MATHEMATICS IA CALCULUS
TECHNIQUES OF INTEGRATION
WORKED EXAMPLES

Find the following integrals:

1. $\displaystyle\int 3x^2 - 2x + 4 \, \mathrm{d}x$. See worked example Page 2.

2. $\displaystyle\int \frac{1}{x^2} + \frac{1}{x^2+1} \, \mathrm{d}x$. See worked example Page 4.

3. $\displaystyle\int x(x+1)^2 \, \mathrm{d}x$. See worked example Page 5.

4. $\displaystyle\int \frac{x+1}{\sqrt{x}} \, \mathrm{d}x$. See worked example Page 6.

5. $\displaystyle\int 2^x \, \mathrm{d}x$. See worked example Page 7.

6. $\displaystyle\int \frac{1}{3x-1} \, \mathrm{d}x$. See worked example Page 8.

7. $\displaystyle\int 3\sec^2(5x) \, \mathrm{d}x$. See worked example Page 9.

8. $\displaystyle\int \frac{2}{\sqrt{1-4x^2}} \, \mathrm{d}x$. See worked example Page 10.

9. $\displaystyle\int x \sin x^2 \, \mathrm{d}x$. See worked example Page 11.

10. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x \, \mathrm{d}x$. See worked example Page 12.

11. $\displaystyle\int x(2x+1)^{52} \, \mathrm{d}x$. See worked example Page 13.

12. $\displaystyle\int_0^4 \frac{\sqrt{x}}{\sqrt{x}+1} \, \mathrm{d}x$. See worked example Page 14.

13. $\displaystyle\int x^2 \cosh x \, \mathrm{d}x$. See worked example Page 16.

14. $\displaystyle\int_1^e \ln x \, \mathrm{d}x$. See worked example Page 18.

15. $\displaystyle\int e^x \cos x \, \mathrm{d}x$. See worked example Page 19.

16. $\displaystyle\int \tan^2 x \, \mathrm{d}x$. See worked example Page 21.

17. $\displaystyle\int \cos^2 x \, \mathrm{d}x$. See worked example Page 22.

18. $\displaystyle\int \cos^4 x \sin^5 x \, \mathrm{d}x$. See worked example Page 23.

19. $\displaystyle\int \cos^2 x \sin^2 x \, \mathrm{d}x$. See worked example Page 24.

20. $\displaystyle\int \cos 4x \sin 5x \, \mathrm{d}x$. See worked example Page 25.

21. $\displaystyle\int \frac{1}{(x^2+1)^3} \, \mathrm{d}x$, using the reduction formula for $n > 1$:

$$\int \frac{1}{(x^2+1)^n} \, \mathrm{d}x = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{1}{(x^2+1)^{n-1}} \, \mathrm{d}x$$

See worked example Page 26.

22. $\displaystyle\int \sec^6 x \, \mathrm{d}x$, by first finding a reduction formula for $\displaystyle\int \sec^n x \, \mathrm{d}x$, $n \geq 3$. See worked example Page 27.

23. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{12} x \, \mathrm{d}x$, by first finding a reduction formula for the *definite* integral $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x$, $n \geq 1$. See worked example Page 30.

24. $\displaystyle\int \frac{5x^2}{\sqrt{9-x^2}} \, \mathrm{d}x$. See worked example Page 33.

25. $\displaystyle\int \frac{x}{\sqrt{x^2-4}} \, \mathrm{d}x$. See worked example Page 35.

26. $\displaystyle\int \frac{1}{x^2+6x+13} \, \mathrm{d}x$. See worked example Page 36.

27. $\displaystyle\int \frac{x^3+2x}{x+3} \, \mathrm{d}x$. See worked example Page 37.

28. $\displaystyle\int \frac{x^2-2x-4}{x^3-2x^2-3x} \, \mathrm{d}x$. See worked example Page 38.

29. $\displaystyle\int \frac{8x^7+47x^6+98x^5+108x^4+106x^3+100x^2+104x+104}{(x-1)(x+2)^3(x^2+2x+2)^2} \, \mathrm{d}x$. See worked example Page 40.

$$\int (3x^2 - 2x + 4)\,dx$$

$$= 3 \times \frac{1}{3}x^3 - 2 \times \frac{x^2}{2} + 4x + C$$

$$= x^3 - x^2 + 4x + C$$

$$\int \frac{1}{x^2} + \frac{1}{x^2+1}\,dx$$

$$= \int x^{-2} + \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{(-1)} x^{-1} + \arctan(x) + C$$

$$= -\frac{1}{x} + \arctan x + C$$

$$\int x(x+1)^2 \, dx$$

$$= \int x(x^2 + 2x + 1) \, dx$$

$$= \int x^3 + 2x^2 + x \, dx$$

$$= \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + C$$

$$\int \frac{x+1}{\sqrt{x}}\,dx$$

$$= \int \sqrt{x} + \frac{1}{\sqrt{x}}\,dx$$

$$= \int x^{\frac{1}{2}} + x^{-\frac{1}{2}}\,dx$$

$$= \tfrac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

$$= \tfrac{2}{3}x\sqrt{x} + 2\sqrt{x} + C$$

$$\int 2^x \, dx$$

$$= \int e^{(\ln 2)x} \, dx$$

$$= \frac{1}{\ln 2} e^{(\ln 2)x} + C$$

$$= \frac{2^x}{\ln 2} + C$$

$$\int \frac{1}{3x-1}\,dx$$

$$= \frac{1}{3}\ln|3x-1| + C$$

$$\int 3 \sec^2(5x)\,dx$$

$$= 3 \times \frac{1}{5} \tan(5x) + C$$

$$= \frac{3}{5} \tan(5x) + C$$

$$\int \frac{2}{\sqrt{1-4x^2}}\,dx$$

$$= \int \frac{2}{\sqrt{1-(2x)^2}}\,dx$$

$$= 2 \times \frac{1}{2} \arcsin(2x) + C$$

$$= \arcsin(2x) + C$$

$$\int x \sin(x^2)\, dx.$$

Let $u = x^2$.

Then $\dfrac{du}{dx} = 2x$

$$\tfrac{1}{2}\, du = x\, dx$$

$$\therefore \int x \sin(x^2)\, dx = \int \tfrac{1}{2} \sin u \quad du$$

$$= -\tfrac{1}{2} \cos u + C$$

$$= -\tfrac{1}{2} \cos(x^2) + C$$

$$\int_0^{\pi/4} \sec^2 x \tan^2 x \, dx$$

$$= \int_0^{\pi/4} \sec^2 x \left(\tan x\right)^2 dx$$

Let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$$du = \sec^2 x \, dx$$

$x = 0 \Rightarrow u = \tan 0 = 0$

$x = \pi/4 \Rightarrow u = \tan \pi/4 = 1$

$$\therefore \int_0^{\pi/4} \sec^2 x \tan^2 x \, dx$$

$$= \int_0^1 u^2 \, du$$

$$= \left[\tfrac{1}{3} u^3\right]_0^1$$

$$= \tfrac{1}{3} \times 1^3 - \tfrac{1}{3} \times 0^3$$

$$= \tfrac{1}{3}$$

$$\int x\,(2x+1)^{52}\,dx.$$

Let $u = 2x+1$

$\frac{du}{dx} = 2$

$\frac{1}{2}\,du = dx$

Also: $u-1 = 2x$

$\frac{1}{2}(u-1) = x$

$$\therefore \int x(2x+1)^{52}\,dx = \int \frac{1}{2} \times \frac{1}{2}(u-1)\,u^{52}\,du$$

$$= \int \frac{1}{4}\left(u^{53} - u^{52}\right) du$$

$$= \frac{1}{4}\left(\frac{1}{54}u^{54} - \frac{1}{53}u^{53}\right) + C$$

$$= \frac{1}{216}u^{54} - \frac{1}{212}u^{53} + C$$

$$= \frac{1}{216}(2x+1)^{54} - \frac{1}{212}(2x+1)^{53} + C$$

$$\int_0^4 \frac{\sqrt{x}}{\sqrt{x}+1}\,dx$$

Let $u = \sqrt{x} + 1$

$= x^{\frac{1}{2}} + 1$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

$2\sqrt{x}\,du = dx$

$2(u-1)\,du = dx$

Also $\sqrt{x} + 1 = u$

$\sqrt{x} = u - 1.$

And:

$x = 0 \Rightarrow u = \sqrt{0} + 1$
$= 1$

$x = 4 \Rightarrow u = \sqrt{4} + 1$
$= 3$

$$\therefore \int_0^4 \frac{\sqrt{x}}{\sqrt{x}+1}\,dx = \int_1^3 \frac{u-1}{u} \cdot 2(u-1)\,du$$

$$= \int_1^3 \frac{2(u-1)^2}{u}\,du$$

$$= \int_1^3 \frac{2(u^2-2u+1)}{u}\,du$$

$$= \int_1^3 \frac{2u^2-4u+2}{u}\,du$$

$$= \int_1^3 2u - 4 + \frac{2}{u}\,du$$

$$\therefore \int_0^4 \frac{\sqrt{x}}{\sqrt{x}+1}\,dx = \left[2\times\frac{1}{2}u^2 \quad - 4u \quad + 2\ln|u| \right]_1^3$$

$$= \left[u^2 \quad -4u \quad + 2\ln|u| \right]_1^3$$

$$= \left(3^2 - 4\times 3 \quad + 2\ln|3| \right)$$

$$- \left(1^2 - 4\times 1 \quad + 2\ln|1| \right)$$

$$= \left(9 - 12 \quad + 2\ln 3 \right)$$

$$- \left(1 - 4 \right)$$

$$= \quad -3 \; + \; 2\ln 3 \quad - -3$$

$$= 2\ln 3$$

$\int x^2 \cosh x \, dx$

Let $f = x^2$ and $g' = \cosh x$

so $f' = 2x$ and $g = \sinh x$.

Then $\int x^2 \cosh x \, dx$

$= \int f g' \, dx$

$= f g - \int f' g \, dx$

$= x^2 \sinh x - \int 2x \sinh x \, dx$

Let $h = 2x$ and $k' = \sinh x$

so $h' = 2$ and $k = \cosh x$.

Then $\int 2x \sinh x \, dx$

$= \int h k' \, dx$

$= h k - \int h' k \, dx$

$$= 2x \cosh x - \int 2 \cosh x \, dx$$

$$= 2x \cosh x - 2 \sinh x + C$$

$$\therefore \int x^2 \cosh x \, dx$$

$$= x^2 \sinh x - \left[2x \cosh x - 2 \sinh x \right] + C$$

$$= x^2 \sinh x - 2x \cosh x + 2 \sinh x + C$$

$$= (x^2 + 2) \sinh x - 2x \cosh x + C$$

$$\int_1^e \ln x \, dx = \int_1^e 1 \cdot \ln x \, dx$$

Let $f' = 1$ and $g = \ln x$

so $f = x$ and $g' = \frac{1}{x}$.

Then $\int_1^e 1 \cdot \ln x \, dx$

$$= \int_1^e f' g \, dx$$

$$= [f g]_1^e - \int_1^e f g' \, dx$$

$$= [x \ln x]_1^e - \int_1^e x \times \frac{1}{x} \, dx$$

$$= [x \ln x]_1^e - \int_1^e 1 \, dx$$

$$= [x \ln x]_1^e - [x]_1^e$$

$$= [e \ln e - 1 \times \ln 1] - [e - 1]$$

$$= [e \times 1 - 1 \times 0] - e + 1$$

$$= e - e + 1$$

$$= 1$$

$$\int e^x \cos x \, dx$$

Let $f' = e^x$ and $g = \cos x$

so $f = e^x$ and $g' = -\sin x$

Then $\int e^x \cos x \, dx$

$$= \int f' g \, dx$$

$$= f g - \int f g' \, dx$$

$$= e^x \cos x - \int e^x (-\sin x) \, dx$$

$$= e^x \cos x + \int e^x \sin x \, dx$$

Let $h' = e^x$ and $k = \sin x$

so $h = e^x$ and $k' = \cos x$

so $\int e^x \sin x \, dx$

$$= \int h' k \, dx$$

$$= h k - \int h k' \, dx$$

$$= e^x \sin x - \int e^x \cos x \, dx$$

$$\therefore \int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x + C$$

$$\therefore \int e^x \cos x \, dx = \tfrac{1}{2} e^x (\cos x + \sin x) + C$$

$$\int \tan^2 x \, dx.$$

$$\cos^2 x + \sin^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$\therefore \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx$$

$$= \tan x - x + C$$

$$\int \cos^2 x \, dx$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$2\cos^2 x = \cos 2x + 1$$

$$\cos^2 x = \tfrac{1}{2}\cos 2x + \tfrac{1}{2}$$

$$\therefore \int \cos^2 x \, dx = \int \tfrac{1}{2}\cos 2x + \tfrac{1}{2} \, dx$$

$$= \tfrac{1}{2} \times \tfrac{1}{2} \sin 2x + \tfrac{1}{2}x + C$$

$$= \tfrac{1}{4} \sin 2x + \tfrac{1}{2}x + C$$

$$\int \cos^4 x \sin^5 x \, dx$$

$$= \int \cos^4 x \sin^4 x \sin x \, dx$$

$$= \int \cos^4 x (\sin^2 x)^2 \sin x \, dx$$

$$= \int \cos^4 x (1 - \cos^2 x)^2 \sin x \, dx$$

Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$-du = \sin x \, dx$$

$$\therefore \int \cos^4 x \sin^5 x \, dx$$

$$= \int -u^4 (1-u^2)^2 \, du$$

$$= \int -u^4 (1 - 2u^2 + u^4) \, du$$

$$= \int -u^4 + 2u^6 - u^8 \, du$$

$$= -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C$$

$$= -\frac{1}{5}(\cos x)^5 + \frac{2}{7}(\cos x)^7 - \frac{1}{9}(\cos x)^9 + C$$

$$\int \sin^2 x \cos^2 x \, dx$$

$$= \int (\sin x \cos x)^2 \, dx$$

$$= \int \left[\tfrac{1}{2}\sin(2x)\right]^2 \, dx$$

$$= \int \tfrac{1}{4}\sin^2(2x) \, dx$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x$$

$$\sin^2(2x) = \tfrac{1}{2} - \tfrac{1}{2}\cos(4x)$$

$$\int \sin^2 x \cos^2 x \, dx$$

$$= \int \tfrac{1}{4}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(4x)\right) dx$$

$$= \int \tfrac{1}{8} - \tfrac{1}{8}\cos(4x) \, dx$$

$$= \tfrac{1}{8}x - \tfrac{1}{8} \times \tfrac{1}{4}\sin(4x) + C$$

$$= \tfrac{1}{8}x - \tfrac{1}{32}\sin(4x) + C$$

$$\int \cos 4x \sin 5x \, dx$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(4x+5x) = \sin 4x \cos 5x + \cos 4x \sin 5x \quad ①$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\sin(5x-4x) = \sin 5x \cos 4x - \cos 5x \sin 4x \quad ②$$

Eq① + Eq② gives:

$$\sin(9x) + \sin(x) = 2\cos 4x \sin 5x$$

$$\therefore \cos 4x \sin 5x = \frac{1}{2}\sin(9x) + \frac{1}{2}\sin(x)$$

$$\therefore \int \cos 4x \sin 5x \, dx$$

$$= \int \frac{1}{2}\sin(9x) + \frac{1}{2}\sin(x) \, dx$$

$$= -\frac{1}{2} \times \frac{1}{9}\cos(9x) + -\frac{1}{2}\cos(x) + C$$

$$= -\frac{1}{18}\cos(9x) - \frac{1}{2}\cos x + C$$

Let $I_n = \int \frac{1}{(x^2+1)^n}\,dx$

Then

$$I_n = \frac{x}{(2n-2)\,(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\, I_{n-1}$$

$$I_1 = \int \frac{1}{x^2+1}\,dx = \arctan x + C$$

$$I_2 = \frac{x}{(2\times2-2)\,(x^2+1)^1} + \frac{2\times2-3}{2\times2-2}\, I_1$$

$$= \frac{x}{2\,(x^2+1)} + \frac{1}{2}\arctan x + C$$

$$I_3 = \frac{x}{(2\times3-2)\,(x^2+1)^2} + \frac{2\times3-3}{2\times3-2}\, I_2$$

$$= \frac{x}{4\,(x^2+1)^2} + \frac{3}{4}\left[\frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x\right] + C$$

$$= \frac{x}{4(x^2+1)^2} + \frac{3}{8}\,\frac{x}{(x^2+1)} + \frac{3}{8}\arctan x + C$$

$$\therefore \int \frac{1}{(x^2+1)^3}\,dx = \frac{x}{4(x^2+1)^2} + \frac{3}{8}\,\frac{x}{(x^2+1)} + \frac{3}{8}\arctan x + C$$

Let $I_n = \int \sec^n x \, dx$

$$= \int \sec^{n-2} x \sec^2 x \, dx$$

Let $f = \sec^{n-2} x$ and $g' = \sec^2 x$

so $f' = (n-2) \sec^{n-3} x \times \sec x \tan x$

$$= (n-2) \sec^{n-2} x \tan x$$

and $g = \tan x$

$$\int \sec^{n-2} x \sec^2 x \, dx$$

$$= \int f g' \, dx$$

$$= f g - \int f' g \, dx$$

$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan x \tan x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

Now $\cos^2 x + \sin^2 x = 1$

$$1 + \tan^2 x = \sec^2 x$$

$$\tan^2 x = \sec^2 x$$

$$\therefore \int \sec^n x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x - \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$$

$$\int \sec^6 x \, dx = I_6$$

$$I_2 = \int \sec^2 x \, dx = \tan x + C$$

$$I_4 = \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} I_2$$

$$= \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \tan x + C$$

$$\therefore I_6 = \frac{1}{5}\sec^4 x \tan x + \frac{4}{5} I_4$$

$$= \frac{1}{5}\sec^4 x \tan x$$

$$+ \frac{4}{5}\left[\frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x\right] + C$$

$$= \frac{1}{5}\sec^4 x \tan x$$

$$+ \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x + C$$

Let $J_n = \int_0^{\pi/2} \sin^n x \, dx$

$$= \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$$

Let $f = \sin^{n-1} x$ and $g' = \sin x$

so $f' = (n-1)\sin^{n-2} x \cos x$ and $g = -\cos x$

$$\therefore J_n = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$$

$$= \int_0^{\pi/2} f g' \, dx$$

$$= \Big[f g \Big]_0^{\pi/2} - \int_0^{\pi/2} f' g \, dx$$

$$= \Big[\sin^{n-1} x (-\cos x) \Big]_0^{\pi/2}$$

$$- \int_0^{\pi/2} (n-1) \sin^{n-2} x \cos x (-\cos x) \, dx$$

$$= \Big[\sin^{n-1}(\pi/2)(-\cos \pi/2) - \sin^{n-1}(0)(-\cos 0) \Big]$$

$$+ (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx$$

$$= [\,0 - 0\,]$$

$$+ (n-1)\int_0^{\pi/2} \sin^{n-2} x \ (1-\sin^2 x)\, dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x - \sin^n x \, dx$$

$$= (n-1)\left[\int_0^{\pi/2} \sin^{n-2} x \, dx - \int_0^{\pi/2} \sin^n x \, dx\right]$$

$$= (n-1)\left[\ J_{n-2} - J_n\ \right]$$

$$\therefore J_n = (n-1)\, J_{n-2} - (n-1)\, J_n$$

$$n\, J_n = (n-1)\, J_{n-2}$$

$$J_n = \frac{n-1}{n} J_{n-2}$$

$$J_0 = \int_0^{\pi/2} \sin^0 x \, dx$$

$$= \int_0^{\pi/2} 1 \, dx$$

$$= \Big[\ x\ \Big]_0^{\pi/2}$$

$$= \pi/2 - 0$$

$$= \pi/2$$

$$J_2 = \frac{2-1}{2} J_0$$

$$= \frac{1}{2} \times \frac{\pi}{2}$$

$$J_4 = \frac{4-1}{4} J_2$$

$$= \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

Continuing this pattern gives:

$$J_{12} = \frac{11}{12} \times \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\therefore \int_0^{\pi/2} \sin^{12} x \, dx = \frac{11}{12} \times \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\int \frac{5x^2}{\sqrt{9-x^2}}\,dx$$

Let $x = 3\sin u$ for $u \in [-\pi/2, \pi/2]$

$$\frac{dx}{du} = 3\cos u$$

$$dx = 3\cos u\,du$$

Also $x^2 = 9\sin^2 u$

$$9 - x^2 = 9 - 9\sin^2 u$$

$$= 9(1 - \sin^2 u)$$

$$= 9\cos^2 u$$

$$\sqrt{9-x^2} = \pm 3\cos u$$

But $\cos u \geqslant 0$ since $u \in [-\pi/2, \pi/2]$

so

$$\sqrt{9-x^2} = 3\cos u$$

Now $$\int \frac{5x^2}{\sqrt{9-x^2}}\,dx = \int \frac{5 \times 9\sin^2 u}{3\cos u} \times 3\cos u\,du$$

$$= \int 45\sin^2 u\,du$$

So $\displaystyle\int \frac{5x^2}{\sqrt{9-x^2}}\,dx = \int 45\sin^2 u\,du$

$$= \int 45\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2u\right) du$$

$$= \int \frac{45}{2} - \frac{45}{2}\cos 2u \;du$$

$$= \frac{45}{2}u - \frac{45}{2} \times \frac{1}{2}\sin 2u + C$$

$$= \frac{45}{2}u - \frac{45}{2} \times \frac{1}{2} \times 2\sin u\cos u + C$$

$$= \frac{45}{2}u - \frac{45}{2}\sin u\cos u + C$$

Now $x = 3\sin u$

so $\frac{x}{3} = \sin u$

$\arcsin\left(\frac{x}{3}\right) = u$

Also $\cos u = \frac{1}{3}\sqrt{9-x^2}$

$$\therefore \int \frac{5x^2}{\sqrt{9-x^2}}\,dx = \frac{45}{2}\arcsin\left(\frac{x}{3}\right) - \frac{45}{2} \times \frac{x}{3} \times \frac{1}{3}\sqrt{9-x^2} + C$$

$$= \frac{45}{2}\arcsin\left(\frac{x}{3}\right) - \frac{5}{2}x\sqrt{9-x^2} + C$$

$$\int \frac{x}{\sqrt{x^2-4}}\,dx$$

Let $u = x^2-4$

$$\frac{du}{dx} = 2x$$

$$\tfrac{1}{2}\,du = x\,dx$$

$$\therefore \int \frac{x}{\sqrt{x^2-4}}\,dx = \int \frac{1}{\sqrt{u}} \cdot \tfrac{1}{2}\,du$$

$$= \int \tfrac{1}{2} u^{-\frac{1}{2}}\,du$$

$$= \tfrac{1}{2} \times 2 u^{\frac{1}{2}} + C$$

$$= u^{\frac{1}{2}} + C$$

$$= \sqrt{x^2-4} + C$$

$$\int \frac{1}{x^2+6x+13}\,dx$$

$$x^2+6x+13 = x^2+6x+9+13-9$$

$$= (x+3)^2+4$$

so $$\int \frac{1}{x^2+6x+13}\,dx = \int \frac{1}{(x+3)^2+4}\,dx$$

Let $2u = x+3$

$$2\frac{du}{dx} = 1$$

$$2du = dx$$

so $$\int \frac{1}{(x+3)^2+4}\,dx = \int \frac{2}{(2u)^2+4}\,du$$

$$= \int \frac{2}{4u^2+4}\,du$$

$$= \int \frac{2}{4(u^2+1)}\,du$$

$$= \int \frac{1}{2}\times\frac{1}{u^2+1}\,du$$

$$= \tfrac{1}{2}\arctan u + C$$

$$= \tfrac{1}{2}\arctan\left(\frac{x+3}{2}\right) + C$$

$$\int \frac{x^3+2x}{x+3}\,dx$$

$$\begin{array}{r}
x^2 \quad -3x \quad +11 \\
x+3 \overline{\smash{)}\, x^3 + 0x^2 + 2x + 0} \\
-(x^3 + 3x^2) \qquad\qquad \\
\hline
-3x^2 + 2x + 0 \\
-3x^2 - 4x \qquad \\
\hline
11x + 0 \\
-(11x + 33) \\
\hline
-33
\end{array}$$

so $\dfrac{x^3+2x}{x+3} = x^2 - 3x + 11 - \dfrac{33}{x+3}$

$$\therefore \int \frac{x^3+2x}{x+3}\,dx = \int x^2 - 3x + 11 - \frac{33}{x+3}\,dx$$

$$= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 11x - 33\ln|x+3| + C$$

$$\int \frac{x^2-2x-4}{x^3-2x^2-3x}\,dx$$

$$x^3-2x^2-3x = x(x^2-2x-3)$$

$$= x(x-3)(x+1)$$

Let $\dfrac{x^2-2x-4}{x^3-2x^2-3x} = \dfrac{A}{x} + \dfrac{B}{x-3} + \dfrac{C}{x+1}$

$$= \frac{A(x-3)(x+1)}{x(x-3)(x+1)}$$

$$+ \frac{B(x)(x+1)}{x(x-3)(x+1)}$$

$$+ \frac{C(x)(x-3)}{x(x-3)(x+1)}$$

$\therefore\ x^2-2x-4 = A(x-3)(x+1) + Bx(x+1) + Cx(x-3)$

if $x=3$,

$$3^2-2\times3-4 = A(3-3)(3+1) + B\times3\times(3+1) + C\times3(3-3)$$

$$-1 = B\times12$$

$$B = -\tfrac{1}{12}$$

If $x = -1$,

$$(-1)^2 - 2(-1) - 4 = A(-1-3)(-1+1) + B(-1)(-1+1) + C(-1)(-1-3)$$

$$-1 = C \times 4$$

$$C = -\frac{1}{4}$$

If $x = 0$,

$$0^2 - 2 \times 0 - 4 = A(0-3)(0+1) + B \times 0(0+1) + C \times 0(0-3)$$

$$-4 = A \times -3$$

$$A = \frac{4}{3}$$

So $$\frac{x^2 - 2x - 4}{x^3 - 2x^2 - 3x} = \frac{\left(\frac{4}{3}\right)}{x} + \frac{\left(-\frac{1}{12}\right)}{x-3} + \frac{\left(-\frac{1}{4}\right)}{x+1}$$

$$\therefore \int \frac{x^2 - 2x - 4}{x^3 - 2x^2 - 3x}\,dx = \int \frac{\left(\frac{4}{3}\right)}{x} + \frac{\left(-\frac{1}{12}\right)}{x-3} + \frac{\left(-\frac{1}{4}\right)}{x+1}\,dx$$

$$= \frac{4}{3}\ln|x| - \frac{1}{12}\ln|x-3| - \frac{1}{4}\ln|x+1| + C$$

$$\int \frac{8x^7 + 47x^6 + 98x^5 + 108x^4 + 106x^3 + 100x^2 + 104x + 104}{(x-1)(x+2)^3(x^2+2x+2)^2}\,dx$$

Let

$$\frac{8x^7 + 47x^6 + 98x^5 + 108x^4 + 106x^3 + 100x^2 + 104x + 104}{(x-1)(x+2)^3(x^2+2x+2)^2}$$

$$= \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

$$+ \frac{Ex+F}{x^2+2x+2} + \frac{Gx+H}{(x^2+2x+2)^2}$$

Substituting values $x = 2, 3, 4, 5, 6, 7, 8, 9$ into the above equation yields 8 equations in 8 unknowns A, B, C, D, E, F, G and H.

Using MATLAB to reduce the augmented matrix for this system to reduced row-echelon form yields this solution:

$A = 1$, $B = 0$, $C = 2$, $D = -2$, $E = 7$, $F = -14$, $G = 18$, $H = 18$.

Thus the original integral becomes:

$$\int \frac{1}{x-1} + \frac{2}{(x+2)^2} + \frac{-2}{(x+2)^3} + \frac{7x-14}{(x^2+2x+2)} + \frac{18x+18}{(x^2+2x+2)^2}\,dx$$

Now: $\int \frac{1}{x-1}\,dx = \ln|x-1| + C$

$$\int \frac{2}{(x+2)^2}\,dx = \int 2\,(x+2)^{-2}\,dx$$

$$= \frac{2}{(-1)}(x+2)^{-1} + C$$

$$= -\frac{2}{x+2} + C$$

$$\int \frac{-2}{(x+2)^3}\,dx = \int -2\,(x+2)^{-3}\,dx$$

$$= \frac{-2}{-2}(x+2)^{-2} + C$$

$$= \frac{1}{(x+2)^2} + C$$

For $\int \frac{7x-14}{x^2+2x+2}\,dx$:

$$x^2 + 2x + 2 = x^2 + 2x + 1 + 2 - 1$$

$$= (x+1)^2 + 1$$

Let $u = x+1$

$\frac{du}{dx} = 1$

$du = dx$

Also $u - 1 = x$

so $7x - 14 = 7(u-1) - 14$

$= 7u - 7 - 14$

$= 7u - 21$

Now $\int \frac{7x-14}{x^2+2x+2}\,dx = \int \frac{7u-21}{u^2+1}\,du$

$$= \int \frac{7u}{u^2+1} - \frac{21}{u^2+1}\,du$$

$$= \int \frac{7}{2}\cdot\frac{2u}{u^2+1} - \frac{21}{u^2+1}\,du$$

$$= \frac{7}{2}\ln|u^2+1| - 21\arctan u + C$$

$$= \frac{7}{2}\ln|x^2+2x+2| - 21\arctan(x+1) + C$$

For $\int \frac{18x+18}{(x^2+2x+2)^2}\,dx$,

let $u = x+1$ so that

$$\int \frac{18x+18}{(x^2+2x+2)^2}\,dx = \int \frac{18u}{(u^2+1)^2}\,du$$

$$= \int 18u\,(u^2+1)^{-2}\,du$$

$$= \int 9\times 2u\,(u^2+1)^{-2}\,du$$

$$= \frac{9}{(-1)}(u^2+1)^{-1} + C$$

$$= \frac{-9}{u^2+1} + C$$

$$= \frac{-9}{x^2+2x+2} + C$$

So the original integral is:

$$\ln|x-1| - \frac{2}{x+2} + \frac{1}{(x+2)^2}$$

$$+ \frac{7}{2} \ln|x^2+2x+2| - 21 \arctan(x+1) - \frac{9}{x^2+2x+2} + C$$

```
>> x=[2;3;4;5;6;7;8;9]

x =

     2
     3
     4
     5
     6
     7
     8
     9

>> f=(8*x.^7+47*x.^6 + 98*x.^5+108*x.^4+106*x.^3+100*x.^2+104*x+104)./((x-1).*(x+2).^3.↙
*(x.^2+2*x+2).^2)

f =

    1307/800
     305/249
     985/937
     394/423
     728/869
     972/1277
    1807/2592
    1240/1929

>> Ac=1./(x-1)

Ac =

       1
       1/2
       1/3
       1/4
       1/5
       1/6
       1/7
       1/8

>> Bc=1./(x+2)

Bc =

       1/4
       1/5
       1/6
       1/7
       1/8
       1/9
       1/10
       1/11

>> Cc=1./((x+2).^2)

Cc =
```

```
       1/16
       1/25
       1/36
       1/49
       1/64
       1/81
       1/100
       1/121

>> Dc=1./((x+2).^3)

Dc =

       1/64
       1/125
       1/216
       1/343
       1/512
       1/729
       1/1000
       1/1331

>> Ec=x./((x.^2+2*x+2))

Ec =

       1/5
       3/17
       2/13
       5/37
       3/25
       7/65
       4/41
       9/101

>> Fc=1./((x.^2+2*x+2))

Fc =

       1/10
       1/17
       1/26
       1/37
       1/50
       1/65
       1/82
       1/101

>> Gc=x./((x.^2+2*x+2).^2)

Gc =

       1/50
       3/289
       1/169
```

```
       5/1369
       3/1250
       7/4225
       2/1681
       9/10201

>> Hc=1./((x.^2+2*x+2).^2)

Hc =

       1/100
       1/289
       1/676
       1/1369
       1/2500
       1/4225
       1/6724
       1/10201

>> A=[Ac,Bc,Cc,Dc,Ec,Fc,Gc,Hc,f]

A =

  Columns 1 through 6

       1              1/4            1/16           1/64           1/5            1/10
       1/2            1/5            1/25           1/125          3/17           1/17
       1/3            1/6            1/36           1/216          2/13           1/26
       1/4            1/7            1/49           1/343          5/37           1/37
       1/5            1/8            1/64           1/512          3/25           1/50
       1/6            1/9            1/81           1/729          7/65           1/65
       1/7            1/10           1/100          1/1000         4/41           1/82
       1/8            1/11           1/121          1/1331         9/101          1/101

  Columns 7 through 9

       1/50           1/100       1307/800
       3/289          1/289        305/249
       1/169          1/676        985/937
       5/1369         1/1369       394/423
       3/1250         1/2500       728/869
       7/4225         1/4225       972/1277
       2/1681         1/6724      1807/2592
       9/10201        1/10201     1240/1929

>> rref(A)

ans =

  Columns 1 through 6

       1              0              0              0              0              0
       0              1              0              0              0              0
       0              0              1              0              0              0
       0              0              0              1              0              0
       0              0              0              0              1              0
```

```
     0     0     0     0     0     1
     0     0     0     0     0     0
     0     0     0     0     0     0

  Columns 7 through 9

     0     0     1
     0     0     *
     0     0     2
     0     0    -2
     0     0     7
     0     0   -14
     1     0    18
     0     1    18
```