Lecture notes on symmetries, topological phases and K-theory

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These are notes for a series of lectures given by the author on the mathematics of topological phases, at the Leiden Summer School on "KK-theory, Gauge Theory and Topological Phases" in February–March 2017. They are meant to introduce some recent developments in the physics of topological phases to the mathematician, and to introduce carefully K-theory and related topics for the physicist. Updated versions of these notes will be available at

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Overview

The precise meaning of *phase* depends on the class of physical systems being studied, e.g. thermodynamic, classical/quantum mechanical, symmetry-constrained,. . . , but there are some general features. There are typically some parameters specifying a physical *state*, e.g. temperature, pressure, . . . , and the parameter space (*phase/moduli space*) divides up into various connected pieces (phases) separated by *phase transitions*.

In many instances, the relevant phase space is a rich topological space, and the *topological phases* are labelled by *topological invariants* familiar from mathematics — winding numbers, homotopy/homology groups, characteristic classes, topological/analytic indices, K-theory. Physicists have been quite creative both in producing models for actual phenomena which realise such invariants, and in identifying general physical principles which allow even mathematically exotic invariants to be relevant. At least three Nobel Prizes have been awarded in direct relation to the idea of topological phases: 2016 (Thouless–Kosterlitz–Haldane), 1985 (von Klitzing), 1998 (Laughlin–Störmer–Tsui).

I will focus on the general principle of quantum mechanical symmetries, and how it leads to complex/real/twisted K-theory invariants labelling the topological phases[1].

1 Quantum mechanical symmetries

Pure states in quantum mechanics (QM) are elements of the projective Hilbert space $\mathbb{P}\mathcal{H}$, usually represented by a normalised vector $|\psi\rangle$. QM symmetries are a bit unusual in that they

[1]In many-body physics, one studies *strongly interacting* particle systems whose description requires techniques from quantum field theory, and which exhibit what is sometimes called *topological order*. These systems are less well-understood, and we will not say anything further about them.

only need to preserve transition probabilities between any pair of states, i.e. the symmetric function

$$p : ([\psi_1], [\psi_2]) \mapsto |\langle\psi_1|\psi_2\rangle|^2 \in [0, 1].$$

I have used Dirac's bra-ket notation for the inner product $\langle\cdot|\cdot\rangle$. A classical theorem of Wigner (see [13] for a modern geometric proof) says that any automorphism of $(\mathbb{P}\mathcal{H}, p)$ is implemented by a unitary or antiunitary[2] operator on $\mathcal{H}$. Modifying the (anti)unitary implementing operator U by an overall phase does not change the automorphism of $(\mathbb{P}\mathcal{H}, p)$, so the target group of "QM automorphisms" is the *projective* unitary-antiunitary (PUA) group of $\mathcal{H}$, and it is "PUA-representation theory" which is needed. This is in contrast to ordinary unitary representation theory (e.g. of locally compact second countable topological groups), for which the target group is the group of unitary operators $\mathcal{U}(\mathcal{H})$.

Example: Complex conjugation κ is the simplest example of an antiunitary operator. A more interesting example of an antiunitary operator is *fermionic* time-reversal T, which squares to -1 instead of $+1$, and is often represented as the operator $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ \kappa$ on the Hilbert space $\mathbb{C}^2$ of internal/spinor degrees of freedom. A more invariant description of T is as a *quaternionic structure* on $\mathbb{C}^2$ which identifies the latter as $\mathbb{H}$. Recall that $\mathrm{Sp}(1) \cong \mathrm{SU}(2) \cong \mathrm{Spin}(3)$ is a double cover of $\mathrm{SO}(3)$, which as groups/manifolds is

$$\{\pm 1\} \cong \mathbb{Z}_2 \hookrightarrow S^3 \cong \mathrm{SU}(2) \twoheadrightarrow \mathbb{RP}^3 \cong \mathrm{SO}(3).$$

The failure of $\mathrm{SO}(3)$ to lift into $\mathrm{SU}(2)$ means that the latter is a *projective representation*[3] of $\mathrm{SO}(3)$. In QM, rotational symmetries *are* allowed to have such projective *spinor* representations. As a general rule, *fermions* (e.g. electrons) behave as spinors, in that they acquire a -1 phase upon a 2π rotation in physical 3D space, since the latter is represented projectively on $\mathbb{C}^2$ by -1 rather than simply the identity operator. In *relativistic* QM, the irreducible PUA representations of the inhomogeneous indefinite group $\mathbb{R}^4 \rtimes \mathrm{SO}(1,3)$ (inhomogeneous *Lorentz* group, whose identity component is also called the *Poincaré* group) correspond to distinct particle species, and *spinor bundles* also come into play.

1.1 Time evolution and dynamical symmetries

Time evolution in QM is a strongly-continuous 1-parameter group of unitaries $\mathbb{R} \ni t \mapsto U_t = e^{-\mathrm{i}Ht}$ on a complex Hilbert space $\mathcal{H}$. The self-adjoint generator H, given by Stone's theorem, is called a *Hamiltonian*.

We are interested in *dynamical* symmetries (for U_t), which roughly means a representation of a group G as QM automorphisms compatible with U_t. More precisely, we need the elements $g \in G$ to be projectively represented by operators g on $\mathcal{H}$ which are unitary or antiunitary according to a continuous homomorphism $\phi : G \to \{\pm 1\}$. Furthermore, $\mathsf{g}U_t\mathsf{g}^{-1} = U_{\tau(g)}t$, where another continuous homomorphism $\tau : G \to \{\pm 1\}$ encodes whether g preserves or reverses the arrow of time. In terms of the Hamiltonian, this means that

$$\mathsf{g}(\mathrm{i}H)\mathsf{g}^{-1} = \tau(g)(\mathrm{i}H) \quad \Rightarrow \quad \mathsf{g}(H)\mathsf{g}^{-1} = \phi \cdot \tau(g)H =: c(g)H.$$

[2]Recall that an antiunitary operator on $\mathcal{H}$ is a complex-antilinear bijection U such that $\langle U\psi_1|U\psi_2\rangle = \langle\psi_2|\psi_1\rangle$ for all $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$.

[3]Projective representations are closely related to central extensions by $\mathrm{U}(1)$, and the latter is the point of view taken in [15].

Often, dynamical symmetries are *defined* to commute with H (so $c \equiv 1$, or equivalently, $\phi = \tau$), but this is not always the case, and we will not assume a priori that $\phi = \tau$. The letter c is meant to suggest "charge-conjugation", or "particle-hole" symmetry: notice that if $c(g) = -1$, then g reflects the spectrum of H about 0. By definition, $\phi \cdot c \cdot \tau \equiv 1$, and any two of these three homomorphisms are independent; we will choose to specify ϕ, c as part of the symmetry data, alongside G and σ.

For simplicity, let us take $c \equiv 1$ first. Given (G, ϕ), we define a PUA-rep on $\mathcal{H}$ to be a map[4] $\theta : g \mapsto \theta_g$ such that θ_g is unitary (resp. antiunitary) if $\phi(g) = +1$ (resp. $\phi(g) = -1$), and such that for all $x, y \in G$, $\theta_x\theta_y$ differs from θ_{xy} only by a phase $\sigma(x, y)$. Associativity of composition means that $\sigma : G \times G \to \mathrm{U}(1)$ is a (generalised) 2-cocycle satisfying

$$\sigma(x, y)\sigma(xy, z) = \sigma(y, z)^{\phi(x)}\sigma(x, yz). \tag{1}$$

The superscript $\phi(x)$ in Eq. (1) is a slight modification to the usual 2-cocycle $\delta\sigma = 1$ condition in the sense of group cohomology. Modifying each operator $\theta_x \mapsto \lambda_x\theta_x$, where $\lambda : x \mapsto \lambda_x$ is a phase function on G, corresponds to multiplying σ by a 2-coboundary $\delta\lambda : (x, y) \mapsto \lambda_x\lambda_y^{\phi(x)}/\lambda_{xy}$. Such phase modifications do not matter physically, so only the cocycle class of σ matters.

An interesting class of QM dynamics is that generated by *gapped* Hamiltonians, and we will shift H by a constant so that 0 lies in a some spectral gap (which is usually distinguished by some physical considerations). Then $\Gamma := \mathrm{sgn}(H)$ gives a $\mathbb{Z}_2$-grading of $\mathcal{H}$ and there is a homotopy from H to Γ through gapped self-adjoint operators (at least for bounded H, otherwise a truncation is assumed to make this homotopy precise). Given the full symmetry data (G, c, ϕ, σ), and interpreting c as a $\mathbb{Z}_2$-grading on G, we are interested in $\mathbb{Z}_2$-graded PUA-reps, i.e. PUA-reps on a $\mathbb{Z}_2$-graded Hilbert space $(\mathcal{H}, \Gamma)$ ("super-Hilbert space"), in which G *graded* commutes with Γ. We call these sPUA-reps, with the letter 's' standing for 'super'.

For a "topological" classification, we will not distinguish gapped H which are homotopic, so the grading operator in a sPUA-rep for (G, c, ϕ, σ) is interpreted as a representative of a "topological class" of symmetry-compatible gapped Hamiltonian. There are several inequivalent ways to state the precise equivalence relation (which at least contains the previous homotopy equivalence) that defines the "moduli" of symmetry-compatible Hamiltonians. We will describe one which leads eventually to a K-theory classification, an idea that originated in [26] and made precise in [15, 42].

2 Clifford algebras and a tenfold way

As an example of fundamental importance in physics, let G be the "CT-group" $\{1, C\} \times \{1, T\}$, so-called because C, T are respectively the **C**harge-conjugation and **T**ime-reversal symmetries. The diagonal element $CT = TC$ is denoted S, for **S**ublattice, as explained later. By convention, τ, c are defined to be

$$\tau(C) = +1,\ c(C) = -1, \qquad \tau(T) = -1,\ c(T) = +1. \tag{2}$$

Note that $\phi(C) = -1 = \phi(T)$, so the representatives C, T are antiunitary, whereas the diagonal element S is represented by a unitary S.

[4]For infinite topological groups, we would technically need θ and the subsequent 2-cocycle σ to be a Borel map, e.g. [34]. For $\sigma \equiv 1$, θ is a homomorphism which is automatically continuous, e.g. [27].

In general, only some subgroup of $A \subset G$ is present for a given physical system. Also, even though $C^2 = 1 = T^2$, the operators C, T only need to be involutions up to a phase, and they also commute only up to a phase — these phase ambiguities are encoded in a 2-cocycle σ satisfying Eq. (1). The symmetry data is (A, σ), with $\phi, c : A \to \{\pm 1\}$ implicitly given by (2).

Q: What are all the possibilities for C, T ("CT-symmetry classes"), i.e. the possible (A, σ)?

A: There are exactly ten classes, corresponding to the $8 + 2$ Morita classes of real + complex (graded) Clifford algebras / super-Brauer group over $\mathbb{R}$ and $\mathbb{C}$ / ten superdivision algebras over $\mathbb{R}$. They are labelled by the squares of C and T (where present).

Sketch of proof: Note that $\mathsf{T}^2 = \lambda$ for some $\lambda \in \mathrm{U}(1)$, so $\lambda\mathsf{T} = \mathsf{T}^3 = \mathsf{T}\lambda = \overline{\lambda}\mathsf{T}$, and $\lambda \in \{\pm 1\}$. Thus $\mathsf{T}^2 = \pm 1$, and similarly $\mathsf{C}^2 = \pm 1$. Note that this sign is invariant under $\mathsf{T} \mapsto \mu\mathsf{T}$. Next, we use the phase freedom in defining $\mathsf{C}, \mathsf{T}, \mathsf{S}$ to "standardize" them; specifically, we can arrange for $\mathsf{TC} = \mathsf{CT}$ and $\mathsf{S}^2 = +1$. Therefore we just need to assign, for each of the five possible subgroups $A \subset G$:

$$\{1\}, \{1, S\}, \{1, T\}, \{1, C\}, \{1, C, T, S\},$$

a ± 1 sign to C, T (where present) — there are ten possibilities in total, see Table 2.2.

Remark: We can identify A with the image of (ϕ, c). More generally, A arises as a *quotient* of the full symmetry group G by the kernel G_0 of (ϕ, c), i.e.

$$1 \to G_0 \to G \xrightarrow{\phi, c} A \to 1,$$

and there is a 2-cocycle $\tilde{\sigma}$ on $G \times G$ which projects onto σ on $A \times A$.

2.1 Clifford algebras

In connection with geometry, it is usual to define the (real) *Clifford algebra* for a vector space V with a quadratic form Q to be the free (tensor) algebra (over $\mathbb{R}$) subject to $v^2 = Q(v), v \in V$. This is a "quantization" of the exterior algebra ($Q \equiv 0$), i.e. the same underlying vector space but modified multiplication law. Complexification $(\cdot) \otimes_{\mathbb{R}} \mathbb{C}$ yields the complex Clifford algebra. Detailed proofs of statements in this section may be found in [28, 24, 37, 42].

We will proceed more concretely (corresponding to taking a standard form for Q, using Sylvester's law of inertia), defining the *complex* Clifford algebra $\mathbb{C}l_n$ to be the complex unital algebra generated by anticommuting elements $f_i, i = 1, \ldots, n$ that square to $+1$. For example, $\mathbb{C}l_0 \cong \mathbb{C}$, $\mathbb{C}l_1 \cong \mathbb{C}[\frac{1+f_1}{2}] \oplus \mathbb{C}[\frac{1-f_1}{2}]$, and $\mathbb{C}l_2 \cong M_2(\mathbb{C})$ with $f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Similarly, the *real* Clifford algebra $Cl_{r,s}$ is generated (over $\mathbb{R}$) by anticommuting elements $e_i, f_j, i = 1, \ldots, r, j = 1, \ldots, s$ such that $e_i^2 = -1, f_j^2 = +1$. For example, $Cl_{0,0} \cong \mathbb{R}, Cl_{1,0} \cong \mathbb{C}, Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$ and $Cl_{1,1} \cong M_2(\mathbb{R})$ with $f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We also have $Cl_{2,0} \cong \mathbb{H}$ and $Cl_{0,2} \cong M_2(\mathbb{R})$. When we complexify, $\mathrm{i}e_i$ squares to $+1$, so $Cl_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}l_{r+s}$ as complex algebras. For example, the complexifications of $Cl_{1,0}$ amd $Cl_{0,1}$ are both $\mathbb{C} \oplus \mathbb{C} \cong \mathbb{C}l_1$, and the complexifications of $Cl_{2,0}, Cl_{1,1}, Cl_{0,2}$ are all $M_2(\mathbb{C}) \cong \mathbb{C}l_2$.

One can also show that $Cl_{0,8} \cong M_{16}(\mathbb{R}) \cong Cl_{8,0}$.

It turns out that there are (algebraic) Bott periodicity identities

$$Cl_{r+1,s+1} \cong Cl_{r,s} \otimes_{\mathbb{R}} Cl_{1,1}$$

$$Cl_{r+8,0} \cong Cl_{r,0} \otimes_{\mathbb{R}} Cl_{8,0}$$

$$Cl_{0,s+8} \cong Cl_{0,s} \otimes_{\mathbb{R}} Cl_{0,8}$$

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2$$

and since $Cl_{1,1}, Cl_{8,0}, Cl_{0,8}, \mathbb{C}l_2$ are each matrix algebras, the Morita class (representation theory) of $\mathbb{C}l_n$ only depends on n (mod 2) while that of $Cl_{r,s}$ only depends on $r-s$ (mod 8). In total, there are 8+2=10 Morita classes of real/complex Clifford algebras, and each is a matrix algebra over $\mathbb{R}/\mathbb{C}/\mathbb{H}$ or a direct sum of two matrix algebras (of the same dimension over the same (skew)-field).

It is mathematically and physically convenient to regard the Clifford algebras as $\mathbb{Z}_2$-graded real/complex C^*-algebras, by requiring e_i, f_j to be odd, e_i to be skew-adjoint, f_j to be self-adjoint, and taking the (unique) C^*-norm e.g. from their matrix realisations. Then we can define $\widehat{\mathscr{M}_{r,s}}$ and $\widehat{\mathscr{M}_n^{\mathbb{C}}}$ to be the free abelian groups generated by the distinct irreducible unitary[5] $\mathbb{Z}_2$-graded representations of $Cl_{r,s}$ and $\mathbb{C}l_n$ respectively. We can regard the grading as an extra Clifford generator f_{s+1} in the *ungraded* sense, so the ungraded versions $\mathscr{M}_{r,s}$ and $\mathscr{M}_n^{\mathbb{C}}$ are related to the graded ones via $\widehat{\mathscr{M}_{r,s}} \cong \mathscr{M}_{r,s+1}$ and $\widehat{\mathscr{M}_n^{\mathbb{C}}} \cong \mathscr{M}_{n+1}^{\mathbb{C}}$. Furthermore, the $\mathbb{Z}_2$-graded tensor product gives an isomorphism $Cl_{r_1,s_1} \hat{\otimes} Cl_{r_2,s_2} \cong Cl_{r_1+r_2,s_1+s_2}$, and the $\mathbb{Z}_2$-graded tensor product $W \hat{\otimes} V$ of modules $W \in \widehat{\mathscr{M}_{r_1,s_2}}, V \in \widehat{\mathscr{M}_{r_2,s_2}}$ is canonically a $Cl_{r_1+r_2,s_1+s_2}$-module. Thus there are natural pairings $\widehat{\mathscr{M}_{r_1,s_1}} \otimes_{\mathbb{Z}} \widehat{\mathscr{M}_{r_2,s_2}} \to \widehat{\mathscr{M}_{r_1+r_2,s_1+s_2}}$ which give $\oplus_{r,s\geq 0}\widehat{\mathscr{M}_{r,s}}$ the structure of a bigraded *ring*; similarly for the complex case.

2.2 Tenfold way

It is now an exercise to get a one-to-one correspondence between the ten C,T possibilities and the Clifford algebras. For example, if $A=\{1,C,T,S\}$ and $\mathsf{C}^2=-1, \mathsf{T}^2=+1$, then $\{\mathsf{C}, \mathrm{i}\mathsf{C}, \mathrm{i}\mathsf{CT}\}$ are odd Clifford generators. From their squares, we see that we have a graded representation of $Cl_{2,1}$, in which $e_1=\mathsf{C}, e_2=\mathrm{i}\mathsf{C}, f_1=\mathrm{i}\mathsf{CT}$. In other words, an element of $\widehat{\mathscr{M}_{r,s}}$ or $\widehat{\mathscr{M}_n^{\mathbb{C}}}$ is nothing but a sPUA-rep for the corresponding (A,σ); or physically, a realisation of a CT-symmetry class in which the grading is a compatible gapped Hamiltonian.

Remark: The two $A=\{1,T\}$ cases are a bit tricky: T gives the graded Hilbert space $\mathcal{H}$ a real or quaternionic structure depending on $\mathsf{T}^2=\pm 1$. We can think of $\mathcal{H}=\mathcal{H}^+\oplus\mathcal{H}^-$ as a formal difference of two ungraded real/quaternionic spaces $[\mathcal{H}^+]-[\mathcal{H}^-]$ (ungraded PUA-reps of (A,σ)). Take $\{\mathrm{i},\mathsf{T},\mathrm{i}\mathsf{T}\Gamma\}$ as generators of the *ungraded* Clifford algebra $Cl_{1,2}$ when $\mathsf{T}^2=+1$ and $Cl_{3,0}$ when $\mathsf{T}^2=-1$. The sPUA-reps are classified by $\mathscr{M}_{1,2}\cong\widehat{\mathscr{M}_{1,1}}\cong\widehat{\mathscr{M}_{0,0}}$ or $\mathscr{M}_{3,0}\cong\mathscr{M}_{4,1}\cong\widehat{\mathscr{M}_{4,0}}$.

Remark: There is another point of view in which there are 10=3+7 *superdivision algebras* over $\mathbb{R}$, with the trivially graded ones $\mathbb{R},\mathbb{C},\mathbb{H}$ being the usual division algebras over $\mathbb{R}$ and the other seven being non-trivially graded; each graded Clifford algebra is graded Morita equivalent to one of these superdivision algebras [15]. The three trivially graded cases correspond

[5] This means *orthogonal* in the real case.

Symmetry	C^2	T^2	Clifford algebra	Graded Morita class
T		$+1$	$Cl_{1,2}$	$Cl_{0,0}$
C,T	-1	$+1$	$Cl_{2,2}$	$Cl_{1,0}$
C	-1		$Cl_{2,1}$	$Cl_{2,0}$
C,T	-1	-1	$Cl_{3,1}$	$Cl_{3,0}$
T		-1	$Cl_{3,0}$	$Cl_{4,0}$
C,T	$+1$	-1	$Cl_{0,4}$	$Cl_{5,0}$
C	$+1$		$Cl_{0,3}$	$Cl_{6,0}$
C,T	$+1$	$+1$	$Cl_{1,3}$	$Cl_{7,0}$
N/A	N/A		$\mathbb{C}l_1$	$\mathbb{C}l_0$
S	$\mathsf{S}^2=+1$		$\mathbb{C}l_2$	$\mathbb{C}l_1$

Table 1: A version of the Tenfold way.

to an old "Threefold way" of Dyson [12], in which another (inequivalent) "Tenfold way" was introduced.

2.3 A glimpse of K-theory

An interesting construction of Atiyah–Bott–Shapiro expresses the K-theory rings of a point to Clifford modules [4]:

$$K^n(\star) \cong \widehat{\mathscr{M}}_n^{\mathbb{C}}/\iota^*\widehat{\mathscr{M}}_{n+1}^{\mathbb{C}}, \qquad KO^{r-s}(\star) \cong \widehat{\mathscr{M}}_{r,s}/\iota^*\widehat{\mathscr{M}}_{r,s+1}$$

where ι^* is the forgetful map which forgets the action of the extra Clifford generator. Let us sketch this construction, which first requires one particular definition of relative K-theory groups of a compact pair $Y \subset X$.

First, the (absolute) complex K-theory groups $K^0(X)$ of a compact Hausdorff space is the Grothendieck completion of the monoid (w.r.t. Whitney sum) of isomorphism classes of complex vector bundles over X. For a compact pair $Y \subset X$, the relative K-theory groups can be defined via triples $(\mathcal{E}_i, \mathcal{F}_i, \alpha_i), i = 1, 2$, where $\mathcal{E}_i, \mathcal{F}_i$ are complex bundles over X and $\alpha_i : \mathcal{E}_i|_Y \to \mathcal{F}_i|_Y$ are bundle isomorphisms on the restrictions to Y. We say that $(\mathcal{E}_1, \mathcal{F}_1, \alpha_1)$ is equivalent to $(\mathcal{E}_2, \mathcal{F}_2, \alpha_2)$ if there are isomorphisms $f : \mathcal{E}_1 \to \mathcal{E}_2$ and $g : \mathcal{F}_1 \to \mathcal{F}_2$, such that $\alpha_2 \circ f|_Y = g|_Y \circ \alpha_1$. An elementary triple is a triple (E, E, α) such that α is homotopic to id_E in the set of bundle isomorphisms. We can form the direct sum of triples in the natural way, and we say that two triples are stably isomorphic if they are isomorphic upon adding elementary triples. The *relative* K-theory group $K^0(X, Y)$ is then defined to be the group of stable isomorphism classes of triples $[\mathcal{E}, \mathcal{F}, \alpha]$, and the absolute groups are recovered by taking $Y = \emptyset$. The real K-theory groups $KO^0(X), KO^0(X, Y)$ are similarly defined but with real vector bundles.

We can extend the definition of $K^0(X)$ to *locally compact* X by taking $K^0(X) := K^0(X^+, \{\infty\})$, where X^+ is the one-point compactification of X. If X is already compact, then X^+ is X with a disjoint point $\{\infty\}$ added. This is "K-theory with compact supports", c.f. I.9 of [28]. Higher K-theory groups are defined by *suspending*, $K^{-n}(X, Y) := K^0((X \setminus Y) \times \mathbb{R}^n)$, and in particular, this gives the higher (absolute) K-theory groups of X as $K^{-n}(X) = K^0(X \times \mathbb{R}^n)$.

For a point $X = \star$, we therefore have

$$K^{-n}(\star) = K^0(\mathbb{R}^n) = K^0(B^n, S^{n-1}),$$

where B^n, S^{n-1} is the unit ball and unit sphere in $\mathbb{R}^n$. In fact, $\oplus_{n\geq 0}K^{-n}(X)$ is a graded *ring*, and furthermore, a graded module over $\oplus_{n\geq 0}K^{-n}(\star)$ through tensor product constructions [28].

Everything works similarly in ordinary real KO-theory, but an interesting and useful generalisation is the Real KR-theory of Atiyah [3] in which there are *two* ways of suspending, and correspondingly a bi-graded theory $KR^{r,s}(\cdot)$, defined on spaces X equipped with involutary self-homeomorphism ς. For this theory, we need to use $\mathbb{R}^{r,s}$, which is $\mathbb{R}^{r+s}$ but with the last s coordinates having the involution $x_j \mapsto -x_j$, as well as the corresponding unit ball and sphere $B^{r,s}, S^{r,s}$.

Let $W = W_0 \oplus W_1$ be a graded $\mathbb{C}l_n$-module, and let $\mathcal{E}_i$ be the trivial vector bundles $B^n \times W_i$ for $i = 1, 2$ over the unit n-ball B^n in $\mathbb{R}^n$. We can associate to W the relative K-group element

$$\phi_n(W) = [\mathcal{E}_0, \mathcal{E}_1, \alpha] \in K^0(B^n, S^{n-1}),$$

where α is the isomorphism over S^{n-1} given by

$$\alpha(x, w) = (x, x \cdot w), x \in S^{n-1}, w \in W_0.$$

One checks that ϕ_n is well-defined on isomorphism classes and is additive, so that there is an induced homomorphism

$$\phi_n : \widehat{\mathscr{M}_n^{\mathbb{C}}} \to K^0(B^n, S^{n-1}) \cong K^{-n}(\star).$$

Now suppose that W extends to a graded $\mathbb{C}l_{n+1}$-module. Let f_{n+1} denote the extra Clifford generator, also regarded as a unit vector of $\mathbb{R}^{n+1}$ orthogonal to $\mathbb{R}^n$. Then we can extend the isomorphism α to the whole of B^n by setting

$$\alpha(x, w) = \left(x, (x + \sqrt{1 - ||x||^2} f_{n+1}) \cdot w\right), \quad x \in B^n, w \in W_0.$$

In this case, $\mathcal{E}_0$ and $\mathcal{E}_1$ are isomorphic, so $\phi_n(W)$ is equivalent to the zero element in $K^0(B^n, S^{n-1}) = K^{-n}(\star)$. The homomorphism ϕ_n thus descends to a homomorphism

$$\phi_n : \widehat{\mathscr{M}_n^{\mathbb{C}}}/\iota^* \widehat{\mathscr{M}_{n+1}^{\mathbb{C}}} \to K^{-n}(\star).$$

That $\oplus_{n\geq 0}\phi_n$ is actually a *graded ring isomorphism* onto $\oplus_{n\geq 0}K^{-n}(\star)$ is one of the main results in [4] (which requires a priori knowledge of $K^\bullet(\star)$ using Bott periodicity). Similarly, in the real case we have isomorphisms $\phi_{r,s} : \widehat{\mathscr{M}_{r,s}}/\iota^* \widehat{\mathscr{M}_{r,s+1}} \to KR^{s,r}(\star) \cong KO^{r-s}(\star)$, c.f. [28] which has some differing sign conventions.

Remark: One physical interpretation for quotienting out $\mathbb{C}l_{n+1}$ is: a $\mathbb{C}l_n$-module (sPUA-rep for some CT-class) which admits a $\mathbb{C}l_{n+1}$-module structure should really be considered sPUA-reps for some *other* CT-class. Another interpretation involves arguing that taking Γ to $-\Gamma$ should be akin to taking an inverse. When an extra Clifford generator is available, it becomes possible to homotope Γ to $-\Gamma$ through compatible grading operators, exhibiting the “triviality” of Γ [24, 42].

3 K-theory and topological insulators

We know that the real/complex K-theory of a point acts on that of a topological space X, and that $K^0(X)$ provides a classification of vector bundles over X. We have already given a physical/representation-theoretic meaning to $K^*(\star)$, using Clifford modules as sPUA-reps for the CT-subgroups. Is there a similar representation-theoretic origin for X (preferably non-discrete), as well as bundles over X, so that all the groups $K^*(X)$ can be given similar physical interpretations as "symmetry protected topological phases"?

It turns out interesting X do appear, e.g. in topological insulators[6], due again to a symmetry principle — they are the Pontryagin dual to some non-compact abelian subgroup $N \subset G$ of symmetries (or the noncommutative analogue).

3.1 Bloch Hamiltonians

Here is a concrete setting. The Hamiltonian H of an electron moving around in a crystalline material in d dimensions has (at least) the symmetry of the lattice translations $N = \mathbb{Z}^d$. It may also have some finite point group symmetries P (or even a compact group of "internal" symmetries), as well as some CT-symmetries.

A typical Hilbert space for the electron is $l^2(\mathbb{Z}^d) \otimes \mathbb{C}^m$ with $\mathbb{C}^m$ a finite-dimensional Hilbert space of internal degrees of freedom, such as spin, orbital angular momentum etc. For simplicity, set $m = 2$. Recall that the dual group (character space) of $\mathbb{Z}^d$ is $\mathbb{T}^d$, called the *Brillouin torus* of *quasimomenta* in physics. Explicitly, for each $k \in \mathbb{T}^d \cong ([0, 2\pi]/_{0\sim 2\pi})^d$, the corresponding character is $\chi_k : n \mapsto e^{\mathrm{i}n\cdot k}$. The Fourier transform diagonalises $\mathbb{Z}^d$ invariant operators on $l^2(\mathbb{Z}^d) \otimes \mathbb{C}^2$ into multiplication operators on $L^2(\mathbb{T}^d \times \mathbb{C}^2) \cong L^2(\mathbb{T}^d) \otimes \mathbb{C}^2$. For example, translation by $n \in \mathbb{Z}^d$ turns into multiplication by $e^{\mathrm{i}n\cdot k}$.

The Hamiltonian becomes a fibered family of *Bloch Hamiltonians* $k \mapsto H(k), k \in \mathbb{T}^d$, each with some eigenvalues. If the Hamiltonian describes a *band insulator*, the spectra assemble into continuous bands over $\mathbb{T}^d$, with a gap at a particular energy called the *Fermi energy* (which we set to zero), see Fig. 1. Therefore, the bundle $\mathcal{E} = \mathbb{T}^d \times \mathbb{C}^2$ is graded by the gapped Hamiltonian into a positive energy sub-eigenbundle and a negative energy sub-eigenbundle $\mathcal{E}_F$. The latter is the line bundle of *valence states*, and when $d \geq 2$, it can be topologically non-trivial! When $m > 2$, $\mathcal{E}_F$ is generally a higher rank vector subbundle. When we add further CT-symmetries, $\mathcal{E}$ has more structure including a natural $\mathbb{Z}_2$ action on the base (see later), and further point group symmetries P turn $\mathcal{E}$ and $\mathcal{E}_F$ into a *twisted P-equivariant bundle* [15].

Remark: A more realistic model involves a *Bloch–Floquet* transform [39] of $L^2(\mathbb{R}^d)$ into the sections of some infinite-dimensional Hilbert bundle over $\mathbb{T}^d$. The fibre at $k \in \mathbb{T}^d$ is the Hilbert space of k-quasiperiodic *Bloch wavefunctions*, i.e. functions $f_k : \mathbb{R}^d \to \mathbb{C}$ which satisfy $f_k(x + n) = e^{\mathrm{i}n\cdot k} f_k(x)$ for all $n \in \mathbb{Z}^d$. Such functions are of course not square-integrable, but we could think of them as sections of a line bundle $\mathcal{L}_k \to \mathbb{R}^d/\mathbb{Z}^d$ with flat connection $\mathrm{i}k$, then the f_k are square-integrable. In this sense, $\mathbb{T}^d$ can be thought of as the moduli space of flat connections, parameterising quasiperiodic boundary conditions. For each k, finding the spectrum of the Bloch Hamiltonian $H(k)$ corresponds to solving the Schrödinger equation for H with k-quasiperiodic boundary conditions. By the compactness of $\mathbb{R}^d/\mathbb{Z}^d$, we expect $H(k)$

[6] See [20] for a physics oriented review.

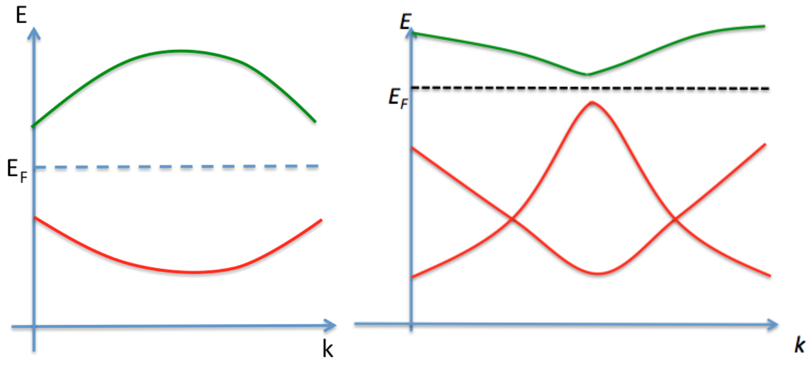

Figure 1: Schematic energy band structures. Horizontal axis labels points k in the Brillouin torus $\mathbb{T}^d$. Vertical axis labels energy eigenvalues for the Bloch Hamiltonian $H(k)$. Green line represents a conduction energy band. Red line(s) represent valence energy bands, which have an associated subbundle of valence states.

to typically have discrete spectrum, and over $k \in \mathbb{T}^d$, these are assumed to form continuous bands labelled by a discrete set (H itself of course need not have discrete spectrum).

There are many energy bands, and for the low energy physics, only the ones lying near the Fermi energy matter. Each band's sub-eigenbundle can be inverse-transformed (but in a gauge-dependent way) into a subspace of wavefunctions spanned by *Wannier functions* centered at lattice sites. There is an interesting relationship between the topological triviality of a band of Bloch eigenstates, and the localisability of such a basis of Wannier functions [33].

3.2 Class A "Chern topological insulator" in 2D

Suppose $G = N = \mathbb{Z}^2$ so there are no CT-symmetries. We seek a classification of topological band insulators in this symmetry class, i.e. classes of complex vector bundles $\mathcal{E}_F \to \mathbb{T}^2$ — these are classified by the (first) Chern class $c_1(\mathcal{E}_F) = c_1(\det \mathcal{E}_F)$.

It suffices to look at line bundles, or principal U(1) bundles, which are classified by $[\mathbb{T}^2, B\mathrm{U}(1)] = \mathbb{CP}^\infty = K(\mathbb{Z}, 2) = H^2(\mathbb{T}^2; \mathbb{Z})$, see eg. [8] for a treatment of classifying spaces and Eilenberg–Maclane spaces in homotopy/cohomology. Because $\mathbb{T}^2$ has low dimension, it suffices to look at maps $\mathbb{T}^2 \to \mathbb{CP}^1 \subset \mathbb{CP}^\infty$. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be the Pauli matrices. Then the homeomorphism $S^2 \cong \mathbb{CP}^1$ is the correspondence

$$\text{unit vector } \mathbf{h} \longleftrightarrow (-1)\text{eigenspace of } \mathbf{h} \cdot \sigma.$$

Physicists usually write 2×2 Bloch Hamiltonians as $H(k) = \mathbf{h}(k) \cdot \sigma$ for some vector field $\mathbf{h} : \mathbb{T}^2 \to S^2$, with $\sigma(k) = \sigma$ taken[7] to be constant over $\mathbb{T}^2$ with respect to some trivialisation of $\mathcal{E} = \mathbb{T}^2 \times \mathbb{C}^2$. It is easy to see that $\mathrm{spec}(H(k)) = \pm|\mathbf{h}(k)|$, so that the Hamiltonian is gapped iff $\mathbf{h}$ has no zeroes. It also follows that the valence line bundle $\mathcal{E}_F$ is nothing but the pullback of the tautological bundle over $\mathbb{CP}^1$, and that its Chern class is the homotopy class

[7] Unfortunately, σ is also standard notation for 2-cocycles, so its meaning in these notes needs to be inferred from the context.

of $\mathbf{h}$ (i.e. Brouwer degree). A Hamiltonian specified by a $\mathbf{h}$ with nonzero degree gives rise to a *Chern insulator* phase — this has been discovered [23].

There is a parallel story in topological K-theory, with classifying space $\mathbb{Z} \times B\mathrm{U}$ (here U is the inductive limit of $\mathrm{U}(n)$), and a Chern character to cohomology. It is perhaps more natural, for later generalisation, to formulate things in operator K-theory. In this picture, the sections of $\mathcal{E}_F$ form a finite projective module over the C^*-algebra $C(\mathbb{T}^2) \cong C^*(\mathbb{Z}^2)$; or equivalently a projection in its matrix algebra, where $C^*(\cdot)$ is the group C^*-algebra. Explicitly, the projection p_F is $p_F(k) = \frac{1}{2}(1 - \hat{\mathbf{h}}(k) \cdot \sigma)$, and defines a class in $K_0(C^*(\mathbb{Z}^2))$. Within $K_0(C^*(\mathbb{Z}^2))$, the equivalence of projections may be defined homotopically, and so provides a precise meaning to the statement that homotopic p_F define the same K-theory class and therefore the same "topological phase" [42].

Noncommutative bundles and the quantum Hall effect

Even though $\mathbb{Z}^d$ is abelian, it may actually act projectively with 2-cocycle σ. This occurs in the integer quantum Hall effect, where electrons in a 2D sample are subject to a perpendicular magnetic field. The Hamiltonian gets modified by a coupling term between the electron charge and magnetic vector potential, and consequently commutes with *magnetic* translations instead of ordinary translations. The magnetic translations in each direction do not commute. Instead, $T_x T_y = \sigma(x, y) T_{x+y}$, $x, y \in \mathbb{Z}^2$ where the cocycle σ is proportional to the magnetic field strength. They generate a *noncommutative torus* $C^*(\mathbb{Z}^2, \sigma)$, which is a sort of "noncommutative" momentum space. Although there is no longer a $\mathcal{E}_F$ in the usual sense, p_F continues to exist as a projection in (the stablised) $C^*(\mathbb{Z}^2, \sigma)$ and has a integrally quantized Chern number in the noncommutative sense [6]. This integer corresponds to the famous quantized transverse Hall conductivity whose discovery in 1980 was rewarded with a Nobel prize. For rational magnetic flux per unit cell, the noncommutative torus thus constructed is Morita equivalent to an ordinary torus, and the commutative approach of [44] suffices in some aspects.

3.3 Class AII: $\mathbb{Z}_2$ topological insulators in 2D and 3D

Suppose $G = N \times A$ with $N = \mathbb{Z}^d$ as before, but now $A = \{1, T\}$ with $\mathsf{T}^2 = -1$. Then T is a quaternionic structure on the Hilbert space $l^2 \otimes \mathbb{C}^m$. Because T is antilinear, it acts on $\mathbb{T}^d$ (the character space of N) by complex conjugation ς. Explicitly, this takes $k \mapsto -k$ corresponding to $\chi_k \mapsto \overline{\chi_k} = \chi_{-k}$. When we Fourier-transform to $L^2(\mathbb{T}^d) \otimes \mathbb{C}^m$, T becomes what is called a "Quaternionic" structure — an antiunitary lift of ς which squares to -1. Thus $\mathcal{E} = \mathbb{T}^d \times \mathbb{C}^m$ is a "Quaternionic" vector bundle, graded by a compatible gapped Hamiltonian, and the valence subbundle $\mathcal{E}_F$ that we want to classify is also a "Quaternionic" vector bundle.

Notice that the involution ς has 2^d fixed points F, so T restricts over these fixed points to an ordinary quaternionic structure. This means that $\mathcal{E}_F$ must be even-dimensional as a complex bundle — this is called Kramers degeneracy in physics (every state has a T partner).

How can we classify such "Quaternionic" bundles? For $d = 2, 3$ Physicists Fu–Kane–Mele [14] came up with some ingenious ad-hoc constructions of $\mathbb{Z}_2$-valued invariants, and predicted that they are manifested on boundaries as pairs of edge states or Dirac cones (mod 2). Remarkably, they were realised experimentally in [21] and this really kick-started the huge interest among physicists in applying topological ideas to condensed matter physics. For mathematicians, this $\mathbb{Z}_2$ invariant is a particular instance of a certain class of topological invariants which had mostly been overlooked, perhaps due to the lack of an obvious geometrical interpretation or concrete application.

Recent work of De Nittis–Gomi [10] show that there is actually a characteristic class $H^2_{\mathbb{Z}_2}(\mathbb{T}^d, F; \mathbb{Z}(1))$ for such bundles which is furthermore bijective in $d \leq 3$ with some mild assumptions (that are satisfied in the physical applications). The meaning of $H^2_{\mathbb{Z}_2}(\mathbb{T}^d, F; \mathbb{Z}(1))$ is the $\mathbb{Z}_2$-equivariant cohomology of $(\mathbb{T}^d, \varsigma)$ with local coefficients $\mathbb{Z}(1)$, relative to the ς-fixed point set F, where $\mathbb{Z}_2$ acts on the local coefficients by $n \mapsto -n$. Further work of Gomi–Thiang [17] provides an easy way to understanding these $\mathbb{Z}_2$ invariants of this using a duality transform.

Let us instead proceed using K-theory. From the topological point of view, we can use Dupont's $\widetilde{KQ}^0(\mathbb{T}^d)$ for the stable classification of "Quaternionic" bundles [11]. This "Quaternionic" K-theory is naturally isomorphic to Atiyah's Real K-theory by a degree-4 shift, so we can also compute $\widetilde{KR}^{-4}(\mathbb{T}^d)$. If one finds the involution ς cumbersome to handle, one can use KO-theory for C^*-algebras instead [42]. In either case, the result is that

$$\widetilde{KQ}^0(\mathbb{T}^2) \cong \mathbb{Z}_2, \quad \widetilde{KQ}^0(\mathbb{T}^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2^3$$

3.3.1 Noncommutative topology — real case

In the complex world, there are Gelfand–Naimark correspondences between commutative C^*-algebras and locally compact Hausdorff spaces, as well as Serre–Swan correspondences between vector bundles over a compact X and f.g.p modules over $C(X)$. In the real world, there is again a correspondence between l.c.h. $\mathbb{Z}_2$-spaces (X, ς) and commutative C^*-algebras over $\mathbb{R}$ [1]. Specifically, the latter are all of the form

$$C_0(X, \varsigma) := \{f \in C_0(X) \,:\, f(\varsigma(x)) = \overline{f(x)} \;\forall x \in X\}.$$

Roughly speaking, (X, ς) is the "Real spectrum" of the algebra. We can understand this from a simple example, the real group C^*-algebra of $\mathbb{Z}^d$, which is $C^*_{\mathbb{R}}(\mathbb{Z}^d) \cong C(\mathbb{T}^d, \varsigma)$. Note that $C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{C} \cong C^*(\mathbb{Z}^d)$, and $C^*_{\mathbb{R}}(\mathbb{Z}^d)$ is the real subalgebra under complex conjugation. The spectrum of $C^*_{\mathbb{R}}(\mathbb{Z}^d)$ is that of the complexification (which is $\mathbb{T}^d$), and the Real structure ς on $\mathbb{T}^d$ is induced from complex conjugation (thus it is conjugation of the characters in $\mathbb{T}^d$). The fixed points are the characters that map into $\mathbb{R}$ (and so take $\mathbb{Z}^d$ into $\{\pm 1\}$).

For $G = \mathbb{Z}^d \times \{1, T\}$ with $\mathsf{T}^2 = -1$, the appropriate group algebra is a *twisted crossed product*[8] [34] in the sense that T acts on $\mathbb{C}$ (as a real algebra) by complex conjugation. So we have

$$C^*(G, \phi, \sigma) = \mathbb{C} \rtimes_{\phi,\sigma} (\mathbb{Z}^d \times \{1, T\}) \cong (C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{C}) \rtimes_{1\otimes\phi, 1\otimes\sigma} \{1, T\} \cong C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{H}.$$

"Quaternionic" bundles over $\mathbb{T}^d$ are thus equivalently (say, right) f.g.p. modules over $C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{H}$, and (their formal differences) are classified by

$$KO_0(C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{H}) \cong KO_0(C^*_{\mathbb{R}}(\mathbb{Z}^d) \hat{\otimes} Cl_{4,0}) \cong KO_4(C^*_{\mathbb{R}}(\mathbb{Z}^d)),$$

where we have used the correspondence between $\mathbb{Z}_2$-graded (i.e. formal differences of) $\mathbb{H}$-modules and graded $Cl_{4,0}$-modules [15].

[8] The real/complex group C^*-algebra of G is equivalently the crossed product of $\mathbb{R}$ or $\mathbb{C}$ by a *trivial* action of G.

Strictly speaking, I have not defined $K(O)_0(\cdot)$ for graded C^*-algebras, and there are several routes to do this. One way is to follow Karoubi and define $K_0(\mathcal{A})$ through triples $(W;\Gamma_1,\Gamma_2)$ where W is an *ungraded* f.g.p. $\mathcal{A}$-module and Γ_1,Γ_2 are compatible grading operators [24]. Such a triple can be interpreted as an obstruction to homotoping the Γ_i (in a stabilised sense). Another approach is due to van Daele [45]. Yet another approach is through Kasparov's KK-theory, which also comes with an in-built index pairing with K-homology, and which was used in an analysis of the *bulk-boundary correspondence* for the IQHE [9].

3.4 Class AIII: Chiral symmetry gapped phases in 1D

Consider $G=\mathbb{Z}\times\{1,S\}$, i.e. 1D band insulators with S symmetry. Without S, this is like the 1D version of the Class A example, so we have $\mathcal{E}\cong\mathbb{T}\times\mathbb{C}^2$. We can choose a trivialisation (gauge) such that S is $\mathsf{S}(k)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $k\in\mathbb{T}$. Since H anticommutes with S, it follows that

$$H(k)=\begin{pmatrix}0&u(k)\\u(k)^*&0\end{pmatrix},\quad u(k)\in\mathbb{C}. \tag{3}$$

If H is furthermore gapped, $(H(k))^2>0$ so we need $u(k)u(k)^*\neq 0$, so that $u(k)\in\mathbb{C}^*\cong\mathrm{GL}(1)$. We can homotope to $\mathrm{sgn}(H)$, which corresponds to replacing $u(k)$ by $u(k)/|u(k))|\in\mathrm{U}(1)$. There is an obvious topological invariant, which is the winding number of $u:\mathbb{T}\to\mathrm{U}(1)$. In terms of K-theory, the classification group is $K^{-1}(S^1)\cong\mathbb{Z}$, where we have used the fact that $K^{-1}(\cdot)$ is equivalently given by homotopy classes of maps into U [24], and for $X=\mathbb{T}$ it suffices to map into U(1). Note, however, that the expression for u is *gauge-dependent*, and so is the winding number, changing by m under conjugation by the "large" gauge transformation[9] $\begin{pmatrix}e^{imk}&0\\0&1\end{pmatrix}$ [43]. Thus we should think of the phases as an *affine* space for $K^{-1}(S^1)$, with no canonical "zero/trivial phase". A typical H with such a symmetry is the Hamiltonian of Su–Schrieffer–Heeger used to model the polymer polyacetylene.

Let us try to understand the physical meaning of the winding number $\mathrm{Wind}(u)$ and the gauge choices involved. Consider $l^2(\mathbb{Z})\otimes\mathbb{C}^2$, representing a degree of freedom at each site of an infinite chain, which is split by the operator S into sublattices A and B (Fig. 2) corresponding to the $\pm$ eigenspaces of S. When a boundary of the chain is specified, the unit cells are well-defined starting from the boundary (each contains one A and one B site) — this corresponds to a "gauge-fixing". The position of a unit cell is labelled by $n\in\mathbb{Z}$, and within a unit cell $\mathbb{C}[\binom{1}{0}],\mathbb{C}[\binom{0}{1}]$ corresponds to the A and B subspace of $\mathbb{C}^2$ respectively. The "hopping term" $T_{\rm blue}$ taking A to B rightwards within a unit cell is represented, after Fourier transform, by $\begin{pmatrix}0&0\\1&0\end{pmatrix}$, whereas the term $T_{\rm red}$ taking B to A rightwards changes unit cell and is represented by $\begin{pmatrix}0&e^{ik}\\0&0\end{pmatrix}$. The general Hamiltonian is a self-adjoint combination of powers of $T_{\rm blue},T_{\rm red}$ which is also required to be gapped and compatible with S, so that after Fourier transform, it has the form in Eq. (3).

Consider the "fully-dimerised" Hamiltonian $H_{\rm blue}=T_{\rm blue}+T^*_{\rm blue}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ which has

[9] This generally refers to an element of a non-identity component of the automorphism group (gauge transformation) of some principal bundle.

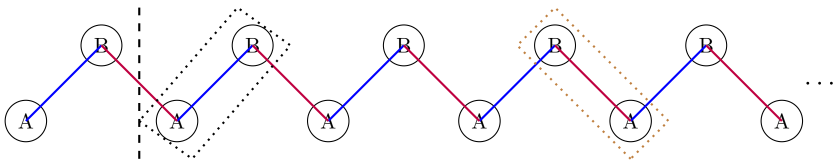

Figure 2: Dashed line represents the boundary of the infinite chain. Black and brown dotted rectangles indicate two choices of unit cells.

winding number 0. Another fully-dimerised Hamiltonian is $H_{\rm red}=T_{\rm red}+T^*_{\rm red}=\begin{pmatrix}0 & e^{{\rm i}k}\\ e^{-{\rm i}k} & 0\end{pmatrix}$ has winding number[10] 1.

Without the boundary, it is clear that $H_{\rm red}$ and $H_{\rm blue}$ are unitarily equivalent by translating A, or equivalently, choosing a different unit cell convention such that $T_{\rm red}$ is an intra-cell hopping term rather than an inter-cell one[11]. With the boundary, the choice of unit cell is fixed, and furthermore, the right-half-space[12] version of $T^*_{\rm blue}$ remains unitary, whereas $T_{\rm red}$ becomes only an isometry since $T_{\rm red}T^*_{\rm red}=1-p_{n_A=0}$ where $p_{n_A=0}$ is the projection onto the A-site at $n=0$. In fact, $T_{\rm red}$ is modified to become a *Toeplitz* operator $\widetilde{T}_{\rm red}$ with symbol the invertible function $u(k)=e^{{\rm i}k}$, which has Fredholm index -1 equal to $-{\rm Wind}(u)$. The latter equality between the Fredholm (analytic) index and the winding number (topological) index is a type of Toeplitz index theorem. When applied to the physical context of the SSH model, identifies the index/winding number with the number of unpaired edge modes — the bulk-with boundary Hamiltonian $\widetilde{H}_{\rm red}$ (Wind $=1$) has an unpaired A site at $n=0$, whereas all sites are paired up for $\widetilde{H}_{\rm blue}$ (Wind $=0$).

4 Bulk-boundary correspondence

The "non-trivial topology" of a (bulk) topological insulator is expected to be manifested on its boundary, because a continuous interpolation between the topological insulator and the outside world (the "vacuum", a topologically trivial insulator) would required a breakdown of the definability of discrete insulator invariants somewhere in between. The idea is that there will be boundary-localised surface states at the Fermi energy (thus breaking the spectral gap condition) which furthermore have some topological properties that are inherited from those of the bulk insulator. More generally, we can consider the interface of two bulk insulators with different bulk invariants, then the surface states are expected to reflect the *difference* between these bulk invariants. This phenomenon was already known and analysed in the context of the quantum Hall effect [25]. We have also given a Class AIII example in which a Toeplitz index implements such a correspondence: there is an unpaired zero energy boundary mode for $\widetilde{H}_{\rm red}$.

[10] Higher winding numbers may be obtained by "dimerising across more unit cells".

[11] This corresponds to the large gauge transformation $\begin{pmatrix}e^{{\rm i}k} & 0\\ 0 & 1\end{pmatrix}$ mentioned earlier.

[12] This is also the *Hardy space* in $L^2(\mathbb{T})$ comprising the functions with non-negative Fourier coefficients.

Later on, bulk-boundary correspondences were found for other classes of topological insulators. For instance, *helical Dirac cones* were detected in Class AII insulators [22]. The precise mathematical nature of these correspondences remains an area of active research. Let us outline one idea which is quite close to the physicists' intuition of "topological boundary states".

4.1 Spectral flow

Recall that when a $\mathbb{Z}^d$-invariant Hamiltonian H on $l^2(\mathbb{Z}^d)$ is assumed to have a spectral gap at the Fermi level E_F, we obtained a projection p_F to the occupied states defining the bulk invariant. In the presence of a boundary, only the quasimomenta $k_{||} \in \mathbb{T}^{d-1}$ parallel to the boundary remain well-defined, so we can still partially Fourier-transformed the bulk Hamiltonian H into $k \mapsto H(k_{||})$, and similarly $p_F(k_{||})$. With boundary conditions imposed, $H(k_{||})$ becomes a *half-space operator* $\widetilde{H}(k_{||})$, whose corresponding $\widetilde{p}_F(k_{||})$ need not remain a projection. We can think of the bulk-with-boundary operators $\widetilde{H}, \widetilde{p}_F$ as living in a *Toeplitz algebra* $\mathcal{T}$, generated by $d-1$ commuting translations together with an extra half-space translation (which is therefore *not* unitary). The Toeplitz algebra fits into a *non-split* extension

$$0 \to C^*(\mathbb{Z}^{d-1}) \otimes \mathcal{K} \to \mathcal{T} \to C^*(\mathbb{Z}^d) \to 1,$$

where $\mathcal{K}$ is the compact operators, and we recognise $C^*(\mathbb{Z}^{d-1}), C^*(\mathbb{Z}^d)$ as, respectively, the boundary and bulk Brillouin tori[13].

Intuitively, the failure of $\widetilde{p}_F(k_{||})$ to remain a projection means that new boundary-localised spectra with some $k_{||}$ is acquired at E_F. Since E_F may be varied within the bulk spectral gap, one sees that the boundary spectra actually fills up this gap and connects a bulk valence band to a conduction band. For $d = 2$, $\widetilde{H}(k_{||})$ is just a periodic family, and we can ask how many times the edge spectra "flows" past the Fermi level from below as the boundary Brillouin circle is traversed around once (Fig. 3). Such edge indices (Bott–Maslov indices) and the connection to the bulk topological invariants were studied for 2D in [16, 5], and using K-theoretic techniques in a more general context in [38], building on the quantum Hall example in [25]. The (topological) bulk-boundary map in [25, 38] is taken to be the connecting homomorphism ∂ in K-theory associated to the above Toeplitz algebra extension, so it computes a kind of Toeplitz index, which in the $d = 2$ case can be understood precisely as computing spectral flow. We can also think of ∂ as a kind of noncommutative Gysin/integration map corresponding to projecting out the transverse momentum $k_\perp$ [32, 2]

In the T-symmetric case (Class AII), the bulk-boundary map can be analogously defined, and has a geometrical interpretation as "T-invariant integration" over the transverse momentum $k_\perp$. The precise interpretation of ∂ as spectral flow in this case, and higher dimensions, is not so well-understood and is the subject of ongoing investigation.

5 Outlook

The K-theoretic approach to topological insulators is formally identical to the D-brane classification in string theory. There is a deep duality in the latter called *T-duality*, which

[13] More generally, a similar Toeplitz extension is canonically associated with some action of an algebra by $\mathbb{Z}$ [36], and the above example is a special case where the algebra is $C^*(\mathbb{Z}^{d-1})$ and the $\mathbb{Z}$-action is trivial.

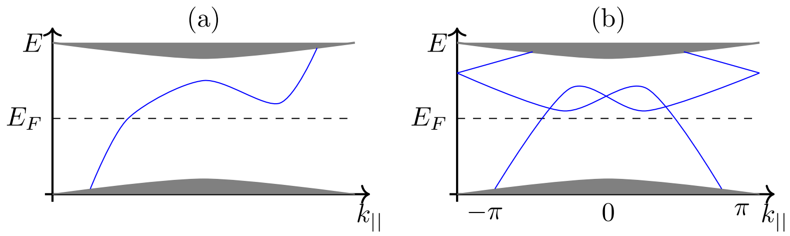

Figure 3: (a) Family of edge states (blue) connecting bulk bands (gray) in Class A case. The number of intersections (chiral edge states) at the Fermi level E_F, counted with signs, stays invariant as E_F is varied in the bulk gap. (b) Edge states in T-symmetric (Class AII) case. Although the signed intersection number at E_F always vanishes, the number of pairs of intersections (edge Kramers pairs) mod 2 is a $\mathbb{Z}_2$-invariant.

n	C^2	T^2	$KO^{d-n}(\star)$ or $K^{d-n}(\star)$			
			$d=0$	$d=1$	$d=2$	$d=3$
0		$+1$	$\mathbb{Z}$	0	0	0
1	$+1$	$+1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
2	$+1$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
3	$+1$	-1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
4		-1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
5	-1	-1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
6	-1		0	0	$\mathbb{Z}$	0
7	-1	$+1$	0	0	0	$\mathbb{Z}$
0	N/A		$\mathbb{Z}$	0	$\mathbb{Z}$	0
1	$\mathsf{S}^2=+1$		0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table 2: A "Periodic Table" of topological phases [26, 15, 42], corresponding through K-theoretic constructions to Bott's table of homotopy groups of the stable classical groups [7]. Only the "strong" phases, corresponding to pullback under $\mathbb{T}^d \to S^d$ are indicated. This idea is frequently used in the physics literature [40], although the precise meaning and construction of these phases is actually quite subtle.

Mathai–Thiang imported into the theory of topological phases to understand geometrically the *bulk-boundary correspondences* [32]. The remarkable effect of T-duality transformations is to convert the map ∂ into a simple geometrical restriction map, as expected from physical intuition. This duality is also linked to many deep mathematical ideas such as the Baum–Connes isomorphism [18, 19].

Although we have spoken only on topological phases of gapped Hamiltonians, there has been a huge surge of interest in the last two years on "semimetallic" phases, following a theoretical prediction of [46]. The latter are, roughly speaking, gapped phases "almost everywhere" in a parameter space (such as $\mathbb{T}^d$), see Fig. 4. Quite dramatically, a "topological semimetal" was discovered in 2015-16 [47, 48, 29], whose quasiparticle excitations realise (a version of) the elusive Weyl fermion from relativistic quantum theory. At first glance, $\mathbb{T}^d$ looks quite boring, and physicists have often "approximated" $\mathbb{T}^d \sim S^d$. The essential difference between the two is the existence of intermediate-degree cohomology, which physicists call *weak invariants*. It turns out that weak invariants are essential for topological semimetals, and that the (careful) topological classification of semimetallic phases involves interesting ideas from differential topology [30], with connections to Turaev's Euler structures and Seiberg–Witten invariants [31]. This is particularly interesting in Class AII, in which "$\mathbb{Z}_2$"-monopoles make an appearance [41].

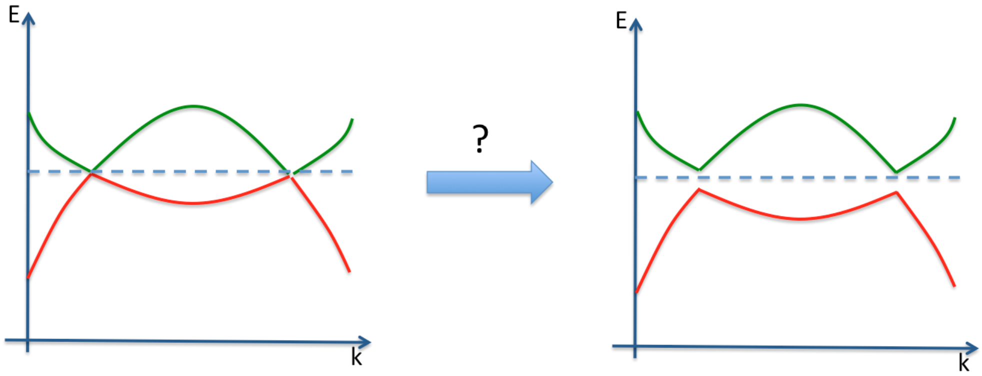

Figure 4: Semimetal band structure, with band crossings at some isolated points in $\mathbb{T}^d$. There are local and global topological obstructions to perturbing a semimetal Hamiltonian into an insulator one. These obstructions can be understood in the language of differential topology [30, 31].

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