K-theory and edge-following states of topological insulators [1]

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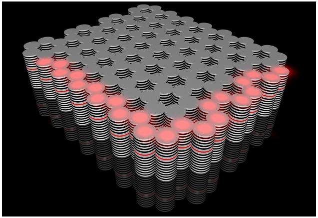
“Topological physics” (Nobel ’16) at the edge

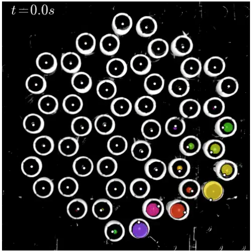
Last five years: physicists produced topological insulators in photonics, acoustics, cold atoms, metamaterials, Floquet systems, exiton-polaritons. . .

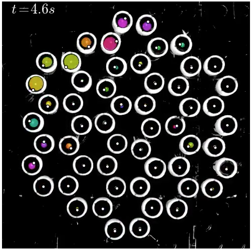
Most examples are (variants of) Chern insulator: a 2D material, described in boundaryless-limit by a $\mathbb{Z}^2$-invariant Hamiltonian $H = H^*$, possessing a “topological spectral gap”.

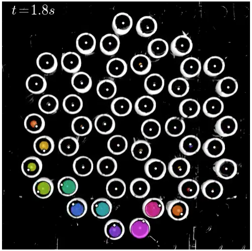


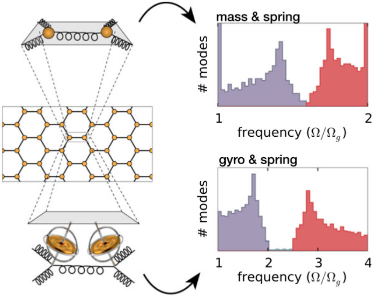
When the (material) boundary is introduced, the spectral gap of H is completely filled up with edge-following topological states!

Experiments[2]: edge-following states



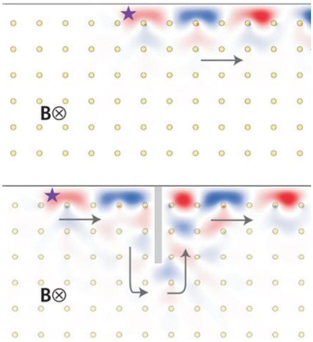


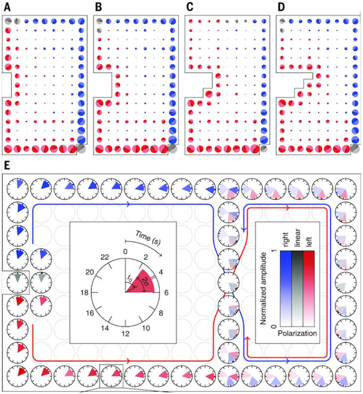


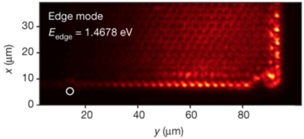


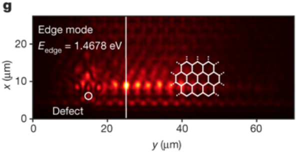
[2] Nash et al, PNAS (2015)

Experiments[3]: edge-following states







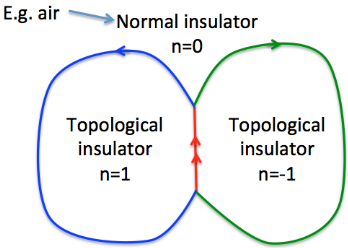


[3] Lu et at, Nature Photonics (2014); Süsstrunk, Huber, Science (2015); Klembt et at, Nature (2018)

Heuristic idea of topological insulators

Insulators can have topological invariants $\rightsquigarrow$ phases.

At a topological insulator's surface, one is at a wall-crossing $\Rightarrow$ insulating condition violated.



Physicist belief / conjecture: There is a "holographic" duality:
"# surface conducting states = topological # of insulator"

cf. **Atiyah–Singer index theorem:**
"# solutions to Dirac equation on M_{spin} = $\hat{A}$-genus of M_{spin}"

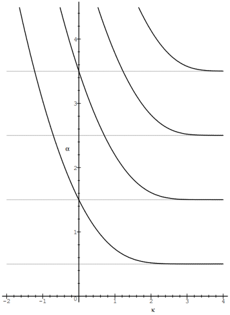
My research: Rigorous formulation and justification of above.

Spectrum of Landau Hamiltonian

Let $X = \mathbb{R}^2$ and $A = x\,dy$. The *Landau Hamiltonian* ['30]

$$H_{\mathrm{Lan},X} = \frac{1}{2}(d - iA)^*(d - iA), \qquad \mathrm{Spec}(H_{\mathrm{Lan},X}) = \frac{1}{2} + \mathbb{N},$$

describes electron on the plane with magnetic field $dA = dx \wedge dy$. Basic model for quantum Hall effect (Nobel '85).



Let $U = \mathbb{R}_+ \times \mathbb{R}$, and $H_{\mathrm{Lan},U}$ the Dirichlet version. Then [De Bièvre, Pulé '02]

$$\mathrm{Spec}(H_{\mathrm{Lan},U}) = [\frac{1}{2}, \infty).$$

I will give a modern view of this *gap-filling phenomenon*, and demonstrate its robustness to various deformations.

Warm-up: Index theory of Toeplitz operators

Under Fourier transform $L^2(S^1) \cong \ell^2(\mathbb{Z})$, the *Hardy subspace* $H^2(S^1) \subset L^2(S^1)$ corresponds to $\ell^2(\mathbb{N}) \subset \ell^2(\mathbb{Z})$.

Let $f \in C(S^1)$. Multiplication operator $M_f \in \mathcal{B}(L^2(S^1))$ compressed to Hardy space is *Toeplitz operator* $T_f \in \mathcal{B}(H^2(S^1))$.

Theorem [F. Noether '21]

T_f is Fredholm iff f is invertible, and its index is $-\mathrm{Wind}(f)$.

"Hole-filling theorem" [Coburn '66]:

$$\underbrace{\bigcap_{K \text{ compact}} \mathrm{Spec}(T_f + K)}_{\text{Weyl spectrum}} = \underbrace{\mathrm{Range}(f)}_{\text{curve in } \mathbb{C}} \cup \underbrace{\text{topological holes}}_{f \text{ winds nonzero times}}.$$

Continuum Toeplitz exact sequence [Ludewig+T:1912.xxxxx]

Let discrete Γ act nicely on Riemannian X, and consider some generic "half-space" $U \subset X$. **Exact sequence:**

$$0 \longrightarrow \underbrace{C^*_U(\partial U)}_{\text{boundary}} \longrightarrow \underbrace{Q^*_\Gamma(U)}_{\text{bulk}-\text{boundary}} \xrightarrow{\pi} \underbrace{C^*_\Gamma(X)}_{\text{bulk}} \longrightarrow 0$$

Bulk: Γ-invariant Roe C^*-algebra[4] in $\mathcal{B}(L^2(X))$.

Bulk-boundary: "Quasi-Γ-invariant" Roe algebra in $\mathcal{B}(L^2(U))$.

Boundary: Roe algebra in $\mathcal{B}(L^2(U))$, *localised at* ∂U.

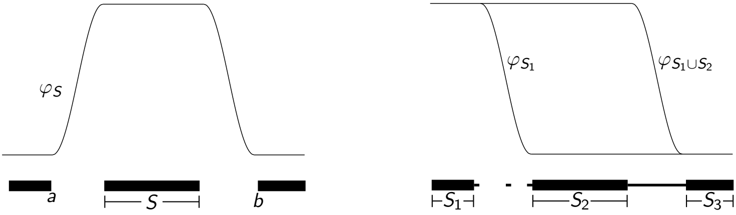
Theorem: Let H_X be Γ-invariant Laplace-type operator on X, and H_U be its compression to U with e.g. Dirichlet bc. For $\varphi \in C_0(\mathbb{R})$,

$$\varphi(H_X) \in C^*_\Gamma(X), \quad \varphi(H_U) \in Q^*_\Gamma(U), \quad \pi(\varphi(H_U)) = \varphi(H_X).$$

[4] Norm-closure of locally compact, finite propagation operators.

Spectral gap filling [T:1908.09559]

If S is a compact separated part of $\mathrm{Spec}(H_X)$, its spectral projection is $\varphi_S(H_X) \in C^*_\Gamma(X)$ for suitable function φ_S:



However, $\varphi_S(H_U)$ may *not* be a projection.

Generally have $\mathrm{Spec}(H_U) \supset \mathrm{Spec}(H_X)$.
$\Rightarrow$ spectral gaps of H_X *partially* filled by new spectra of H_U.

"Topological" question: Could a whole spectral gap close?

Topology, K-theory, and gap-filling

Can classify spectral projections $\varphi_S(H_X) \in C^*_\Gamma(X)$ abstractly classified by stable homotopy class in $K_0(C^*_\Gamma(X))$.

K_1 functor is stable homotopy classes of invertibles.

K-theory is a (co)homology theory with cyclic long exact sequences due to Bott periodicity. For each $U \subset X$, have

$$\begin{array}{ccccc}
K_0\big(C^*_U(\partial U)\big) & \longrightarrow & K_0\big(Q^*_\Gamma(U)\big) & \xrightarrow{\pi_*} & K_0\big(C^*_\Gamma(X)\big) \\
{\scriptstyle \mathrm{Ind}_U}\uparrow & & & & \downarrow{\scriptstyle \mathrm{Exp}_U} \\
K_1\big(C^*_\Gamma(X)\big) & \xleftarrow[\pi_*]{} & K_1\big(Q^*_\Gamma(U)\big) & \longleftarrow & K_1\big(C^*_U(\partial U)\big).
\end{array}$$

Theorem: If $\mathrm{Exp}_U[\varphi_S(H_X)] \neq 0$, then $\varphi_S(H_U)$ cannot be a projection. *Then H_U has spectrum in the whole gap above S.*

Topology, K-theory, and gap-filling by edge states

True significance of $\overbrace{[\varphi_S(H_X)] \neq 0}^{\text{top. insulator}}$**:** its K-theory exponential gives a *topological obstruction* for H_U to maintain spectral gap!

Compute $\quad 0 \overset{?}{\neq} \mathrm{Exp}_U : K_0(C^*_\Gamma(X)) \to K_1(C^*_U(\partial U)).$

Strategy: Combine physics intuition with Roe's cobordism-invariant *partitioned manifold index theorem*, reducing problem to standard $U =$half-space.

Remark: Persistence of edge states (the "extra spectra of H_U") under deformations of U, is precisely the most remarkable feature of topological insulators. *This had never been shown rigorously*!

“Cobordism” invariance of Exp_U

Take $U \subset X = \mathbb{R}^2$ example, with $\Gamma = \mathbb{Z}^2$.

- Partition $U = U_+ \cup U_-$ with $N = U_+ \cap U_-$ a hypersurface.

- For invertible $A \in C^*_U(\partial U)^+$, show that $\Pi_{U_+} A$ is invertible modulo compacts, thus Fredholm.

- Get index morphism $\theta_{U_+} = \mathrm{Ind}(\Pi_{U_+} \cdot\,) : K_1(C^*_U(\partial U)) \to \mathbb{Z}$.

- Show that U_- can be replaced by U'_- without affecting θ_{U_+}.

- Swap U_+ and U_- roles. So U may be replaced by $U' = \mathbb{R}_+ \times \mathbb{R}$.

$\Rightarrow K_0(C^*_\Gamma(X)) \xrightarrow{\mathrm{Exp}_U} K_1(C^*_U(\partial U)) \xrightarrow{\theta_{U_+}} \mathbb{Z}$ is nonzero, from explicit computation in $U' = \mathbb{R}_+ \times \mathbb{R}$ case.

Chern insulator example

For $\Gamma = \mathbb{Z}^2$ acting freely on $X = \mathbb{R}^2$, have

$$K_0(C^*_{\mathbb{Z}^2}(\mathbb{R}^2)) \cong K_0(C^*_r(\mathbb{Z}^2)) \cong K^0(\mathbb{T}^2) \cong \mathbb{Z}[1] \oplus \mathbb{Z}[P_{\rm Chern}].$$

First equality is Morita invariance, second is Fourier transform.
Third identification is $\;\; {\rm Chern} : K^0(\mathbb{T}^2) \cong H^0(\mathbb{T}^2) \oplus H^2(\mathbb{T}^2)$.

"Chern insulator with Chern number $k \neq 0$"
$\Leftrightarrow H_X = H_{{\rm Chern},X}$ has $[\varphi_S(H_{{\rm Chern},X})] = k \cdot [P_{\rm Chern}]$.

Proposition: $\quad K_1(C^*_U(\partial U)) \cong K_1(C^*(\partial U \sim \mathbb{R})) \cong \mathbb{Z}$.
We may represent generator by "edge-travelling operator" w (see later).

Edge-following topological states

Cobordism reduction & Künneth theorem & $\theta_{U_+}([w]) = 1$ shows

$$\theta_{U_+}(\mathrm{Exp}_U([\varphi_S(H_{\mathrm{Chern},X})]) = k \neq 0.$$

Conclusion: For a Chern insulator, the gap above S becomes filled by new "edge spectra" of $H_{\mathrm{Chern},U}$, regardless of shape of U.

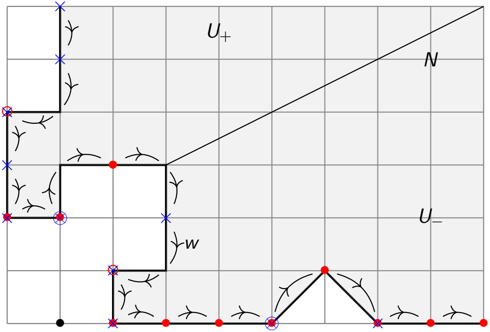
Example: $H_{\mathrm{Lan},X}$ is an example of a Chern insulator ($k = 1$ is folk theorem since mid 80s).

The spectral projection class $[\varphi_S(H_{\mathrm{Chern},X})]$ exponentiates to k-units of $[w] \in K_1(C^*_U(\partial U))$.

. . . Does this say something more about the gap-filling edge states?

Edge-following topological states

θ_{U_+} is actually a cyclic 1-cocycle pairing with $K_1(C^*_U(\partial U))$, measuring "edge current flowing across (any) partition N".



Lemma: $\theta_{U_+}([w]) \equiv \mathrm{Ind}(\Pi_{U_+} w) = \mathrm{Ind}(\mathrm{Shift}) = 1.$

Philosophical points

Topologists work on *abstract homotopy classification* problem, not so much on analytic consequences.

Analysts want to solve spectral problem for H_U, but only possible for *special* U on *case-by-case* basis.

Physicists know empirically that *edge currents are robust to deformations*. Don't care about precise eigenfunctions for specific H_U, only *qualitative quantised features* common to *all* H_U.

K-theory, NC and coarse (*new!*) geometry and index theory, provides efficient mathematical setting to study the most important features of topological matter.

"Topology justifies extrapolation from special case".

Some other ongoing projects

I am interested in *good* mathematical dualities in/from physics of topological matter and string theory.

With V. Mathai (Adelaide), used T-duality to study non- Euclidean bulk-boundary correspondences [CMP '16, AHP '17, LMP '18, ATMP '19]. Application/interpretation/analogues of *Baum–Connes conjecture*.

With K. Gomi (Tokyo Tech), I discovered crystallographic T-duality [JGP '19] of flat orbifolds. Now investigating algebraic topology consequences.

Mathematics of *topological semimetals* [CMP '17]. Collaborated with physicists [PRL '19] to explain topological origin of MSSWs (used in discovery of GMR, Nobel '07.)