Basic Trigonometry and   *Mathematics IMA*
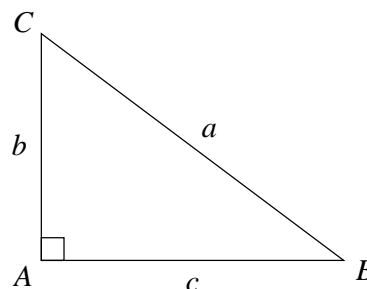# Radians

Pythagoras' Theorem

In a right-angled triangle

$$a^2 = b^2 + c^2,$$

where a is the side *not* involved with the the right-angle.
This is also the longest side and is called the *hypotenuse*.

Example: What is the length of the hypotenuse of a right-angled triangle whose other sides have lengths 1.7 cm and 2.3 cm?

If the hypotenuse has length a cm, then by Pythagoras,

$$\begin{aligned} a^2 &= (1.7)^2 + (2.3)^2 \\ &= 8.18 \\ \therefore \quad a &= 2.86 \text{ cm (to 2 d.p.)} \end{aligned}$$

Exercises

(1)   What is the length of the hypotenuse of a right-angled triangle whose other sides have lengths 4.5 cm and 5.6 cm?

(2)   What is the length of the diagonal of a rectangle whose sides are 3 m and 4 m?

(3)   The largest side of a right-angled triangle is 27.0 cm, and the other sides are equal. What is their length?

Radian Measure of Angles

Up until now you have probably seen angles measured in *degrees*. It turns out that the alternative unit of *radians* are more suitable for most mathematical applications (particularly calculus).

Radians: To define a radian, draw a circle of radius 1 unit (known as the *unit circle*) as shown.

If we start on the circle at $(1,0)$ and measure around anti-clockwise until we have an *arc length* = 1 unit (as shown by the thick line), the angle formed at the centre is 1 radian, or 1^c.
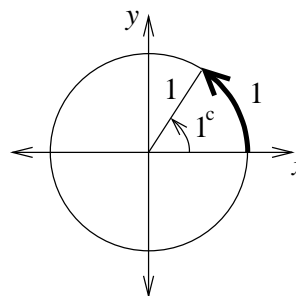
Since the circumference of the circle $= 2\pi r = 2\pi \times 1 = 2\pi$, we can say that

$$\begin{aligned} 2\pi^c &= 360° \\ \therefore \quad \pi^c &= 180°. \end{aligned}$$

Other commonly used angles are easily converted as well:

$$\frac{\pi^c}{2} = 90°, \quad \frac{\pi^c}{4} = 45°, \quad \frac{\pi^c}{3} = 60°, \quad \frac{\pi^c}{6} = 30°.$$

From above, a useful conversion factor is

$$\frac{\pi^c}{180°} \text{ for degrees to radians} \quad \left(\text{or } \frac{180°}{\pi^c} \text{ for radians to degrees}\right).$$

Examples:

(i) $\dfrac{\pi^c}{4} = \dfrac{180°}{4} = 45°$ (ii) $3\dfrac{1}{3}\pi^c = \dfrac{10}{3} \times 180° = 600°$

(iii) $60° = \dfrac{60°}{1} \times \dfrac{\pi^c}{180°} = \dfrac{\pi^c}{3}$ (iv) $720° = \dfrac{720°}{1} \times \dfrac{\pi^c}{180°} = 4\pi^c$

(v) $2^c = \dfrac{2^c}{1} \times \dfrac{180°}{\pi^c} = 114.59°$

Your calculator should have a $\boxed{\text{DRG}}$ or $\boxed{\text{Mode}}$ button. This allows you to tell the calculator which units you are going to use for angles.
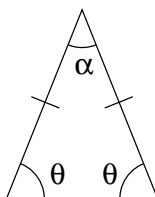
Triangles

In any triangle the angles add to 180°. (The angles in a straight line also add to 180°.)
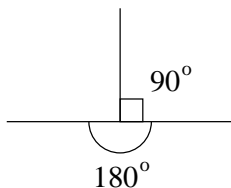
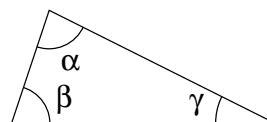The three types of triangle are:
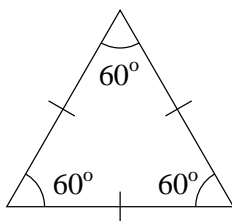
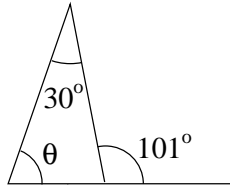|		
equilateral	isosceles	scalene
(3 sides, 3 angles equal)	(2 sides, 2 angles equal)	(no sides, no angles equal)

Exercises

(4)   A triangle has two angles of 31°. What is the other angle?

(5)   Is it possible for a right-angled triangle to have another angle of 105°?

(6)   In the diagram find the angle θ.

(7)   Convert to radians: (do not use a calculator for questions marked * in (7) and (8)).
     (a)   37°   (b)   45° *   (c)   88°   (d)   270° *   (e)   30° *

(8)   Convert to degrees:
     (a)   1 radian   (b) $\frac{\pi}{3}$ radians *   (c)   2.41 radians
     (d) $\frac{\pi}{2}$ radians *   (e)   0.12 radians

Trigonometric Ratios

In a right-angled triangle, let θ be one of the remaining angles.

Recall that the *hypotenuse* is the side not involving the right angle.

Of the other two sides, one is *adjacent* (or touching) θ and the other is *opposite*.
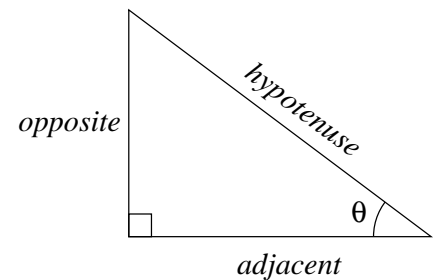
Using these, we make the following definitions:

The *sine* of θ, abbreviated "sin θ", is given by

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Similarly, the *cosine* of θ is

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

and the *tangent* of θ is

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{hypotenuse}} \div \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{opposite}}{\text{adjacent}}.$$

Using the first letter of each definition and the sides involved produces the 'word'

$$\text{"SohCahToa"},$$
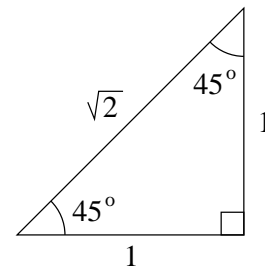
which may help you remember them.

The following two simple triangles allow us to work out these trigonometric ratios *exactly* for certain angles.

1. (First check that the side lengths satisfy Pythagoras' theorem.)

$$\sin 45° = \sin\frac{\pi^c}{4} = \frac{1}{\sqrt{2}} = 0.707 \text{ (to 3 dec. pl.)}$$

$$\cos 45° = \sin\frac{\pi^c}{4} = \frac{1}{\sqrt{2}} = 0.707$$

$$\tan 45° = \tan\frac{\pi}{4} = \frac{1}{1} = 1.$$

2. (Again, check that the lengths of the sides of the right-angled triangles satisfy Pythagoras' theorem.)
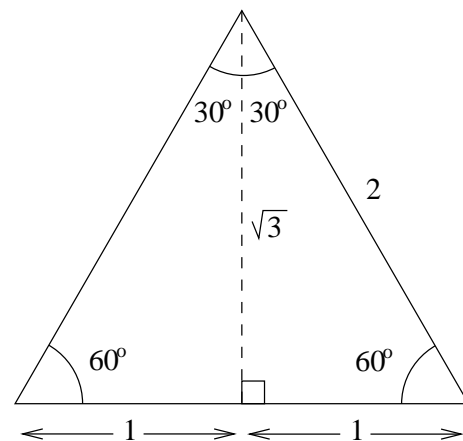
$$\sin 30° = \sin\frac{\pi^c}{6} = \frac{1}{2}$$

$$\cos 30° = \cos\frac{\pi^c}{6} = \frac{\sqrt{3}}{2} = 0.866 \text{ (to 3 d.p.)}$$

$$\tan 30° = \tan\frac{\pi^c}{6} = \frac{1}{\sqrt{3}} = 0.577 \text{ (to 3 d.p.)}$$

$$\sin 60° = \sin\frac{\pi^c}{3} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \cos\frac{\pi^c}{3} = \frac{1}{2}$$

$$\tan 60° = \tan\frac{\pi^c}{3} = \sqrt{3} = 1.732 \text{ (to 3 d.p.)}.$$

For other angles, we usually use a calculator to find sines, cosines and tangents, for example

$$\tan 50° = 1.192$$

$$\cos 1.1 = 0.454$$

If the degree symbol ° is not present, it is assumed that the angle is measured in radians.

Check these results on your calculator and remember to use the $\boxed{\text{DRG}}$ or $\boxed{\text{Mode}}$ button where necessary.

Example: Find the acute angle whose sine is 0.743. (Note: the use of 'acute' is important as there are obtuse angles (ones greater than 90°) which also have a sine of 0.743.)

In other words, find θ such that

$$\begin{aligned} \sin\theta &= 0.743 \\ \Rightarrow \theta &= \sin^{-1}0.743 \end{aligned}$$

This is evaluated on your calculator by using the $\boxed{\text{INV}}$ $\boxed{\text{sin}}$ keys (or $\boxed{\sin^{-1}}$ key, or $\boxed{\text{arcsin}}$ key). In this case

$$\theta = 48.0° \quad \text{(to 1 d.p.)}$$

Notes   (1) $\theta = \sin^{-1}(\text{number})$
        $\uparrow$ $\uparrow$
      angle   ratio of 2 side *lengths*

     (2)   "$\sin^{-1}$" is an operation name and, despite it's index appearence, is *not* $(\sin\theta)^{-1}$

        e.g. $\sin^{-1}(0.5) = 30°$, *but* $(\sin 30°)^{-1} = \dfrac{1}{\sin 30°} = \dfrac{1}{\frac{1}{2}} = 2$

For any angle θ,

$$\begin{aligned} (\sin\theta)^2 + (\cos\theta)^2 &= \left(\frac{\text{opposite}}{\text{hypotenuse}}\right)^2 + \left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)^2 \\[2mm] &= \frac{(\text{opposite})^2 + (\text{adjacent})^2}{(\text{hypotenuse})^2} \\[2mm] &= 1, \text{ by Pythagoras' theorem.} \end{aligned}$$

It is usual to write $(\sin\theta)^2$ as $\sin^2\theta$, so that

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

Exercises

(9)   Evaluate, using a calculator

     (a) $\sin 22°$
     (b) $\cos 22°$ and verify that $\sin^2 22° + \cos^2 22° = 1$.

(10)   Evaluate

     (a) $\sin 0.29$ (b) $\cos 43°$ (c) $\tan 65°$
     (d) $\sin 86°$ (e) $\cos \frac{4\pi}{9}$ (f) $\sin 0$
     (g) $\cos 0$ (h) $\tan 0$ (i) $\sin 90°$
     (j) $\cos 90°$ (k) $\tan 90°$
     Can you explain the response you get from the calculator in (j) and (k)?

(11)   Use a calculator to find the angles (to 3 sig. fig.) in (i) degrees and (ii) radians for

     (a) $\sin^{-1}(0.800)$ (b) $\sin^{-1}(-0.800)$ (c) $\cos^{-1}(0.707)$
     (d) $\cos^{-1}(-0.707)$ (e) $\tan^{-1}(1.00)$
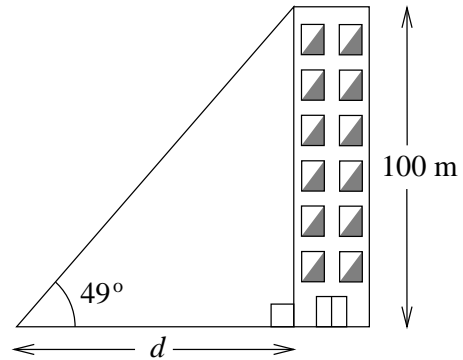
Problems Using Trigonometry

Example: For a person viewing a tall build-
ing, the angle of elevation of the top of the
building is 49°. If the building is 100 m tall,
how far is the man from the building?

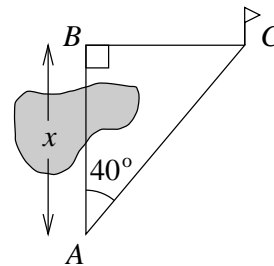**Solution**: Let the distance be d m. Then

$$\frac{100}{d} = \tan 49°$$

$$\therefore \quad d = \frac{100}{\tan 49°} = 86.9 \text{ m.}$$

Exercises

(12)   A small boat 500 m out to sea observes a cliff top on the shore at an angle of elevation
of 21°. What is the height of the cliff?

(13)   A surveyor wishes to establish the distance
x from point A to point B on the other side
of a lake. The distance AC in the diagram is
measured to be 275 m and the angle $\angle BAC$
is 40°. Find x.

(14)   [A bit harder] A forestry worker wishing to
establish the height h of a tall tree on the
other side of a river bank measures the angle
of elevation of the top of the tree to be 61° at
A, but 43° at a point B 15 m from A. Find
the height of the tree.
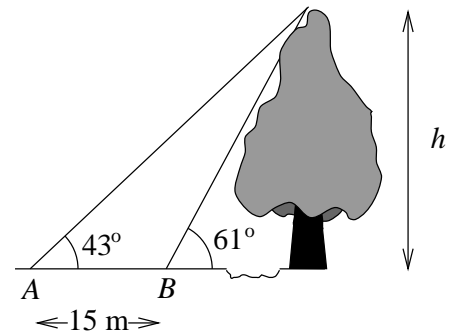
Notice from the exercises that trigonometry is often useful in situations where a direct mea-
surement is difficult or impossible. If we can evaluate other sides and angles of a triangle
involving the unknown quantity, we can often use trigonometry to find the length or angle
that we need.

Answers to Exercises

(1) 7.18 cm   (2) 5 m   (3) 19.1 cm

(4) 118°   (5) No   (6) 71°

(7) (a) 0.65   (b) $\pi/4$ or 0.79   (c) 1.54   (d) $3\pi/2$ or 4.71   (e) $\pi/6$ or 0.52

(8) (a) 57.3°   (b) 60°   (c) 138.1°   (d) 90°   (e) 6.9°

(9) (a) 0.375   (b) 0.927

(10) (a) 0.286   (b) 0.731   (c) 2.145   (d) 0.998   (e) 0.174   (f) 0
     (g) 1   (h) 0   (i) 1   (j) 0   (k) $\tan 90°$ does not exist.

(11) (a) 53.1°, 0.927^c (b) $-53.1°, -0.927^c$ (c) 45.0°, 0.786^c
     (d) 135°, 2.36^c (e) 45.0°, 0.785^c

(12) 192 m   (13) 210.7 m   (14) 28.95 m