). From the point s_2 where it intersects the curve $c = c_1$ we draw the trajectory of the vector field (1.6) until it intersects the curve $c = c_2$ at the point s_3. We connect the points s_3 and $[-a'(c_2), 0]$ by a section of straight line. We join the point s_4 where that section intersects the curve $c = c_*$ to the point (s_*, c_*) (Fig.1).

The solution has the form

$$\begin{aligned}
&s = s^\circ,\quad c = c^\circ;\quad -\infty < \xi < 0 \\
&\xi = \frac{dF}{ds} = F_s',\quad s(0) = s^\circ,\quad c(0) = c^\circ \\
&0 < \xi < V_1 = F_s'(s_1, c^\circ) = \frac{F(s_1, c^\circ)}{s_1 + a'(c_1)} \\
&\xi = \frac{dF}{ds} = \frac{F}{s + a'},\quad s(V_1) = s_2,\quad c(V_1) = c_1 \\
&V_1 = \frac{F(s_2, c_1)}{s_2 + a'(c_1)} < \xi < V_2 = \frac{F(s_3, c_2)}{s_3 + a'(c_2)} \\
&s = s_4,\quad c = c_*,\quad V_2 = \frac{F(s_4, c_*)}{s_4 + a'(c_2)} < \xi < D = \frac{F(s_4, c_*)}{s_4 - s_*} \\
&s = s_*,\quad c = c_*,\quad D < \xi < \infty
\end{aligned} \tag{2.3}$$

The solution consists of a section of centered s-wave $(s^\circ, c^\circ) - (s_1, c^\circ)$, a c-jump $(s_1, c^\circ) \to (s_2, c_1)$, a section of centered c-wave $(s_2, c_1) - (s_3, c_2)$, a c-jump $(s_3, c_2) \to (s_4, c_*)$, a quiescent region, $s = s_4$, $c = c_*$, and an s-jump $(s_4, c_*) \to (s_*, c_*)$ (Fig.1). Each of the three jumps appearing in the solution are Lax stable. The form of solution is shown in Figs.1 and 3 by the solid line.

To derive the second solution of problem (2.2) we draw from the point $\{-[a(c^\circ) - a(c_*)]\cdot(c^\circ - c_*)^{-1}, 0\}$ a tangent to the curve $c = c^\circ$ and connect its point of intersection s_6 with the curve $c = c_*$ to the point (s_*, c_*). The solution has the form

$$\begin{aligned}
&s = s^\circ,\ c = c^\circ,\ -\infty < \xi < 0 \\
&\xi = \frac{dF}{ds} = F_s',\quad s(0) = s^\circ,\quad c(0) = c^\circ \\
&0 < \xi < V_1 = F_s'(s_5, c^\circ) = \frac{F(s_5, c^\circ)}{s_5 + [a(c^\circ) - a(c_*)](c^\circ - c_*)^{-1}} \\
&s = s_6,\quad c = c_* \\
&V_1 = \frac{F(s_6, c_*)}{s_6 + [a(c^\circ) - a(c_*)](c^\circ - c_*)^{-1}} < \xi < D = \frac{F(s_6, c_*)}{s_6 - s_*} \\
&s = s_*,\quad c = c_*,\quad D < \xi < \infty
\end{aligned} \tag{2.4}$$

and consists of a section of centred s-wave $(s^\circ, c^\circ) - (s_5, c^\circ)$, a c-jump $(s_5, c^\circ) \to (s_6, c_*)$, a quiescent region $s = s_6$, $c = c_*$, and an s-jump $(s_6, c_*) \to (s_*, c_*)$. Both jumps are Lax stable.

Thus two stable generalized solutions have been constructed for the problem of discontinuity decay for the system of equations (1.1), (1.2). Both provide a plausible flow picture of petroleum displacement by an active-admixture solution (Fig.3).

3. Criterion of the admissibility of the discontinuity. We will consider the discontinuous solution of system (1.1), (1.2) to be admissible, if it is the limit of solutions of a two-phase filtration system with active admixture, taking into account the capillary jump of pressure between phases, and the non-equilibrium of the sorption process

$$\frac{\partial s}{\partial t}+\frac{\partial F}{\partial x}=hA^{\circ}\frac{\partial}{\partial x}\left[A(s,c)\frac{\partial s}{\partial x}\right] \tag{3.1}$$

$$\frac{\partial}{\partial t}(cs+a)+\frac{\partial}{\partial x}(cF)=hA^{\circ}\frac{\partial}{\partial x}\left[cA(s,c)\frac{\partial s}{\partial x}\right]$$

$$\frac{\partial a}{\partial t}=\frac{c-y}{h\tau},\quad a=a(y)$$

when the characteristic time $h\tau$ of the establishment in the system of thermodynamic equilibrium and the characteristic value hA° of the pressure jump approach zero. In this formula y is the admixture equilibrium concentration for which the dissolved admixture is with the sorbed admixture, whose current concentration is equal to c, in a state of thermodynamic equilibrium, i.e., $a=a(y)$ and $A(s,c)>0$. Compared with system (1.1), (1.2) the equation $a=a(c)$ of the sorption isotherm is replaced by the equation of sorption kinetics, and in the equation of phase-mass balance the "vanishing viscosity" term is added.

Theorems of the existence and uniqueness of the generalized solution of system (1.1), (1.2) do not exist at present. Let us, therefore, clarify how the proposed test of selecting the "true" solution is linked with the conditions of the existence and uniqueness of the generalized Cauchy problem for one first-order hyperbolic equation.

If in system (1.1), (1.2) we set $s=F=1$, we obtain the equation of equilibrium of sorption from the flow

$$\frac{\partial}{\partial t}[c+a(c)]+\frac{\partial c}{\partial x}=0 \tag{3.2}$$

If, however, we set $s=F=1$ in system (3.1), we obtain the system of equations of non-equilibrium sorption from the flow /12/

$$\frac{\partial}{\partial t}(c+a)+\frac{\partial c}{\partial x}=0 \tag{3.3}$$

$$\frac{\partial a}{\partial t}=\frac{c-y}{h\tau},\quad a=a(y) \tag{3.4}$$

Let us assume that the generalized solution of Eq.(3.2) is the limit of the solutions of system (3.3), (3.4) as $h\to 0$. At some point (x_0, t_0) let the solution (3.2) be discontinuous, $c(x_0-0,t_0)=c^-$, $c(x_0+0,t_0)=c^+$, and let the discontinuity velocity at that point be V.

We will write the Hugoniot condition at the discontinuity in the solution of the equilibrium sorption equation (3.2)

$$V=(1+[a]\cdot[c]^{-1})^{-1}$$

For the system of non-equilibrium sorption equations (3.3), (3.4) the Hugoniot conditions at the discontinuity have the form

$$[c+a]\cdot V=[c],\quad [a]\cdot V=0$$

It follows from them that the discontinuity velocity in system (3.3), (3.4) is equal either to zero or unity. The discontinuity velocity in (3.2) is less than unity. Hence, in spite of the fact that the non-equilibrium sorption system admits of discontinuous solutions, its solution at the point (x_0, t_0) is continuous. Then in a small neighbourhood of the point (x_0, t_0), the solution is approximately represented by

$$c(x,t)=\begin{cases} c^-, & x-x_0-V(t-t_0)<0\\ c^+, & x-x_0-V(t-t_0)>0\end{cases} \tag{3.5}$$

We will seek the solution of system (3.3), (3.4) in a small neighbourhood of the point (x_0, t_0) in the form of a travelling wave

$$c(x,t)=c(\xi),\quad a(x,t)=a(\xi),\quad \xi=[x-x_0-V(t-t_0)]/h$$

Substituting these functions into the system, we obtain, taking into account the condition for discontinuity admissibility, the boundary value problem

$$c(\pm\infty)=c^{\pm},\ a(\pm\infty)=a(c^{\pm}) \tag{3.6}$$

for the system of ordinary differential equations

$$-V\frac{d}{d\xi}(c+a)+\frac{dc}{d\xi}=0 \tag{3.7}$$

$$-V\frac{da}{d\xi}=\frac{c-y}{\tau}, \quad a=a(y) \tag{3.8}$$

If a solution of this boundary value problem exists, then as $h\to 0$ it becomes the discontinuous solution (3.5) of Eq.(3.2).

We integrate (3.7) from $-\infty$ to ξ taking (3.6) into account

$$(c-c^-)-V\left[c-c^-+a-a(c^-)\right]=0 \tag{3.9}$$

Substituting the function $y=y(a)$ the inverse of $a=a(y)$, into Eq.(3.8) and, then the value of a expressed in terms of c from formula (3.9)

$$\frac{dc}{d\xi}=\frac{y(a^-+(1-V)(c-c^-)V)-c}{\tau(1-V)}=\Phi(c) \tag{3.10}$$

The boundary value problem (3.6) for Eq.(3.10) is solvable when and only when

a) the points c^- and c^+ are singular for the vector field (3.10):

b) the sign of $\Phi(c)$ in the interval between c^- and c^+ is the same as that of the reminder (c^+-c^-) /9/.

We substitute $c=c^+$ into (3.10). From condition a) we obtain

$$y(a(c^-)+(1-V)(c^+-c^-)/V)=c^+$$

By virtue of the monotonicity of the function $y(a)$ we have

$$a(c^-)+(1-V)(c^+-c^-)/V=a(c^+)$$

from which follows the Hugoniot condition at the discontinuity (3.5) for Eq.(3.2)

$$c^+-c^-=V\left[c^++a(c^+)-c^--a(c^-)\right]$$

Substituting the expression obtained for the discontinuity velocity V into (3.10), from condition b) we obtain that in the interval between c^- and c^+ the sign of expression

$$\{a(c^-)+[a(c^+)-a(c^-)](c^+-c^-)^{-1}(c-c^-)-a(c)\}$$

is the same as that of the difference (c^+-c^-). In the plane (c,a), when $c^+<c^-$ the curve $a(c)$ lies in the interval between c^- and c^+ above the segment that joins the points $[c^-,a(c^-)]$ and $[c^+,a(c^+)]$. These points are neighbouring points of intersection of the curve with the segments. Hence the inequality

$$a'(c^-)\leqslant[a(c^+)-a(c^-)](c^+-c^-)^{-1}\leqslant a'(c^+)$$

holds. From this follows the discontinuity-evolution condition (the discontinuity is reached by two characteristics)

$$\frac{1}{1+a'(c^+)}\leqslant V=\frac{1}{1+[a]\cdot[c]^{-1}}\leqslant\frac{1}{1+a'(c^-)}$$

Hence from the condition for a discontinuity to be admissible the Hugoniot condition, the condition for discontinuity evolution, and the supplementary condition at the discontinuity follow. When these conditions are satisfied, the Cauchy problem for Eq.(3.2) has a generalized solution which is unique.

Note that for an arbitrary quasilinear first-order hyperbolic equation

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0$$

by passing to the "non-linear" system

$$\frac{\partial u}{\partial t}+\frac{\partial f}{\partial x}=0,\quad \frac{\partial f}{\partial t}=\frac{u-w}{h\tau};\quad f=f(w)$$

we also obtain the conditions that ensure the existence and uniqueness of the generalized solution. For one equation the proposed approach yields the same results as the method of vanishing viscosity /9/.

Let us find under what conditions the discontinuous solution of system (1.1), (1.2) can be obtained as the limit of the solutions of system (3.1), for this we will analyze the behaviour of the solutions of system (3.1) in the neighbourhood of the point (x_0,t_0). System (3.1) admits of discontinuous solutions with concentration jumps. It follows from the Hugoniot conditions that the concentration discontinuities in systems (1.1), (1.2) and (3.1) propagate at different velocities. Hence in the neighbourhood of the point (x_0,t_0) the solution of system (3.1) is generally continuous with appropriate boundary conditions.

As in the case of one equation, we confine ourselves in the neighbourhood of the point of discontinuity of the solution of system (1.1), (1.2)

$$s(x,t)=\begin{cases} s^-, x-x_0-V(t-t_0)<0 \\ s^+, x-x_0-V(t-t_0)>0 \end{cases} \tag{3.11}$$
$$c(x,t)=\begin{cases} c^-, x-x_0-V(t-t_0)<0 \\ c^+, x-x_0-V(t-t_0)>0 \end{cases}$$

to solutions of system (3.1) in the form of a travelling wave

$$c(x,\ t)=c(\xi),\quad s(x,\ t)=s(\xi),\quad a(x,\ t)=a(\xi)$$
$$\xi=[x-x_0-V(t-t_0)]/h$$

We obtain the following boundary value problem:

$$s(\pm\infty)=s^{\pm},\ c(\pm\infty)=c^{\pm},\ a(\pm\infty)=a(c^{\pm}) \tag{3.12}$$

for the system of ordinary differential equations

$$-V\frac{ds}{d\xi}+\frac{dF}{d\xi}=A^\circ\frac{d}{d\xi}\left[A(s,c)\frac{ds}{d\xi}\right] \tag{3.13}$$
$$-V\frac{d}{d\xi}(cs+a)+\frac{d}{d\xi}(cF)=A^\circ\frac{d}{d\xi}\left[cA(s,c)\frac{ds}{d\xi}\right] \tag{3.14}$$
$$-V\frac{da}{d\xi}=\frac{c-y}{\tau},\quad a=a(y) \tag{3.15}$$

Since the solution of the boundary value problem (3.12) for system (3.13)—(3.15) approaches the discontinuous solution (3.11) of system (1.1), (1.2) as $h\to 0$, the existence of a continuous solution of the boundary value problem is the condition for the discontinuty to be admissible. Below we separate the types of jumps that have the structure (3.1). It is assumed that non-evolutionary jumps, which have an infinite number of structures, are not admissible /8/.

Theorem. The boundary value problem (3.12) has a unique continuous solution when and only when

1) the Hugoniot conditions at the discontinuity (3.11) are satisfied for the system of laws of conservation (1.1), (1.2)

$$V=\frac{[F]}{[s]}=\frac{[cF]}{[cs+a(c)]} \tag{3.16}$$

2) in the interval between c^- and c^+ the sign of the expression

$$a(c^-)+[a][c]^{-1}(c-c^-)-a(c)$$

is the same as that of the difference (c^+-c^-)

3) The over-all number of characteristics in the zone ahead of the jump at a velocity not higher than V and in the zone behind the jump at a velocity not less than V, is equal to three.

Proof. Necessity. Let the continuous solution of the boundary value problem (3.12) exist. We integrate Eqs.(3.13) and (3.14) form $-\infty$ to ξ taking into account the boundary conditions

$$A^\circ A(s,c)\frac{ds}{d\xi}=F-F^--V(s-s^-) \tag{3.17}$$

$$A^\circ cA(s,c)\frac{ds}{d\xi}=cF-c^-F^--V[cs+a-c^-s^--a(c^-)] \tag{3.18}$$

The points $[s^-,c^-,a(c^-)]$ and $[s^+,c^+,a(c^+)]$ are singular for system (3.15), (3.17), (3.18). Substituting the values $c=c^+$, $s=s^+$ and $a=a(c^+)$ into them, we obtain the Hugoniot conditions (3.16) from which formulas (1.7) and (1.8) follow.

Condition 2) has meaning when $c^-\neq c^+$. At the discontinuity condition (1.8) is satisfied. We subtract Eq.(3.17) multiplied by c from Eq.(3.18). For $c^-\neq c^+$ we obtain the following equation: $F^-=V[s^-+(a-a(c^-))(c-c^-)^{-1}]$. Comparing it with (1.8), we have

$$[a-a(c^-)]\cdot(c-c^-)^{-1}=[a]\cdot[c]^{-1} \tag{3.19}$$

The set of points in the plane (c,a) that satisfy this relation lies on the segment connecting the points $(c^-,a(c^-))$ and $(c^+,a(c^+))$. Substituting the expression for a from (3.19) into (3.15), and using $y=y(a)$ we obtain

$$\frac{[a]}{[c]}\cdot\frac{dc}{d\xi}=\frac{y\{a(c^-)+[a]\cdot(c-c^-)[c]^{-1}\}-c}{\tau V} \tag{3.20}$$

As in the case considered above of one hyperbolic equation, the condition for a continuous solution of the boundary value problem for Eq. (3.20) to exist is equivalent to conditions a) and b). Condition a) is satisfied. When $c^- > c^+$ condition b) for Eq. (3.20) means that in the interval (c^+, c^-) since the function $a(y)$ is monotonic the inequality $a(c^-) + [a](c - c^-)[c]^{-1} < a(c)$ is satisfied. When $c^+ > c^-$ the inequality sign is reversed. Condition 2) of the theorem is satisfied.

From this we have the inequalities $a'(c^-) \leqslant [a]\cdot[c]^{-1} \leqslant a'(c^+)$. Then

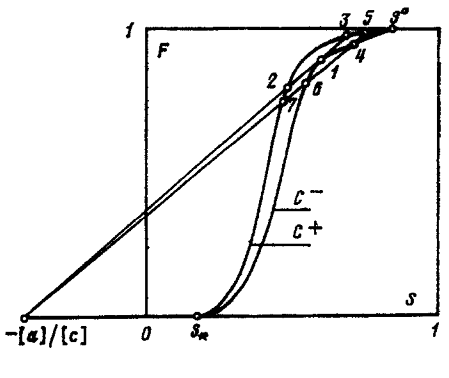

Fig.4

$$\frac{F^+}{s^+ + a'(c^+)} \leqslant \frac{F^+}{s^+ + [a][c]^{-1}} = V = \frac{F^-}{s^- + [a][c]^{-1}} \leqslant \frac{F'}{s^- + a'(c^-)}$$

i.e. the velocity of the c-characteristic in the zone ahead of the discontinuity is not greater than V, and in the zone behind it, it is not less than V. Two c-characteristics reach the c-jumps. It remains to prove that only one c-characteristic reaches the c-jumps.

Let $c^- > c^+$. Depending on the number of of s-characteristics reaching the jump from curve $c = c^-$ to curve $c = c^+$, (various types of jumps can be seen in Fig.4), we have

$$V_{12} = F_s'(s_1, c^-), \quad F_s'(s_3, c^+) < V_{13} = F_s'(s_1, c^-)$$
$$F_s'(s_5, c^+) < V_{45}, \quad V_{67} < F_s'(s_6, c^-)$$
$$F_s'(s_5, c^+) < V_{65} < F_s'(s_6, c^-)$$

where V_{ij} is the velocity of the jump $(s_i, c^-) \to (s_j, c^+)$.

Let us assume the opposite, i.e. the discontinuity which is not reached by a single s-characteristic, or by two s-characteristics is admissible. Then the discontinuity belongs to one of the types (Fig.4)

$$(s_4 \to s_7), \quad (s_6 \to s_5), \quad (s_1 \to s_3)$$

Let a trajectory exist that links points 4 and 7. Let us consider the system of two ordinary differential equations (3.17) and (3.20), in the phase plane (s, c). For the trajectory considered we have

$$s(-\infty) = s_4, \quad c(-\infty) = c^-, \quad s(\infty) = s_7, \quad c(\infty) = c^+$$

Let us consider the behaviour of the system trajectories in the neighbourhood of the singular point (s_4, c^-). The matrix of the linearized system has two eigenvalues

$$\lambda_1 = \frac{F_s'(s_4, c^-) - V}{A^\circ A(s_4, c^-)} < 0, \quad \lambda_2 = \frac{[a][c]^{-1} - a'(c^-)}{V\tau[a][c]^{-1}a'(c^-)} > 0$$

to which correspond the eigenvectors

$$h_1 = (1, 0), \quad h_2 = \left(-\frac{F_c'(s_4, c^-)}{A^\circ A(s_4, c^-)}, \ \frac{F_s'(s_4, c^-) - V}{A^\circ A(s_4, c^-)} - \frac{[a]\cdot[c]^{-1} - a'(c^-)}{V\tau[a][c]^{-1}\cdot a'(c^-)}\right)$$

i.e. the singular point (s_4, c^-) is a saddle point. From the theorem of the behaviour of trajectories near a saddle point /13/ it follows that the required field trajectory is an unsteady whisker of the saddle and touches the eigenvector h_2. By virtue of condition 2) proved above, $c^\cdot < 0$ and the motion along the trajectory in the neighbourhood of the singular point is in the direction of vector h_2 when $s^\cdot > 0$, i.e., in some neighbourhood of the singular point $c_s^\cdot < 0$.

Let us consider the trajectory pattern in the plane (s, F). Since $c^\cdot < 0$, the trajectory lies above the straight line $F - F_4 = V(s - s_4)$. Hence, as follows from (3.17), along the whole trajectory we have $s^\cdot < 0$. The contradiction obtained shows that a trajectory connecting points 4 and 7 does not exist.

The pairs of points 6 and 5, and 1 and 3 connect an infinite multiplicity of trajectories, i.e. the respective jumps are not admissible. The case of $c^- < c^+$ is considered similarly.

Let us consider the s-jumps $c^- = c^+$. In the solution of the boundary value problem (3.12) for system (3.17), (3.20) we have $c(\xi) = c^\pm$. Problem (3.12) reduces to the boundary value problem $s(-\infty) = s^-, s(+\infty) = s^+$ for one equation (3.17). Since the curve $F = F(s, c^\pm)$ has only one point of inflection, condition b) for a solution of the boundary value problem to exist for one ordinary differential equation, is equivalent to the conditions

$$F_s'(s^+, c^\pm) \leqslant V \leqslant F_s'(s^-, c^\pm)$$

i.e. there are two s-characteristic reaching the saturation jumps.

Sufficiency. We shall prove that when conditions 1)−3) are satisfied, a unique solution of the boundary value problem $s(\pm\infty)=s^{\pm}, c(\pm\infty)=c^{\pm}$ exists for system (3.17)−(3.20). That solution together with $a(\xi)$ expressed in (3.19) is the solution of the boundary value problem (3.12) for system (3.12)−(3.15).

From the Hugoniot conditions 1) it follows that the points (s^+, c^+) and (s^-, c^-) are singular in system (3.17), (3.20). Two *s*-characteristics must reach the *s*-jumps. This implies condition b) for Eq.(3.17), when $c(\xi)=c^{\pm}$, i.e., a solution $s(\xi)$ of the boundary value problem $s(\pm\infty)=s^{\pm}$ exists. This solution together with $c(\xi)=c^{\pm}$ for system (3.17), (3.20) is the solution required.

Let $c^-\neq c^+$. Condition 2) implies condition b) for Eq.(3.20), i.e. a solution of the boundary value problem $c(-\infty)=c^-, c(\infty)=c^+$ exists for it. From condition 2) it also follows that the *c*-characteristic velocity in the zone ahead of the jump is not greater than *V* and in the zone behind the jump it is not less than *V*. Consequently, it follows from condition 3) that one *s*-characteristic reaches the *c*-jump. The jump belongs to one of the types (Fig. 4)

$$(s_7\to s_7),\quad (s_4\to s_5),\quad (s_1\to s_2)$$

The proof that a solution exists for all types of jumps involves investigating the behaviour of system trajectories in the phase plane (s, c).

Consider the type $(s_6\to s_7)$. The matrix of the linearized system in the neighbourhood of the point (s_7, c^+) has two eigenvalues

$$\lambda_1=\frac{F_s'(s_7,c^+)-V}{A^\circ A(s_7,c^+)}>0,\quad \lambda_2=\frac{[a][c]^{-1}-a'(c^+)}{V\tau[a][c]^{-1}a'(c^+)}<0$$

to which correspond two eigenvectors

$$h_1=(1,0),\quad h_2=\left(-\frac{F_c'(s_7,c^+)}{A^\circ A(s_7,c^+)},\ \frac{F_s'(s_7,c^+)-V}{A^\circ A(s_7,c^+)}-\frac{[a][c]^{-1}-a'(c^+)}{V\tau[a][c]^{-1}\cdot a'(c^+)}\right)$$

i.e. the singular point (s_7, c^+) is a saddle. Consider the steady whisker along which $c^{\cdot}<0$. We direct $\xi\to-\infty$ along it. Then $c\to c^-$.

It is necessary to prove that $s\to s_6$. Let us assume the opposite. If $V_{67}<(s^\circ+[a][c]^{-1})^{-1}$, then point 6 is a unique singular point of the system when $c=c^-$. Hence from the above assumption it follows that the limit of $s(\xi)$ does not exist as $\xi\to-\infty$. If $V_{67}>(s^\circ+[a][c]^{-1})^{-1}$, then for $c=c^-$ apart from point 6, one more singular point 4 of the system exists. However, as shown above, a continuous trajectory connecting points 4 and 7 does not exist. In this case the assumption implies that the limit of $s(\xi)$ as $\xi\to-\infty$ does not exist. Since the quantity $s(\xi)$ is limited, it follows from here that it is not monotonic in any neighbourhood $|c-c^-|<\varepsilon$, i.e. in any such neighbourhood (or $\xi<-N_\varepsilon$) there exist on the trajectory a point at which $s^{\cdot}=0$.

Let us consider the pattern of the trajectory in the (s, F) plane. It follows from the above that in any neighbourhood of point 6 a point of intersection of the trajectory with the straight line $F-F(s_6,c^-)=V(s-s_6)$ exists. This means that in the phase plane of the system, in any neighbourhood of point 6, a point exists that belongs to the trajectory. The matrix of the linearized system in the neighbourhood of point 6 has two eigenvalues

$$\lambda_1=\frac{F_s'(s_6,c^-)-V}{A^\circ A(s_6,c^-)}>0,\quad \lambda_2=\frac{[a][c]^{-1}-a'(c^-)}{V\tau[a][c]^{-1}a'(c^-)}>0$$

i.e. that point is an unstable node. Then the statement that the limit of $s(\xi)$ does not exist as $\xi\to-\infty$ contradicts the theorem on the behaviour of the trajectories in the neighbourhood of the unsteady node /13/.

It follows from the above reasoning that the trajectory considered here is a unique trajectory connecting points 6 and 7.

The existence and uniqueness of the trajectories linking points 4 and 5, and 1 and 2 is proved similarly.

The conditions for the discontinuity to be admissible obtained above are not only necessary, but also, sufficient to construct the unique selfsimiliar solution of the Riemann problem of the decay of an arbitrary discontinuity for the system of equations (1.1) and (1.2).

In solution (2.3) of the problem of the decay of discontinuity (2.1) all discontinuities are admissible (2.4) the jump $(s_5\to s_6)$ is present. The segment connecting points $[c^-, a(c^-)]$ and $[c^+, a(c^+)]$ in the (c, a) plane intersects the curve $a(c)$, which contradicts condition 2) of the admissibility of the discontinuity (Fig.2). Solution (2.3) is true.

The criterion obtained for the admissibility of the discontinuity is a well-known generalization of the concept of a wave adiabatic compared with the condition for the stability of the discontinuity in the Lax form /5,6/. Condition 3) of the theorem is the well-known condition for the evolution of the discontinuity, which ensures the existence of solution of the linearized problem of the interaction of a small perturbation with the discontinuity /8,11/.

Condition 2) of the theorem is a supplementary condition at the discontinuity obtained as the condition for the structure (3.1) to exist. This condition means that the sorption process, is undirectional, i.e. that in the neighbourhood of a discontinuity either a sorption process $a_t > 0$ or a desorbtion process $a_t < 0$ occurs. Condition 2) of the theorem can be rephrased thus: the discontinuity $(s^-, c^-) \to (s^+, c^+)$ is admissible, if the points (s^-, c^-) and (s^+, c^+) can be connected by a continuous curve (s_η, c_η) on which the Hugoniot conditions for the jump $(s^-, c^-) \to (s_\eta, c_\eta)$ are satisfied, and the velocity V for any η is not less than the velocity V of the jump $(s^-, c^-) \to (s^+, c^+)$, i.e.

$$V_\eta \geqslant V \tag{3.21}$$

If condition (3.21) is not satisfied, then in solving the non-linearized problem of the interaction of the discontinuity with small perturbations an inversion of the perturbation front occurs before the perturbation reaches the discontinuity.

4. Admissibility of a discontinuity in more complicated systems. We can now consider a thermodynamically unstable system in order to obtain the conditions for a discontinuity of many systems of equations of underground physico-chemical hydro-gasdynamics to be admissible. We shall give a few examples.

The process of two-phase filtration with an active admixture, soluble in both phases, is defined by the system of equations of phase-mass balance (1.1) and admixture-mass balance.

$$\frac{\partial}{\partial t}[cs + \varphi(c)(1-s)] + \frac{\partial}{\partial x}[cF + \varphi(c)(1-F)] = 0 \tag{4.1}$$

where $\varphi(c)$ is the equilibrium concentration of admixture in the second phase. If $\varphi(c)$ has the form shown in Fig.2, problem (2.1) admits of two selfsimilar solutions, when $\varphi(c)$ is more complex it has three or more solutions.

To obtain the supplementary conditions at the discontinuity we introduce into Eqs.(1.1) and (1.2) a capillary pressure jump similar to (3.1), and replace the equation $\varphi = \varphi(c)$ by the equation of the kinetics of the distribution of the admixture among the phases

$$\frac{\partial \varphi}{\partial t} = \frac{c - y}{h\tau}, \quad \varphi = \varphi(y) \tag{4.2}$$

When $h \to 0$ on the discontinuity (3.11) of system (1.1), (4.1) we obtain the Hugoniot conditions and the stability conditions. They are equivalent to conditions 1) and 3) of the theorem and to equality of the sign of the equilibrium and running concentration φ and the sign of the difference $(c^- - c^+)$.

The process of petroleum displacement by a solvent is described by a system of equations of two-phase three-component filtration /14/

$$\begin{gathered}\frac{\partial}{\partial t}[cs + \sigma(c)(1-s)] + \frac{\partial}{\partial x}[cF + \sigma(c)(1-F)] = 0 \\ \frac{\partial}{\partial t}[\varphi(c)s + \psi(c)(1-s)] + \frac{\partial}{\partial x}[\varphi(c)F + \psi(c)(1-F)] = 0\end{gathered} \tag{4.3}$$

where c and σ are the concentrations of the solvent in the water and petroleum phases, and φ and ψ are the concentrations of admixture in the water and petroleum phases. Changing to the unknowns $C_W = \varphi(c)s + \psi(c)\ (1-s)$ and $U_W = \varphi(c)F + \psi(c)(1-F)$, we obtain

$$\frac{\partial C_W}{\partial t} + \frac{\partial U_W(C_W, c)}{\partial x} = 0, \quad \frac{\partial c}{\partial t} + \frac{U_W + \beta'/\alpha'}{C_W + \beta'/\alpha'}\frac{\partial c}{\partial x} = 0$$

$$\alpha(c) = (c - \sigma)(\varphi - \psi)^{-1}, \ \beta(c) = \sigma - \psi(c - \sigma)(\varphi - \psi)^{-1}$$

If the form of the function $\beta(\alpha)$ is that shown in Fig.2, the problem of discontinuity decay has two Lax stable selfsimilar solutions. Supplementary conditions at the discontinuity are obtained by introducing into system (4.3) a capillary jump and taking into account the kinetics of the solution process.

The discontinuity in system (4.3) is admissible, if conditions 1) and 3) of the theorem are satisfied and, also, the sign of the difference $c^- - c^+$ is the same as that of the expression $\beta^- + [\beta](\alpha - \alpha^-)[\alpha]^{-1} - \beta(\alpha)$.

For the systems considered here the Hugoniot conditions and the stability conditions ensure the existence and uniqueness of a selfsimilar solution of the problem of arbitrary discontinuity decay. The proof of this reduces to the problem of classifying the types of configurations for an arbitrary discontinuity decay /3/ and to proving the uniqueness for each type as in /6/.

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ON THE ASYMPTOTIC THEORY OF THE THREE-DIMENSIONAL FLOW OF A HYPERSONIC STREAM OF RADIATING GAS AROUND A BODY*

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The three-dimensional flow of a hyersonic stream of ideal gas round bodies of arbitrary thickness allowing for radiation at high temperatures is investigated using the method of a thin optically transparent shock layer, which is a generalization of the well-known method of a thin shock (boundary) layer /1/. Using the fundamental property of the gas in the thin shock layer, which expresses the conservation of the ratio of the stream component of vorticity along streamlines to the density of the gas /2,3/, an analytic solution is obtained of the non-linear problem of the flow round a body bounded by a surface of zero total curvature. The distribution of the radiation heat flux to the body is determined. The effect of radiation on the flow of gas is considered, as an example, in the neighbourhood of the plane of symmetry of a conical body at the angle of attack.

The flow of a hypersonic stream of radiating gas round a body for the plane and axisymmetric cases has been studied in numerous papers (see /4,5/ and the bibliography there). Recently the first results of a numerical calculation of the three-dimensional hypersonic flow of a selectively radiating gas mixture over a blunted body were obtained in /6/. Two-dimensional flow round bodies was considered in /7,8/ using the method of a thin shock layer /1/.

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