Topic 5

Complex Numbers

\[ y = x^2 + 2x + 2 \]

\[ x^2 + 2x + 2 = 0 \implies x = -1 \pm \sqrt{-1} \]
This topic introduces the complex number system and methods for representing and working with complex numbers.

The main number systems used in mathematics are:

- natural numbers $\mathbb{N}$
- integers $\mathbb{Z}$
- rational numbers $\mathbb{Q}$
- real numbers $\mathbb{R}$
- complex numbers $\mathbb{C}$

Each number system was developed through the need to find solutions to new types of equations. The discovery of the complex number system $\mathbb{C}$ completed this evolutionary process as no further numbers are needed to solve any equation constructed out of elementary functions.

Complex numbers are found everywhere in mathematics and its applications to science. They are used in engineering, electronics, hydrodynamics and quantum physics.

The topic has 2 chapters:

**Chapter 1** begins by showing how the introduction of the new number $i = \sqrt{-1}$ enables all quadratic equations having real coefficients to be solved. The set of all complex numbers $\mathbb{C}$ is explored and the concept of a complex conjugate is introduced. The remainder of the chapter is explores the arithmetic of complex numbers.

**Chapter 2** looks at two ways of representing complex numbers: the Cartesian form and the polar form. Complex numbers that are expressed in Cartesian form are seen to be closely related to vectors. The argument and modulus of a complex number are introduced and are used to represent complex numbers in polar form. De Moivre’s theorem can be used to solve quadratic equations having complex coefficients.

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1. see Appendix A
Chapter 1

Complex Numbers

1.1 A new number

Some quadratic equations have no solutions.

Example

The equation $x^2 + 2x + 2 = 0$ has no solutions.

a. The quadratic formula\(^1\) gives

$$x = \frac{-1 \pm \sqrt{-4}}{2}$$

...but $-4$ does not have a square root.

b. The graph of $y = x^2 + 2x + 2$ does not meet the $x$-axis ($y = 0$).

\[\begin{array}{c}
\text{The parabola } y = x^2 + 2x + 2 \text{ does not meet the } x\text{-axis, so the equation } x^2 + 2x + 2 = 0 \\
\text{has no solution in real numbers. But can this equation have solutions in another number system?}
\end{array}\]

---

\(^1\)The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ whenever this formula makes sense.
CHAPTER 1. COMPLEX NUMBERS

If we invent a new number $i$ with the property that

$$i = \sqrt{-1} \quad \text{or} \quad i^2 = -1,$$

then we can express the square root of any negative number in terms of $i$.

**Example**

*a.* $x = \sqrt{-4} = \sqrt{4} \times \sqrt{-1} = 2i$

*b.* $x = \sqrt{-11} = \sqrt{11} \times \sqrt{-1} = \sqrt{11}i$

We can now solve quadratic equations that couldn’t be solved before.

**Example**

*a.* $x^2 + 4 = 0 \implies x^2 = -4 \implies x = \pm \sqrt{-4} = \pm 2i$.

*b.* $x^2 + 2x + 2 = 0 \implies x = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

*c.* $x^2 + x + 1 = 0 \implies x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

The square root of any negative number can be expressed as a multiple of $i$, so we can use the quadratic formula to solve all quadratic equations.

**Definition**

A complex number is a number that can be written in the form $a + bi$, where $a$ and $b$ are real numbers and $i = \sqrt{-1}$. The set of all complex numbers is represented by the symbol $\mathbb{C}$.

**Exercise 1.1**

1. Write the following in terms of $i$:

   a. $\sqrt{-9}$  
   b. $\sqrt{-16}$  
   c. $\sqrt{-\frac{1}{4}}$  
   d. $\sqrt{-11}$

2. Solve the equations below, writing your answers in the form $a + bi$:

   a. $x^2 + 25 = 0$  
   b. $x^2 + 2x + 3 = 0$  
   c. $x^2 - x + 2 = 0$

---

$^2$Physicists traditionally use $j$ instead of $i$ to represent $\sqrt{-1}$. 

1.2 The complex plane

The real numbers can be represented as points on a number line (the \textit{real line}). This enables us to visualise negative numbers, which are not visible in the physical world.\textsuperscript{3}

The Swiss mathematician Argand was the first to represent complex numbers as points on a number plane (the \textit{complex plane}).\textsuperscript{4} This representation allowed complex numbers to be visualised, even though there were strong doubts about the actual existence of these ‘imaginary’ numbers.\textsuperscript{5}

The diagram below shows the complex plane.

- the horizontal axis is called the \textit{real axis}
- the vertical axis is called the \textit{imaginary axis}
- the complex number \( z = a + bi \) is represented by the point \((a, b)\) on the plane.

Terminology . . .

- \( a \) \textit{is the real part of} \( z \) \textit{and is written as} \( a = \text{Re}(z) \)
- \( b \) \textit{is the imaginary part of} \( z \) \textit{and is written as} \( b = \text{Im}(z) \)
- \( z \) \textit{is real when} \( \text{Im}(z) = 0 \), \textit{and is purely imaginary when} \( \text{Re}(z) = 0 \)

Example

\textbf{a.} \( \text{Re}(5 + 2i) = 5 \) and \( \text{Im}(5 + 2i) = 2 \)

\textbf{b.} \( \text{Re}(1 + i) = 1 \) and \( \text{Im}(1 + i) = 1 \)

\textbf{c.} \( \text{Re}(-1 - i) = -1 \) and \( \text{Im}(-1 - i) = -1 \)

\textsuperscript{3}See appendix A.
\textsuperscript{4}The complex plane is commonly called the \textit{Argand diagram} in older texts.
\textsuperscript{5}Complex numbers were later found to be useful for solving problems in electronics.
d. \( \text{Im}(2) = 0 \) and \( \text{Re}(3i) = 0 \)

We need to specify when complex numbers are equal.

**Definition**

*Complex numbers are equal when they have the same real and imaginary parts.*

**Example**

- Using equality
  a. \( \frac{2}{2} + \sqrt{4}i = 1 + 2i \)
  b. If \( x \) and \( y \) are real numbers, solve \( x + (x + y)i = 2 + i \)
    
    \[
    \begin{cases} 
    x &= 2 \\
    x + y &= 1 \implies x = 2 \text{ and } y = -1 
    \end{cases}
    \]

*The set of complex numbers \( \mathbb{C} \) is a new number system that contains and extends the real number system (the real axis).*

**Exercise 1.2**

1. Represent the following complex numbers on a complex plane:
   a. 1
   b. \(-1\)
   c. \(i\)
   d. \(-i\)
   e. \(1 + i\)
   f. \(1 - i\)
   g. \(-1 + i\)
   h. \(-1 - i\)

2. Find:
   a. \(\text{Re}(2 + 3i)\)
   b. \(\text{Im}(2 + 3i)\)
   c. \(\text{Re}(-i)\)
   d. \(\text{Im}(-i)\)

3. Find real \(x\) and \(y\) for which \(x - y + 2(x + y - 1)i = 5 + 4i\).
1.3 Complex conjugates

The word ‘conjugate’ refers to being joined in a pair.

Definition
The complex conjugate of \( z = a + bi \) is \( \overline{z} = a - bi \).

Example
The quadratic equation \( x^2 + x + 1 = 0 \) has two solutions which are complex conjugates:

\[
\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad \frac{1}{2} - \frac{\sqrt{3}}{2}i
\]

In the general case . . .

Theorem
The quadratic equation \( ax^2 + bx + c = 0 \) has two solutions

\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}
\]

when \( a, b \) and \( c \) are real numbers. These are either

- real solutions, or
- complex conjugate solutions

Properties of complex conjugates

a. Complex conjugates are reflections of each other across the real axis in the complex plane, eg.

b. If \( z = a + bi \), then \( \overline{z} = a - bi \) and \( \overline{\overline{z}} = a - bi = a + bi \).
1. Represent the pairs of complex conjugates below on a complex plane:
   a. \(1 + 2i\) and \(1 - 2i\)  
   b. \(-2 + i\) and \(-2 - i\)

2. Check that the solutions of the following equations are complex conjugates.
   a. \(x^2 + 25 = 0\)  
   b. \(z^2 + 2z + 3 = 0\)

3. Suppose \(z = a + bi\) where \(a\) and \(b\) are real. Prove that \(\overline{z} = z\).
1.4 Arithmetic with complex numbers

1.4.1 Addition and subtraction

We add or subtract expressions like $2 + 3x$ and $5 - 4x$ by combining like terms.

$$(2 + 3x) + (5 - 4x) = 7 - x \quad \text{and} \quad (2 + 3x) - (5 - 4x) = -3 + 7x$$

We can add and subtract complex numbers in the same way, by combining their real and imaginary parts separately.

$$(2 + 3i) + (5 - 4i) = 7 - i \quad \text{and} \quad (2 + 3i) - (5 - 4i) = -3 + 7i$$

**Definition**

To add two complex numbers, add their real parts and their imaginary parts separately:

$$(a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 + b_2)i$$

**Definition**

To subtract two complex numbers, subtract their real parts and their imaginary parts separately:

$$(a_1 + b_1i) - (a_2 + b_2i) = (a_1 - a_2) + (b_1 - b_2)i$$

We can add and subtract any number of complex numbers by combining their real and imaginary parts separately. We use the ordinary rules of arithmetic on the separate parts.

**Example**

$$(1 + 2i) + (3 + 4i) - (5 + 6i) = (1 + 3 - 5) + (2 + 4 - 6)i = -1 + 0i$$

**Exercise 1.4**

1. If $z_1 = 2 + 3i$, $z_2 = 1 + 2i$ and $z_3 = 2 - 2i$, find the following in the form $a + bi$ where $a$ and $b$ are real:

   - a. $z_1 + z_2$
   - b. $z_1 + z_3$
   - c. $z_2 + z_3$
   - d. $2z_1$
   - e. $z_1 - z_2$
   - f. $z_1 - z_3$
   - g. $z_1 - z_2 - z_3$
   - h. $-z_2 + z_3$

2. Suppose $z_1 = a + bi$ and $z_3 = c + di$ where $a$ and $b$ are real. Prove that:

   - a. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
   - b. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

3. Prove that $z + \overline{z}$ is real for every complex number $z$. 


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1.4.2 Multiplication

To multiply \((2 + 3x)(5 - 4x)\), we multiply out the brackets then simplify by collecting like terms:

\[
(2 + 3x)(5 - 4x) = 2(5 - 4x) + 3x(5 - 4x) \\
= 10 - 8x + 15x - 12x^2 \\
= 10 + 7x - 12x^2
\]

We can multiply complex numbers in the same way, by multiplying out the brackets and collecting like terms. But we can simplify the answer further because \(i^2 = -1\).

\[
(2 + 3i)(5 - 4i) = 2(5 - 4i) + 3i(5 - 4i) \\
= 10 - 8i + 15i - 12i^2 \\
= 10 + 7i - 12(-1) \\
= 22 - 7i
\]

Definition

To multiply two complex numbers, multiply out the brackets in the usual way and replace \(i^2\) by \(-1\) where possible.

Exercise 1.4

4. If \(z = 1 + i\) and \(w = 2 - 3i\), find the following in the form \(a + bi\) where \(a\) and \(b\) are real.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>a.</td>
<td>(1.7z)</td>
</tr>
<tr>
<td>b.</td>
<td>(iw)</td>
</tr>
<tr>
<td>c.</td>
<td>(zw)</td>
</tr>
<tr>
<td>d.</td>
<td>(2wz)</td>
</tr>
<tr>
<td>e.</td>
<td>(z^2)</td>
</tr>
<tr>
<td>f.</td>
<td>(iw^2)</td>
</tr>
<tr>
<td>g.</td>
<td>(5z - z\bar{w})</td>
</tr>
<tr>
<td>h.</td>
<td>((\bar{z} + 1)(1 - w))</td>
</tr>
</tbody>
</table>

5. Find real numbers \(x\) and \(y\) for which:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>((2 - 3i)(x + yi) = 5 - i)</td>
</tr>
<tr>
<td>b.</td>
<td>((x + yi)^2 = -3 + 4i)</td>
</tr>
</tbody>
</table>

6. Suppose \(z = a + bi\) and \(w = c + di\) where \(a\) and \(b\) are real. Prove that \(z\bar{w} = \bar{z}w\).

7. Prove that \(z\bar{z}\) is real for every complex number \(z\).
1.4.3 Simplifying fractions

We can use conjugates to simplify the denominators of complex fractions. This makes use of the fact that if \( z = a + bi \), then

\[
z\bar{z} = (a + bi)(a - bi) = a^2 + b^2
\]

is a real number.

**Example**

Write the complex fraction \( \frac{1 + 2i}{2 - 3i} \) in the form \( a + bi \).

**Answer**

To change the denominator to a real number, multiply both the numerator and the denominator by \( 2 + 3i \):

\[
\frac{1 + 2i}{2 - 3i} = \frac{1 + 2i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i}
\]

\[
= \frac{(1 + 2i)(2 + 3i)}{(2 - 3i)(2 + 3i)}
\]

\[
= \frac{(1 + 2i)(2 + 3i)}{4 + 9}
\]

\[
= \frac{-4 + 7i}{13}
\]

\[
= \frac{-4}{13} + \frac{7}{13}i
\]

**Exercise 1.4**

8. If \( z = 1 - 2i \) and \( w = 3 + 4i \), represent the following in the form \( a + bi \) where \( a \) and \( b \) are real.

   a. \( \frac{z}{w} \)  
   b. \( \frac{w}{z} \)  
   c. \( \frac{z}{w} \)  
   d. \( w^{-1} \)

9. Find real numbers \( x \) and \( y \) for which

\[
\frac{1}{x + yi} = 1 + i.
\]

10. *Prove* that \( \left( \frac{\bar{z}}{w} \right) = \frac{\bar{z}}{w} \) for any complex numbers \( z \) and \( w \neq 0 \).
1.5 Properties of complex conjugates

If $z = a + bi$ is any complex number:

- $\overline{(z)} = z$
- $z + \overline{z}$ is real ($= 2a$)
- $z - \overline{z}$ is purely imaginary ($= 2b$)
- $z\overline{z} = a^2 + b^2$

If $z$ and $w$ are any complex numbers, then:

- $\overline{z \pm w} = \overline{z} \pm \overline{w}$.
- $\overline{zw} = \overline{z} \overline{w}$.
- $\overline{\left( \frac{z}{w} \right)} = \frac{\overline{z}}{\overline{w}}$ when $w \neq 0$. 
Chapter 2

Geometry of Complex Numbers

2.1 Complex numbers and vectors

There is a strong connection between complex numbers, co-ordinate geometry and vectors in a number plane.

The diagram below shows that the complex number \( a + bi \) corresponds to the point with co-ordinates \((a, b)\), and to the vector with components \([a, b]\). Adding complex numbers is similar to adding vectors.

A complex number is said to be in *Cartesian form* when it is has the form \( a + bi \). This is because \((a, b)\) are its Cartesian co-ordinates in the complex plane.

**Example**

The fraction \( \frac{1 + i}{2 + 3i} \) is represented in Cartesian form as \( \frac{5}{13} - \frac{1}{13}i \).

The next sections investigate the modulus and argument of a complex number. These are counterparts to the length and direction of the corresponding vector in the complex plane.
2.1.1 Modulus

The absolute value of a real number is its distance from 0. The modulus of a complex number extends this concept.

**Definition 2.1.1**
The modulus (pl. moduli) of a complex number \( z = a + bi \) is equal to its distance from 0. It is written as

\[
|z| = |a + bi| = \sqrt{a^2 + b^2},
\]

and is referred to simply as ‘mod \( z \)’.

Sometimes the modulus of \( z \) is referred to as the length of \( z \), since it represents the length of \( z \) when drawn as an arrow on the complex plane. As can be seen in the diagram below, this can be calculated using Pythagoras’ Theorem.

![Diagram showing the modulus of a complex number as its distance from 0](image)

**Example**

1. \( |1| = |-1| = |i| = |-i| = 1 \)
2. \( |5 - 12i| = \sqrt{5^2 + (-12)^2} = 13 \)
3. \( |z| = |\overline{z}| \) for all complex numbers, by symmetry.

The modulus has the useful property that the modulus of a product is the product of the moduli, i.e.

**Theorem**

If \( z_1, z_2, \ldots \) are complex numbers, then

\[
|z_1 z_2 \ldots| = |z_1| \times |z_2| \times \ldots ,
\]

**Example**

If \( z = 3 + 4i \), then

\[
|z^{10}| = |z|^{10} = |3 + 4i|^{10} = (\sqrt{3^2 + 4^2})^{10} = 5^{10}
\]
Exercise 2.1

1. Plot the following numbers on the complex plane and find their moduli:
   
   a. 1   b. −1   c. i   d. −i
   e. 1 + i   f. 1 − i   g. −1 + i   h. −1 − i

2. Is $|z| = |\bar{z}|$ true for all complex numbers $z$?

3. Check that $|(1 + i)(1 + 2i)| = |1 + i| \times |1 + 2i|$.

4. Evaluate $|(1 + i)^8|$.

5. Find $|z|$ if $z$ is a complex number with $z^2 = i$.

6. Draw the following sets of complex numbers on different diagrams.
   
   a. $\{z : |z| = 1\}$   b. $\{z : |z| \leq 1\}$   c. $\{z : 1 \leq |z| \leq 2\}$
CHAPTER 2. GEOMETRY OF COMPLEX NUMBERS

2.1.2 Argument

The sign of a real number indicates its direction from O along the real line. The argument of a complex number extends this concept to the whole complex plane.

Definition 2.1.2

The argument of a complex number \( z = a + bi \) is equal to the angle turned from the positive real axis to give \( z \). It is written as \( \arg z \) and is referred to simply as ‘\( \arg z \)’.

An anticlockwise turn gives a positive angle. A clockwise turn gives a negative angle.

Example

If \( z = 3 + 4i \), then it can be seen below that \( z \) is at an angle of \( \theta = 53.13^\circ \) (0.93 radians) to the positive real axis.\(^1\) This indicates its direction from O and we write \( \arg z = 53.13^\circ = 0.93 \).

\[
\begin{align*}
\tan \theta &= 4/3 \\
\theta &= \arctan(4/3) \\
&= 53.13^\circ \\
&= 0.93
\end{align*}
\]

We could also have described the direction of \( z \) from O using the angle \( \phi = -306.87^\circ \) (−5.36 radians), measured clockwise from the positive real axis.

\(^1\)When the \( ^\circ \) symbol is used, this indicates that the angle is measured in degrees. If no symbol is used, the angle is assumed to be measured in radians.
The direction of \( z \) from \( O \) can be indicated using any of the angles

\[
53.13 + 360k^\circ (0.93 + 2\pi k),
\]

where \( k \) is any positive or negative integer. To avoid these infinitely many possibilities, when we refer to the argument of \( z \), we mean the angle \( \theta \) with \(-180^\circ < \theta \leq 180^\circ \) \((-\pi < \theta \leq \pi)\).

Example

\[
\begin{array}{c}
\text{Im} \\
3 \\
\hline
\text{Re} \\
\theta \\
\hline
3 + 4i \\
4 \\
\hline
3 - 4i \\
\end{array}
\]

\[
\text{arg } 3 + 4i = 53.13^\circ = 0.93 \\
\text{arg } 3 - 4i = -53.13^\circ = -0.93
\]

Exercise 2.1

7. Plot the following numbers on the complex plane and find their arguments:
   a. \( 1 \)  
   b. \( -1 \)  
   c. \( i \)  
   d. \( -i \)  
   e. \( 1 + i \)  
   f. \( 1 - i \)  
   g. \( -1 + i \)  
   h. \( -1 - i \)

8. What is the argument of:
   a. a positive number?  
   b. a negative number?  
   c. 0?

9. Is \( \text{arg } z = -\text{arg } \overline{z} \) true for all complex numbers \( z \)?

10. Draw the following sets of complex numbers on different diagrams:
    a. \( \{ z : \text{arg } z \geq 0 \} \)  
    b. \( \{ z : \text{arg } z \leq 0 \} \)  
    c. \( \{ z : |\text{arg } z| \leq \pi/2 \} \)
2.1.3 Polar Form

The position of a complex number \( z = x + yi \) in the complex plane can be described by using either cartesian co-ordinates \((x, y)\) or polar co-ordinates \((r, \theta)\), where \( r = |z| \) and \( \theta = \arg z \).

\[
\begin{align*}
& z = x + yi \\
& r = |z| = \sqrt{x^2 + y^2} \\
& y = r \sin \theta \\
& \tan \theta = \frac{y}{x}
\end{align*}
\]

You can see from this diagram that the Cartesian and polar co-ordinates of \( z \) satisfy:

\[
x = r \cos \theta \\
y = r \sin \theta
\]

The complex number \( z \) can be represented solely in terms of \( r \) and \( \theta \) as:

\[
z = x + yi = r(\cos \theta + i \sin \theta).
\]

This is called the polar form of \( z \). It is abbreviated as \( r \cis \theta \) and is referred to simply as \( r \cis \theta \).2

**Example**

- If \( z = 3 + 4i \), then \( |z| = 5 \) and \( \arg z = 53.13^\circ \), so \( z = 5 \cis 53.13^\circ \)

**Example**

- If \( z_1 = 5 \cis 45^\circ \), then \( |z_1| = 5 \) and \( \arg z_1 = 45^\circ \).
- If \( z_2 = \cis \frac{\pi}{6} \), then \( |z_2| = 1 \) and \( \arg z_2 = \frac{\pi}{6} \).
- If \( z_3 = 10 \cis 260^\circ \), then \( |z_3| = 10 \) and \( \arg z_3 = -100^\circ \).

---

2Note that “cis” is usually pronounced “sis”. Another abbreviation used by engineers is \( r \angle \theta \)
11. Convert the following to polar form:

   a. 1  
   b. $-1$  
   c. $i$  
   d. $-i$  
   e. $1 + i$  
   f. $1 - i$  
   g. $-1 + i$  
   h. $-1 - i$  
   i. $\sqrt{3} + i$  
   j. $1 - \sqrt{3}i$  
   k. $-5 \text{cis} 45^\circ$  
   l. $\text{cis} 45^\circ + \text{cis} (-45^\circ)$

12. Convert the following to Cartesian form:

   a. $\text{cis} \frac{\pi}{2}$  
   b. $\text{cis} (-90^\circ)$  
   c. $3 \text{cis} 0$  
   d. $2 \text{cis} \pi$  
   e. $8 \text{cis} \frac{\pi}{4}$  
   f. $\sqrt{2} \text{cis} 45^\circ$  
   g. $2 \text{cis} \frac{2\pi}{3}$  
   h. $5 \text{cis} (-30^\circ)$
CHAPTER 2. GEOMETRY OF COMPLEX NUMBERS

2.2 Calculations using polar form

The polar form has three properties that are useful when multiplying and dividing complex numbers:

Theorem

1. \(\text{cis} \theta \times \text{cis} \phi = \text{cis} (\theta + \phi)\)

2. \(\frac{\text{cis} \theta}{\text{cis} \phi} = \text{cis} (\theta - \phi)\)

3. \(\text{cis} (\theta + 2k\pi) = \text{cis} \theta\) for any integer \(k\).

The first two properties are similar to the rules for simplifying powers:

\[ A^\theta \times A^\phi = A^{(\theta+\phi)} \quad \text{and} \quad \frac{A^\theta}{A^\phi} = A^{(\theta-\phi)} \]

The trigonometric identities\(^3\) can be used to prove properties 1 - 3. Properties 1 and 2 are also good ways of remembering the trig identities!

Proof of properties

- \(\text{cis} \theta \times \text{cis} \phi = (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi)\)
  \[= \left[\cos \theta \cos \phi - \sin \theta \sin \phi\right] + i\left[\sin \theta \cos \phi + \cos \theta \sin \phi\right]\]
  \[= \cos(\theta + \phi) + i \sin(\theta + \phi)\]
  \[= \text{cis} (\theta + \phi)\]

- \(\frac{\text{cis} \theta}{\text{cis} \phi} = \frac{\text{cis} \theta}{\text{cis} \phi} \times \frac{\text{cis} (-\phi)}{\text{cis} (-\phi)}\)
  \[= \frac{\text{cis} \theta}{\text{cis} \phi} \times \frac{\text{cis} (-\phi)}{\text{cis} (\theta - \phi)}\]
  \[= \frac{\text{cis} \theta}{\text{cis} (-\phi)} \quad \text{... by property 1}\]
  \[= \text{cis} (\theta - \phi) \quad \text{... as \text{cis} 0 = 1}\]

Example

If \(z_1 = 2 \text{cis} 120^\circ\) and \(z_2 = 3 \text{cis} 80^\circ\), then

\[z_1z_2 = 2 \text{cis} 120^\circ \times 3 \text{cis} 80^\circ = 6 \text{cis} 200^\circ,\]

so \(|z_1z_2| = 6\) and \(\arg z = -160^\circ\).

---

\(^3\)The trig identities used are (i) \(
\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi\) and (ii) \(
\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi\).
Example
If \( z_1 = 2 \text{cis} 120^\circ \) and \( z_3 = \text{cis} 80^\circ \), then

\[
\frac{z_1}{z_2} = \frac{2 \text{cis} 120^\circ}{3 \text{cis} 80^\circ} = \frac{2}{3} \text{cis} 40^\circ,
\]

so \( \left| \frac{z_1}{z_2} \right| = \frac{2}{3} \) and \( \arg \left( \frac{z_1}{z_2} \right) = 40^\circ \).

Exercise 2.2

1. Use properties 1 and 2 to simplify:
   a. \( \text{cis} 2\theta \text{cis} 3\theta \)
   b. \( \frac{\text{cis} 3\theta}{\text{cis} 2\theta} \)
   c. \( [\text{cis} 2\theta]^3 \)

2. Find the argument of the following complex numbers:
   a. \( \text{cis} 500^\circ \)
   b. \( \text{cis} (-480^\circ) \)
   c. \( 3 \text{cis} 5\pi \)
   d. \( \left[ 2 \text{cis} \frac{4\pi}{5} \right]^3 \)

3. Write the following as a single complex number in polar form:
   a. \( 2 \text{cis} \frac{\pi}{2} \times 3 \text{cis} \frac{\pi}{3} \times 4 \text{cis} \frac{\pi}{4} \)
   b. \( \frac{2 \text{cis} 20^\circ \times 3 \text{cis} 40^\circ}{4 \text{cis} 60^\circ \times 5 \text{cis} 80^\circ} \)
2.3 De Moivre’s theorem

2.3.1 Powers of complex numbers

Powers of complex numbers in polar form can be calculated easily using the properties in section 2.2.

Example
If \( z = 2 \text{ cis } 30^\circ \), then

- \( z^2 = 2^2 \text{ cis } (2 \times 30^\circ) = 4 \text{ cis } 60^\circ \)
- \( z^3 = 2^3 \text{ cis } (3 \times 30^\circ) = 8 \text{ cis } 90^\circ \)
- \( z^4 = \ldots \text{ etc} \)

This pattern is true for all integer powers:

De Moivre’s Theorem If \( z = r \text{ cis } \theta \) and \( n \) is an integer, then \( z^n = r^n \text{ cis } n\theta \).

Example evaluating a power
Use De Moivre’s theorem to find the value of \( (1 + i)^7 \).

Answer
As \( |1 + i| = \sqrt{2} \) and \( \arg(1 + i) = \frac{\pi}{4} \),

\[
1 + i = \sqrt{2} \text{ cis } \frac{\pi}{4}
\]

and

\[
(1 + i)^7 = \left( \sqrt{2} \text{ cis } \frac{\pi}{4} \right)^7 = 2^{7/2} \text{ cis } \frac{7\pi}{4}.
\]

Example simplifying a negative power
Simplify \( \left[ 2 \text{ cis } \frac{2\pi}{3} \right]^{-2} \).

Answer
\[
\left[ 2 \text{ cis } \frac{2\pi}{3} \right]^{-2} = 2^{-2} \text{ cis } (-2 \times \frac{2\pi}{3}) = \frac{1}{4} \text{ cis } (-\frac{4\pi}{3}) = \frac{1}{4} \text{ cis } \frac{2\pi}{3}.
\]

Exercise 2.3

1. Use DeMoivre’s theorem to find the value of
   a. \( (1 + i)^9 \)   b. \( (1 - \sqrt{3}i)^4 \)   c. \( (1 - i)^{-4} \)   d. \( (\sqrt{3} + i)^{-2} \)
2.3. DE MOIVRE’S THEOREM

2.3.2 Quadratic equations with complex coefficients

The new number \( i = \sqrt{-1} \) was introduced to enable equations like \( x^2 + 4 = 0 \) to be solved. Are there any other missing new numbers which can be used to solve equations that we can not currently solve?

It can be shown that all equations made up from the elementary functions have complex number solutions, so this is the end of our journey which began with the counting numbers.

While it is extremely hard to show that all such equations have solutions in complex numbers, we can show that any quadratic equation with complex coefficients can be solved in \( \mathbb{C} \). This is best done by example. If we can find the square root of any complex number, then the quadratic formula can be used to solve any quadratic equation with coefficients in \( \mathbb{C} \).

Example

Solve the equation \( z^2 = 1 + i \).

Answer

Step 1: Write \( z \) and \( 1 + i \) in polar form

\[ z = r \cis \theta \quad \text{and} \quad 1 + i = \sqrt{2} \cis \frac{\pi}{4} \]

Step 2: Use De Moivre’s theorem to obtain \( z^2 \)

\[ z^2 = r^2 \cis 2\theta \]

Step 3: As \( z^2 \) and \( 1 + i \) are equal . . .

\[ r^2 = \sqrt{2} \quad \text{and} \quad \cis 2\theta = \cis \frac{\pi}{4} \]

These equations have solutions

\[ r = 2^{1/4} \quad \text{and} \quad 2\theta = \frac{\pi}{4} + 2\pi n \] for any integer \( n \),

that is,

\[ r = 2^{1/4} \quad \text{and} \quad \theta = \frac{\pi}{8} + \pi n \] for any integer \( n \).

Step 4: Substitute back into \( z = r \cis \theta \) to find the values of \( z \) . . .

\[ z = r \cis \theta = 2^{1/4} \cis \left( \frac{\pi}{8} + \pi n \right) \] for any integer \( n \).

and . . .

as there are only two distinct values for \( z \), corresponding to even and odd values of \( n \), the solutions of the equation \( z^2 = 1 + i \) are

\[ z_1 = 2^{1/4} \cis \left( \frac{\pi}{8} \right) \quad \text{and} \quad z_2 = 2^{1/4} \cis \left( \frac{9\pi}{8} \right) \]

Note. As expected \( z_2 = -z_1 \).
Exercise 2.3

2. Solve \( z^2 = i \).

3. Solve \( z^2 + 2z + i = 0 \)
Appendix A

Number Systems

A number system is a collection of ‘numbers’ for which

- there is a rule for what it means to be a number in the system
- numbers can be combined by adding and multiplying
- the system is closed under addition and multiplication
- addition and multiplication obey the commutative, associative and distributive rules of arithmetic.

A.1 Natural Numbers ($\mathbb{N}$)

The ‘counting’ numbers 1, 2, 3, ... were discovered independently in many countries. They are answers to questions concerned with ‘How many?’ They are called the natural numbers.

The set of all natural numbers is traditionally represented by the letter $\mathbb{N}$, so we can write $\mathbb{N} = \{1, 2, 3, \ldots \}$. It is closed under addition and multiplication.

However, $\mathbb{N}$ is not closed under subtraction, and equations like $x + 5 = 4$ have no solutions in this number system.

A.2 Integers ($\mathbb{Z}$)

Negative numbers gained acceptance in the 16th century. They are possible answers to questions asking ‘How many more does A have than B have?’

There was originally some doubt as to whether negative numbers really existed. What does ‘-3 apples’ mean? This doubt was resolved by interpreting numbers as positions on a number line. Numbers to the right of the origin were taken as positive and numbers to the left as negative.

---

1. This means that the sum and product of two numbers are numbers in the system.
2. These rules are described in Matrices, Module 3, Appendix A
3. Curly brackets {} are used to indicate a set or collection of numbers.
Definition
An integer is a natural number, the negative of a natural number, or zero.

The set of all integers is traditionally represented as \( \mathbb{Z} = \{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \} \). It contains the set of natural numbers \( \mathbb{N} \), and is closed under addition, subtraction and multiplication.

However \( \mathbb{Z} \) is not closed under division and equations like \( 2x + 5 = 4 \) have no solutions in this number system.

A.3 Rational Numbers (\( \mathbb{Q} \))
Simple fractions like \( \frac{1}{2}, \frac{3}{4}, \ldots \) were discovered independently in many countries and are used to describe ratios, eg. ‘Please give me three quarters.’ These are examples of rational numbers.

Definition
A rational number is a number that can be written in the form \( \frac{m}{n} \) where \( m \) and \( n \) are integers with \( n \neq 0 \).

Example
a. \( \frac{3}{4} \) is a rational number as 3 and 4 are integers.

b. \( -\frac{3}{4} \) is a rational number as it can be written as \( -\frac{3}{4} \).

c. 1 is rational as it can be written as \( \frac{1}{1} \).

d. 1.234 is a rational number as it can be written as \( \frac{1234}{1000} \).

The set of all rational numbers is represented by the letter \( \mathbb{Q} \). It contains the integers \( \mathbb{Z} \), and is closed under \( + \), \( - \), \( \times \) and \( \div \).

The ancient Greeks believed that every number was a rational number and that this demonstrated the logic and perfection of the universe. These ideals collapsed when a member of Pythagoras’s secret society of mathematicians and mystics showed that the equation \( x^2 = 2 \) had no solutions in the rational number system.

Numbers which are non-rational are called irrational numbers. Other examples of irrational numbers are \( \sqrt{3}, \sqrt{5} \) and \( \pi \).
A.3 Real Numbers ($\mathbb{R}$)

The real numbers are the measuring numbers. They are all the numbers that can be found on a number line (commonly called a real line) and are used to describe measurements of continuous quantities such as length, weight, quantity of fluid, etc.

The set of all real numbers is represented by the letter $\mathbb{R}$. It contains the set of integers $\mathbb{Z}$ and the set of rational numbers $\mathbb{Q}$, and is closed under $+$, $-$, $\times$ and $\div$.

The square of any real number is positive, so the equation $x^2 = -1$ does not have a solution in the real number system.
Appendix B

Answers

Exercise 1.1
1(a) 3i  1(b) 4i  1(c) $\frac{1}{2}i$  1(d) $\sqrt{11}i$

2(a) $x = \pm 5i$  2(b) $x = -1 \pm \sqrt{2}i$  2(c) $x = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

Exercise 1.2
1.

2(a) 2  2(b) 3  2(c) 0  2(d) $-1$

3. $x = 4$ and $y = -1$
Exercise 1.3

1. 

\[
\begin{align*}
1 + 2i & \\
-2 - i & \\
-2 - i & \\
-1 & \\
i & \\
O & \\
1 & \\
\end{align*}
\]

2. Self-checking
3. Check with lecturer.

Exercise 1.4

1(a) 3 + 5i  
1(b) 4 + i  
1(c) 3  
1(d) 4 + 6i  
1(e) 1 + i  
1(f) 5i  
1(g) -1 + 3i  
1(h) 1 - 4i

2. Check with lecturer.
3. Check with lecturer.

4(a) 1.7 + 1.7i  
4(b) 3 + 2i  
4(c) 5 - i  
4(d) 10 - 2i  
4(e) 2i  
4(f) 12 - 5i  
4(g) 6  
4(h) 1 + 7i

5(a) x = y = 1  
1(b) x = 1, y = 2 and x = -1, y = -2

6. Check with lecturer.
7. Check with lecturer.

8(a) -1/5 - 2/5i  
8(b) -1 + 2i  
8(c) 11/25 - 2/25i  
8(d) 3/25 - 4/25i

9. x = 1/2 and y = -1/2
10. Check with lecturer.

Exercise 2.1

1(a) |1| = 1  
1(b) |-1| = 1  
1(c) |i| = 1  
1(d) |-i| = 1  
1(e) |1 + i| = \sqrt{2}  
1(f) |1 - i| = \sqrt{2}  
1(g) |-1 + i| = \sqrt{2}  
1(h) |-1 - i| = \sqrt{2}

2. Both are equal to \sqrt{a^2 + b^2}.
3. Both are equal to \sqrt{10}.

4. |(1 + i)^8| = |(1 + i)|^8 = (\sqrt{2})^8 = 16.
5. $|z| = 1$

6(a) Boundary of circle with centre $(0, 0)$ and radius 1.
6(b) All points inside the circle with centre $(0, 0)$ and radius 1.
6(c) All points between the two circles with centres $(0, 0)$ and radii 1 and 2.

7(a) $\arg 1 = 0$  
7(b) $\arg(-1) = \pi$  
7(c) $\arg i = \pi/2$  
7(d) $\arg(-i) = -\pi/2$  
7(e) $\arg(1 + i) = \pi/4$  
7(f) $\arg(1 - i) = -\pi/4$  
7(g) $\arg(-1 + i) = 3\pi/4$  
7(h) $\arg(-1 - i) = -3\pi/4$

8(a) 0  
8(b) $\pi$  
8(c) not defined

9. It is true for all complex numbers except for $-1$.

10(a) All points on and above the real axis, excluding 0.
10(b) All points below the real axis, but not including the negative real numbers and 0.
10(c) All points on and to the right of the imaginary axis, but not including 0.

11(a) $1 \cis 0$  
11(b) $1 \cis \pi$  
11(c) $1 \cis \pi/2$  
11(d) $1 \cis(-\pi/2)$  
11(e) $\sqrt{2} \cis(\pi/4)$  
11(f) $\sqrt{2} \cis(-\pi/4)$  
11(g) $\sqrt{2} \cis(3\pi/4)$  
11(h) $\sqrt{2} \cis(-3\pi/4)$  
11(i) $2 \cis(\pi/6)$  
11(j) $2 \cis(-\pi/3)$  
11(k) $5 \cis(-135^\circ) = 5 \cis(-3\pi/4)$  
11(l) $\sqrt{2} \cis 0$

12(a) $i$  
12(b) $-i$  
12(c) 3  
12(d) $-2$  
12(e) $4\sqrt{2}(1 + i)$  
12(f) $1 + i$

12(g) $-1 + \sqrt{3}i$  
12(h) $5(\sqrt{3} - i)$

**Exercise 2.2**

1(a) $\cis 5\theta$  
1(b) $\cis \theta$  
1(c) $\cis 6\theta$

2(a) $140^\circ = 7\pi/9$  
2(b) $-120^\circ = -2\pi/3$  
2(c) $\pi = 180^\circ$  
2(d) $2\pi/5 = 72^\circ$

3(a) $24 \cis(-11\pi/12)$  
3(b) $\frac{3}{10} \cis(-80^\circ)$

**Exercise 2.3**

1(a) $16\sqrt{2} \cis(\pi/4) = 16 + 16i$  
1(b) $16 \cis(2\pi/3)$ or $16 \cis 120^\circ$

1(c) $-\frac{1}{4}$  
1(d) $\frac{1}{2} \cis(-\pi/3)$ or $\frac{1}{2} \cis(-60^\circ)$

2. $z = \pm\frac{1}{2}(1 + i)$

3. $z = -1 \pm 2^{1/4} \cis(-\pi/8)$