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Upscaling of Stochastic Micro Model for Suspension Transport in Porous Media

P. Bedrikovetsky

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Abstract Micro scale population balance equations of suspension transport in porous media with several particle capture mechanisms are derived, taking into account the particle capture by accessible pores, that were cut off the flux due to pore plugging. The main purpose of the article is to prove that the micro scale equations allow for exact upscaling (averaging) in case of filtration of mono dispersed suspensions. The averaged upper scale equations generalise the classical deep bed filtration model and its latter modifications.

Keywords Colloid · Suspension · Porous media · Transport · Averaging · Upscaling · Size distribution · Retention · Size exclusion · Straining · Accessibility · Stochastic model

Nomenclature

C	Suspended particle concentration distribution by sizes, L^{-4}
C_t	Concentration distribution for suspended particles trapped in cut-off accessible pores, L^{-4}
c	Total suspended particle concentration, L^{-3}
f	Fractional flow function, dimensionless
H	Pore concentration distribution, L^{-3} or L^{-4}
h	Total pore concentration (density), L^{-2} or L^{-3}
j	Jamming ratio, dimensionless
j_0	Maximum value of jamming ratio for non-zero accessibility
$k(\sigma)$	Absolute permeability, L^2
$k_a(\sigma)$	Permeability of accessible part of the porous medium, L^2
$k_{na}(\sigma)$	Permeability of inaccessible porous medium, L^2
$k_c(\sigma)$	The total of pore accessible conductivities weighted with capture probability, L^2
k_1	Conductivity of a single pore, L^4
l	Characteristic microscopic length, L
L	Length of the core, L

P. Bedrikovetsky (✉)
Faculty of Engineering and Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia
e-mail: pavel.russia@gmail.com

P	Pressure, M/LT
$p(r_s/r_p)$	Overall capture probability, dimensionless
$p_a(r_s/r_p)$	Attachment capture probability, dimensionless
$q(r_p)$	Total flow rate in a single pore, $L^{-3}T^{-1}$
$q_a(r_p, r_s)$	Flow rate through accessible cross section of a single pore, $L^{-3}T^{-1}$
r	Size of a particle or of a pore, L
s_1	Pore cross-section, L^2
t	Time, T
T	Fast dimensionless time
U	Total velocity of the flux, LT^{-1}
U_a	Velocity of the accessible flux, LT^{-1}
U_{na}	Velocity of the inaccessible flux, LT^{-1}
v	Concentration front velocity in 1d filtration flow, LT^{-1}
x	Coordinate, L
X	Dimensionless coordinate
y	Independent variable in system of differential-integral equations

Greek letters

α	Critical porosity fraction, dimensionless
ε	Small parameter that equals to the ratio between the injected concentration and the critical porosity fraction
$\lambda'(\sigma)$	Filtration coefficient, L^{-1}
$\lambda(\sigma)$	Dimensionless filtration coefficient
μ	Dynamic viscosity, $ML^{-1}T^{-1}$
η	Collision efficiency
$\nu(r_s/r_p)$	Single pore flux reduction factor, dimensionless
σ	Volumetric concentration of captured particles, L^{-3}
Σ	Size distribution of the captured particle concentration, L^{-4}
$\underline{\Sigma}$	Distribution of the captured particle concentration over the pore and particle radii, L^{-5}
τ	Slow dimensionless time
$\phi(x,t)$	Porosity, dimensionless
$\phi_a(r_s, x, t)$	Accessible porosity for a particle of the size r_s, dimensionless
$\phi_{na}(r_s, x, t)$	Inaccessible porosity for a particle with size r_s, dimensionless
$\chi(r_s, r_p)$	Accessible fraction of a single pore cross-section, dimensionless

Subscripts

a	Accessible
na	Inaccessible
s	Suspended (solid) particle
p	Pore
v	Volumetric
0	Initial condition
1	Single pore (cross section, conductivity)
*	Lower percolation threshold corresponding to connectivity of accessible pores

Superscripts

0	Boundary condition
*	Upper percolation threshold corresponding to connectivity of inaccessible pores

1 Introduction

Transport of particulate suspensions and colloids in porous media is accompanied by particle capture and consequent permeability decline. It occurs in oil reservoirs during sea- or produced water injection, produced water disposal in aquifers with subterranean water contamination, drilling fluid invasion causing formation damage, filtration of completion fluid, fines migration during production of heavy oils in low consolidated reservoirs resulting in productivity decline and during sand production control by gravel packs and sand screens (Khilar and Fogler 1998; Civan 2006). The reliable prediction of these processes helps in design of injected water treatment, in particle sizing of drilling and completion fluids, in sizing the gravel pack and designing the sand screens.

Flow of suspensions and colloids in natural porous formations with particle capture by matrix occurs also in environmental engineering: viruses, protozoa and bacteria are transported by subterranean water in aquifers and contaminate water resources (Elimelech et al. 1995; Logan 2001). Filtering of water and industrial liquids, membrane technologies and size exclusion chromatography are traditional areas of chemical engineering (Mays and Hunt 2005).

Prediction of particle propagation and retention by mathematical modelling is an essential stage during planning and design of above-mentioned industrial and environmental processes. The classical suspension-colloid filtration theory is the most commonly used approach for predicting particle deposition behaviour and consequent rock alterations.

The classical deep bed filtration model consists of two equations for particle balance and for capture kinetics (Herzig et al. 1970; Payatakes et al. 1974). The model contains the filtration coefficient that is a function of retained particle concentration. Various forms of filtration function for different particle capture mechanisms have been presented in the literature (Herzig et al. 1970; Kuhnen et al. 2000). Several analytical solutions for different forms of filtration coefficient are also presented in the literature (Herzig et al. 1970).

Usually the outlet suspension concentration is measured during laboratory tests on suspension flow in cores. The particle breakthrough curve allows calculation of filtration function (Soo et al. 1986; Pang and Sharma 1994; Bolster et al. 1998; Foppen et al. 2005; Foppen and Schijven 2006). The solution of inverse problem exists, is unique and stable with respect to small perturbations of measured data (Alvarez et al. 2006). The solution of the direct problem using the filtration coefficient, as determined from breakthrough curve by solution of the inverse problem, allows predicting the retention profile.

Recently several articles have presented measured retention profile together with breakthrough curve during corefloods by suspensions (Bradford et al. 2002, 2003, 2004; Bradford and Bettahar 2006; Tufenkji and Elimelech 2004; Al-Abduwani et al. 2004; Al-Abduwani 2005). It gives an opportunity to verify the classical deep bed filtration model by determination of filtration coefficient from the breakthrough curve, prediction of the retention profile and its comparison with that measured. The authors report that it is impossible to match simultaneously the breakthrough curve and the retention profile by tuning the filtration function.

The explanation of the deviation phenomenon was presented for the case of electric particle attachment (Tufenkji and Elimelech 2005). The deviation was not explained for the case of particle straining.

Disagreement between the modelled and measured deep bed filtration data makes it necessary to analyse the fundamentals of the classical model for suspension transport in porous media, including its derivation from micro scale equations and formulation of possible generalisations.

Another motivation for generalisation of the classical deep bed filtration model is different breakthrough times as obtained from laboratory and mathematical modelling. Laboratory tests on deep bed filtration exhibit the value for breakthrough time that varies from 0.4–0.8 to 10–100 pore volumes injected (Harvey and Garabedian (1991); Roque et al. 1995; Kretzschmar et al. 1997; Chauveteau et al. 1998; Camesano et al. 1999; Veerapen et al. 2001; Massei et al. 2002) while the breakthrough time as obtained by the classical model is equal to unity. Introduction of porosity decrease due to particle retention into the classical model allows predicting only less-than-unity breakthrough times (Corapcioglu and Choi 1996).

The retention rate, for the size exclusion particle capture mechanism, depends on particle and pore sizes that widely vary in natural rocks and suspensions. The particle capture rate, caused by the electric and molecular forces that retain the particles, is also particle-size dependent. The classical model operates with mean concentrations and does not account for micro characteristics like size, surface energy and electric charge distributions.

Several studies have proposed to consider probabilistic distribution of filtration coefficient to account for local variations in the particle collection efficiency (Elimelech et al. 1995; Khilar and Fogler 1998). Distribution function was matched in order to adjust the experimental data (Baygents et al. 1998; Simoni et al. 1998; Bolster et al. 1998, 1999). In particular, the bimodal distribution can fairly well adjust several data on simultaneous breakthrough curve and deposition profile for the cases where the data cannot be adjusted by the classical model (Tufenkji et al. 2003; Tufenkji and Elimelech 2005).

Sharma and Yortsos (1987a,b,c) derived basic population balance equations for transport of particulate suspensions in porous media. The model accounts for particle and pore size distribution variation due to different particle capture mechanisms. It is assumed that an overall pore space is accessible for particles and the particle population moves with the average flow velocity of the carrier water. In the case of a porous medium with a uniform pore size distribution, this assumption results in independent deep bed filtration of different particle size populations. Nevertheless, during deep bed filtration with size exclusion mechanism, particles smaller than the pore radii should pass the rock without being captured and particles larger than the pore radii should not enter the rock.

Introduction of accessibility and flux reduction factors into population balance equations results in capture-free transport of smaller particles in the porous medium with single- sized pores; large particles do not enter the medium (Santos and Bedrikovetsky 2006).

In the current article, the suspension storage in cut-off part of accessible pore volume and varying porosity, are introduced into the micro scale population balance model of suspension transport in porous media. The exact averaging is performed for the case of mono dispersed suspension. The resulting equations generalise the classical deep bed filtration model and its latter modifications. So, the large-scale model, for suspension transport in rocks, was obtained from micro scale population balance model only in the case of single size particles.

The structure of the article is as follows. The main model parameters for porous media and suspension (colloid) are defined in Sect. 2. Section 3 contains derivation of governing equations. The particular case of mono dispersed suspension is discussed in Sect. 4 where it is proven that the vacant pore concentration distribution is a function of the total vacant pore concentration. The averaged system of suspension flow is derived in Sect. 5. Section 6 presents simplifications of the model for cases of low retention and long time. The structure of the solution for suspension injection into the clean rock is given in Sect. 7. Analysis of the cases, where the particle breakthrough happens before and after the injection of one pore volume, is presented in Sect. 8.

2 Definition of Model Parameters

In this section the parameters and variables for population balance model of suspension flow in porous media are briefly described. With minor modifications, the description follows Sharma and Yortsos (1987a,b,c) and Shapiro et al. (2007).

The statistical theory of suspension transport in porous media is based upon the definition of size distribution functions for pores and particles (Sharma and Yortsos 1987a,b,c).

First we describe the porous space geometry and define rock transport and volumetric properties.

The surface pore concentration distribution $H(r_p, x, t)$ is defined via number of pores $H\mathrm{d}r_p$ with sizes varying from r_p to $r_p + \mathrm{d}r_p$, and crossing the unitary cross section. This definition is convenient since the total pore concentration is simply obtained as an integral of the pore concentration distribution

$$h\,(x,t) = \int_0^\infty H\left(r_p, x, t\right)\mathrm{d}r_p \tag{1}$$

Further in the text, H and h are called the pore concentration distribution and the total pore concentration, respectively.

The surface pore concentration distribution H is used to determine porosities and fluxes. The expression for particle retention rate uses the volumetric pore concentration distribution $H^v(p_r, x, t)$ defined via the number of pores $H^v\mathrm{d}r_p$ with sizes varying from r_p to $r_p + \mathrm{d}r_p$ per rock volume unit. The surface and volumetric concentration distributions are assumed to be related by equality

$$H^v = \frac{H}{l} \tag{2}$$

where l is a reference length parameter of porous media that has an order of magnitude of a medium pore length.

Consider the porous media model consisting of a parallel bunch of capillaries alternated by mixing chambers. Different scenarios of particle flow and capture in the model porous media are shown in Figs. 1–4. Each particle flows in capillary with a size that exceeds the particle size (Figs. 2, 4). Different size particles are transported to chambers via the accessible capillaries, and complete mixing of different size particles occurs in the chambers (Fig. 2).

Two processes of particle transport and capture, occurring simultaneously in natural rocks, are separated in the proposed model. The size exclusion particle capture occurs at the thin pore inlet, where large particles arrive (Fig. 2). So, an inlet cross section of each parallel capillary section acts as a sieve, i.e. large particles do not enter thin pores and are captured at chamber outlets. The attachment particle capture occurs inside capillary (see Fig. 3).

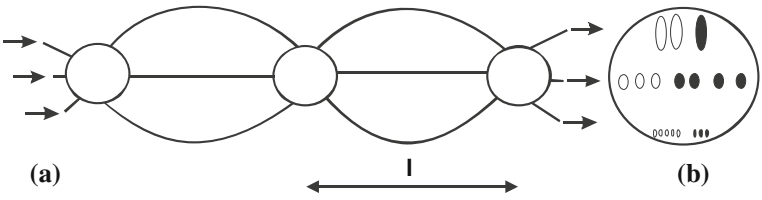

Fig. 1 Geometric model of porous media—bundle of parallel capillaries alternated by mixing chambers: (**a**) "vertical" cross section of the rock in plane (x, z); (**b**) cross section of the geometric rock model in plane (y, z)

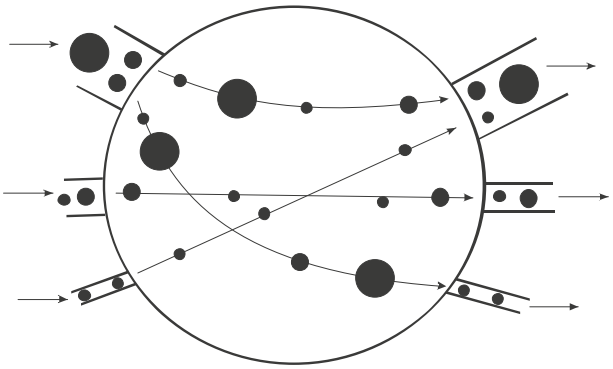

Fig. 2 Schema of complete suspension mixing in a single chamber

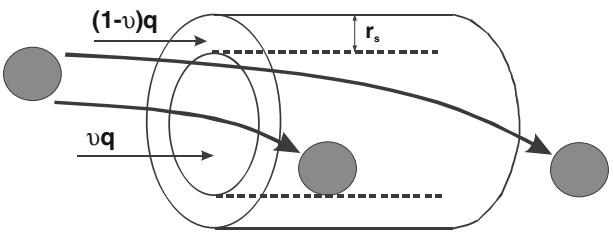

Fig. 3 Particle retention at the pore entrance due to size exclusion and in situ the pore due to attachment

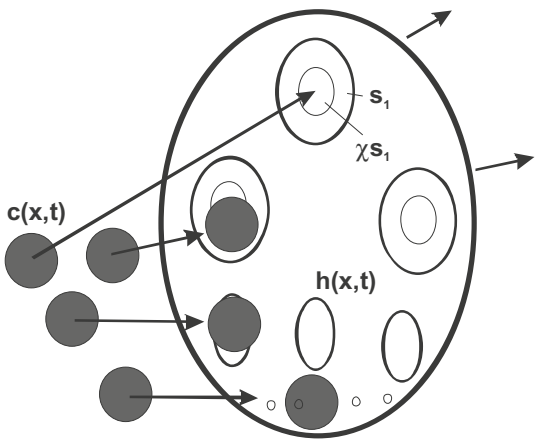

Fig. 4 Schematic for transversal-to-flow cross-section of the geometrical model of porous media

It is assumed that the chamber volume is negligibly small if compared to the capillary (pore) volume.

The phenomenological length parameter l for this model, (2), is the distance between chambers (Fig. 1).

The porosity is defined as a surface of the total pore cross-section via unitary rock cross section

$$\phi = \int_0^\infty s_1\left(r_p\right) H\left(r_p, x, t\right) \mathrm{d}r_p \tag{3}$$

Here, $s_1(r_p)$ is a cross section of a single pore. For a parallel bunch of capillaries with mixing chambers, s_1 is equal to πr_p^2.

The porous space is divided into two parts: that accessible for particles with size r_s and that inaccessible for r_s-particles

$$\phi = \phi_a + \phi_{na} \tag{4}$$

Introduce the fraction χ of a single pore cross-section accessible for r_s particles (Fig. 4). Further in the text, χ is called the single pore accessibility factor. It depends on both sizes r_p and r_s. Since χ is dimensionless, it depends on the ratio r_s/r_p that is called the jamming ratio j

$$j = r_s/r_p \tag{5}$$

The accessible cross section of a single pore is equal to χs_1.

Particles with size r_s are transported via accessible pores and are captured by size exclusion in inaccessible pores. Three upper pores in Fig. 4 are larger than particles, so the cross sections of both accessible and inaccessible volumes are shown for these pores. Seven lower pores are smaller than particles and, therefore, are inaccessible. Depending on maximum pore size and shapes of pores and particles, complete and incomplete plugging of pore by particle may occur, both cases are shown in the figure.

Let j_0 be a minimum value of j for which all pores are inaccessible, i.e. $\chi(j) = 0$ for all $j > j_0$. Further j_0 is called the critical jamming ratio.

For the case of parallel bunch of round capillaries with mixing chambers and of spherical particles, the single pore accessibility factor χ is

$$\chi\left(j\right) = \begin{cases} 0, & j > 1 \\ (1-j)^2, & j < 1 \end{cases} \tag{6}$$

Overall pore cross-section is accessible for fine particles with $r_s << r_p$, so $\chi(0) = 1$, like in (6). The particle does not enter the smaller pore, so $\chi(j) = 0$ for jamming ratios that exceed the critical value j_0, i.e. for $j > j_0 = 1$, also like in (6).

If the pores are elliptic and the particles are spherical, the capture happens if the particle size exceeds the lower axes of the ellipsis. The average pore size exceeds the lower axes of the ellipsis, so the critical jamming ratio $j_0 < 1$.

For the case of cylindrical pores and elliptic particles, the capture happens if the particle low axes size exceeds the pore size. Since the particle low axes size is lower than the particle average size, the critical jamming ratio $j_0 > 1$.

The dependency of single pore accessibility factor χ versus jamming ratio for an ensemble of non-ideal pores and particles is determined by averaging over their probability shape distributions.

The single pore accessibility factor $\chi = \chi(j)$ is a given function for fixed pore space geometry and particle shapes. In a general case of any arbitrary pore and particle geometry, the properties of the $\chi(j)$-function are as follows:

– $\chi = 1$ for fine particles with $r_s << r_p$, i.e. for $j = 0$;

– χ decreases monotonically as j increases;
– for large particles, $j > j_0$, single pore accessibility factor χ becomes zero.

The critical jamming ratio j_0 can either exceed or be less than unity.

The accessible and inaccessible porosities for the model of parallel capillary with mixing chambers (Fig. 1) are defined as

$$\phi_a = \int_0^\infty s_1\left(r_p\right)\chi\left(r_s/r_p\right)H\left(r_p,x,t\right)\mathrm{d}r_p \tag{7}$$

$$\phi_{na} = \int_0^\infty s_1\left(r_p\right)(1-\chi)H\left(r_p,x,t\right)\mathrm{d}r_p \tag{8}$$

Since the chamber volume is assumed to be negligible, the formulae (7), (8) for accessible and inaccessible porosities are the same as those for the model of the bundle of parallel capillary.

Introduction of the single pore accessibility factor χ allows for simultaneous capture of two effects: of the inaccessible pore volume in accessible pores with $j < j_0$ (particle centre cannot approach the pore wall closer than the particle radius) and of thin pores with $j > j_0$ where larger particles cannot enter. These two effects have been reported earlier for polymer flow in porous media (Dawson and Lantz 1972; Bartelds et al. 1997).

So, inaccessible porous space (8), can be divided in two parts: the volume of inaccessible pores where $\chi = 0$, and the inaccessible volume near to walls of accessible pores

$$\phi_{na} = \int_0^{r_s/j_0} s_1\left(r_p\right)H\left(r_p,x,t\right)\mathrm{d}r_p + \int_{r_s/j_0}^\infty s_1\left(r_p\right)(1-\chi)H\left(r_p,x,t\right)\mathrm{d}r_p,$$

respectively.

For the model of parallel capillaries with mixture chambers, (6), the expressions for accessible and inaccessible porosities become:

$$\phi_a = \int_{r_s}^\infty H\left(r_p,x,t\right)\pi r_p^2\left(1-\frac{r_s}{r_p}\right)^2\mathrm{d}r_p \tag{9}$$

$$\phi_{na} = \int_0^{r_s} H\left(r_p,x,t\right)s_1\left(r_p\right)\mathrm{d}r_p + \int_{r_s}^\infty H\left(r_p,x,t\right)\pi r_p^2\left[1-\left(1-\frac{r_s}{r_p}\right)^2\right]\mathrm{d}r_p \tag{10}$$

Opposite to routine dependence of function on variables, which is usually denoted by parentheses, the functional dependence (a functional operator as applied to a function) is denoted by square brackets. Porosity depends on pore concentration distribution function, while the accessible and inaccessible porosities depend also on particle size

$$\phi\left[H\right],\phi_a\left[H,r_s\right],\phi_{na}\left[H,r_s\right] \tag{11}$$

It is assumed that viscous pressure losses along the capillary are much higher than those in "small" chambers, so the rock permeability for porous space models with and without chambers are the same.

The Poiseuille-type flow is assumed for each single pore as:

$$q\left(r_p\right) = -\frac{k_1\left(r_p\right)}{\mu}\frac{\partial P}{\partial x} \tag{12}$$

For cylindrical pores

$$k_1\left(r_p\right) = \frac{\pi r_p^4}{8} \tag{13}$$

The overall suspension flux is calculated by integrating (12)

$$U = \int_0^\infty q\left(r_p\right) H\left(r_p, x, t\right) \mathrm{d}r_p \tag{14}$$

Comparison between Eqs. (12, 14) and the Darcy law for the overall flux

$$U = -\frac{k}{\mu}\frac{\partial p}{\partial x} \tag{15}$$

allows obtaining the expression for absolute permeability

$$k\left[H\right] = \int_0^\infty k_1\left(r_p\right) H\left(r_p, x, t\right) \mathrm{d}r_p \tag{16}$$

It is assumed that pressure losses on chambers are negligibly small if compared with viscous pressure losses along the capillary. Therefore, formula (16) for permeability is valid for the bundle of parallel capillary with and without mixing chambers.

From (12, 14) and (15) follows the expression for flux through given size pores, via the overall suspension flux U

$$q\left(r_p\right) = \frac{k_1\left(r_p\right)}{k} U \tag{17}$$

The overall flux is divided into that via accessible pore cross-sections and that via inaccessible surface

$$U = U_a + U_{na} \tag{18}$$

Introduce the flux $q_c(r_s, r_p)$ in a single pore that carries particles with size r_s, i.e. water flux via accessible fraction of pore cross-section. The fraction ν

$$q_a\left(r_p, r_s\right) = \nu q\left(r_p\right) \tag{19}$$

is called the single pore flux reduction factor. It depends on both sizes r_p and r_s. Since ν is dimensionless, it depends on the jamming ratio r_s/r_p.

Figure 3 shows the structure of the overall flux in accessible pore. The particle, transported by the accessible flux νq, passes the pore throat (exit of the mixing chamber) and enters the capillary. The inaccessible flux $(1-\nu)q$ does not transport particles.

Consider Poiseuille velocity profile in each pore for parallel bunch of round capillaries with mixing chambers and spherical particles:

$$v\left(r\right) = -\frac{r_p^2 - r^2}{4\mu}\frac{\partial p}{\partial x} \tag{20}$$

Calculation of flux via an overall pore cross-section and via its accessible fraction

$$q = \int_0^{r_p} 2\pi r v\,(r)\,\mathrm{d}r, \quad q_a = \int_0^{r_p - r_c} 2\pi r v\,(r)\,\mathrm{d}r \tag{21}$$

results in single pore flux reduction factor ν

$$\nu\,(j) = \begin{cases} 0, & j > 1 \\ (1-j)^2\left(1+2j-j^2\right), & j < 1 \end{cases}; j = r_s/r_p \tag{22}$$

The single pore flux reduction factor ν is equal to unity for particles with negligibly small size—$\nu(0) = 1$; it monotonically decreases as jamming ratio increases and becomes zero for large particles—$\nu(j) = 0$, $j > j_0$. The critical jamming ratio j_0 is the same for the flux reduction factor ν and for the accessibility factor χ—both factors are zero for $j > j_0$, i.e. χ and ν are vanish where large particle cannot enter the fine pore.

By definitions, the flux reduction factor in single pore is analogous to fractional flow of non-wetting phase with two-phase displacement, and the accessibility factor is analogous to non-wetting phase saturation (Lake 1989; Bedrikovetsky 1993).

In a large pore with $j < j_0$, the particle centres move via the central parts of pore being carried by ν-th flux fraction, while particle-free water moves near to pore walls with the flux which equals the $(1-\nu)$-th fraction of the overall flux in the pore.

The accessible flux is a flux via accessible cross sections of all pores:

$$U_a = \int_0^{\infty} \nu\left(r_s/r_p\right) q\left(r_p\right) H\left(r_p, h\right) \mathrm{d}r_p \tag{23}$$

Comparing (12) and (23) with the Darcy's law for an accessible flux

$$U_a = -\frac{k_a}{\mu}\frac{\partial P}{\partial x} \tag{24}$$

allows obtaining the expression for accessible permeability

$$k_a\,[r_s, H] = \int_0^{\infty} \nu\left(r_s/r_p\right) k_1\left(r_p\right) H\left(r_p, h\right) \mathrm{d}r_p \tag{25}$$

The expressions for inaccessible flux and permeability are obtained in the same way as those (23) and (25)

$$\begin{aligned} U_{na} &= \int_0^{\infty} (1-\nu)\, q\left(r_p\right) H\left(r_p, h\right) \mathrm{d}r_p \\ k_{na}\,[r_s, H] &= \int_0^{\infty} (1-\nu)\, k_1\left(r_p\right) H\left(r_p, h\right) \mathrm{d}r_p \end{aligned} \tag{26}$$

The inaccessible flux of particle-free water consists of those via fine inaccessible pores and via dead volumes of accessible pores:

$$U_{na} = \int_0^{r_s/j_0} q\left(r_p\right) H\left(r_p, h\right) \mathrm{d}r_p + \int_{r_s/j_0}^{\infty} (1-\nu)\, q\left(r_p\right) H\left(r_p, h\right) \mathrm{d}r_p$$

Division of the overall suspension flux into its accessible and inaccessible parts permits introduction of fractional flow via accessible pore space

$$U_a = fU, \quad U_{na} = (1 - f)\, U \tag{27}$$

The expression for fractional flow function follows from (25) and (27):

$$f = \frac{\int_o^\infty \nu\left(r_s/r_p\right) k_1\left(r_p\right) H\left(r_p, x, t\right) \mathrm{d}r_p}{\int_0^\infty k_1\left(r_p\right) H\left(r_p, x, t\right) \mathrm{d}r_p} = f\left[H, r_s\right] \tag{28}$$

The fractional flow (28) is analogous to that for non-wetting phase with two-phase displacement, and the accessible porosity is analogous to non-wetting phase saturation. The fractional flow formulae for bundle of parallel capillary with and without mixing chambers are the same.

Now we describe the populations of particles in suspensions (colloids) in porous media.

The particles are present in porous media as those suspended and retained. The total concentrations for suspended and retained particles are denoted as $c(x, t)$ and $\sigma(x, t)$, respectively. The definition of size distribution for suspended particles $C(r_s, x, t)$ is such that $C(r_s, x, t)\mathrm{d}r_s$ is the number of particles per unitary volume, with particle radii between r_s and $r_s + \mathrm{d}r_s$, located in point x at time t. The retained particle size distribution $\Sigma(r_s, x, t)$ is also defined via the number of retained particles $\Sigma(r_s, x, t)\mathrm{d}r_s$ with radii varying between r_s and $r_s + \mathrm{d}r_s$.

The total concentration is a sum of concentrations distributed by particle size

$$c\left(x, t\right) = \int_0^\infty C\left(r_s, x, t\right) \mathrm{d}r_s \tag{29}$$

$$\sigma\left(x, t\right) = \int_0^\infty \Sigma\left(r_s, x, t\right) \mathrm{d}r_s \tag{30}$$

The retained concentration is also distributed by the size of pores where the particle is captured

$$\Sigma\left(r_s, x, t\right) = \int_0^\infty \underline{\Sigma}\left(r_p, r_s, x, t\right) \mathrm{d}r_p \tag{31}$$

Retained particles plug the pores and cause the decrease of the accessible pore volume.

For each moment, the suspended particles with size r_s are present in the current accessible pore volume and also in the fraction of initial accessible pore volume cut off the flux by plugging. The cut off porosity is equal to the difference between the initial and the current accessible porosities

$$\phi_a\left[H\left(p_p, x, 0\right), r_s\right] - \phi_a\left[H\left(p_p, x, t\right), r_s\right] \tag{32}$$

The suspended concentration distribution in cut off accessible volume is denoted as $C_t(r_s, x, t)$.

Consider an individual interaction between pore and particle that already entered the pore. Following Shapiro et al. (2007), let us introduce the probability $p(r_s/r_p)$ of particle to

be captured by the pore, and the attachment probability of particle capture inside the pore $p_a(r_s/r_p)$. The attachment probability is proportional to so called collision efficiency η from the classical filtration theory (Elimelech et al. 1995).

Introduction of the attachment probability p_a allows describing both size exclusion and attachment mechanisms of particle retention (Fig. 3). The capture probability depends not only on the jamming ratio $j = r_s/r_p$ but also on the ionic strength of the aqueous solution, its pH, mineralogical composition of grain surface, temperature, etc.

Figure 3 shows that straining occurs at the pore inlet while the attachment takes place inside the pore body (Siqueira et al. 2003). The particle can either pass or be captured by size exclusion at the entrance to the pore body. Being passed, it may be attached in the pore or pass via the pore exit entering the next mixing chamber.

Let us calculate the overall particle capture probability p by either straining or attachment. The probability for particle to be strained at the pore entrance is $1 - \nu$, see (19). Here, we assume that if the particle just "touches" the wall, it would be attached by low-range attraction DLVO forces (Derjagin and Landau 1941; Landau and Lifshitz 1980).

The probability that particle enters the accessible pore volume is equal ν, see (19). The attachment probability p_a is a conditional probability for particle that already entered the pore.

The overall capture probability is given by the Bayes' theorem

$$p = 1 - \nu + \nu p_a \tag{33}$$

In inaccessible pores, where the flux reduction factor is equal zero, $p = 1$, i.e. all particles that arrive to smaller pores are captured.

Figure 5 exhibits several dependencies $p = p(r_s/r_p)$.

If the pores are elliptic, the spherical particle passes if its radius is lower than the low axes of the ellipsis. Since the pore size r_p is an average determined from the pore cross-section area and is larger than its low axes size, the critical jamming ratio $j_0 < 1$.

For the case of cylindrical pores and elliptic particles, the capture happens if the particle low axes size exceeds the pore size. The particle size r_s is an average calculated from the particle volume and exceeds the low ellipsis size. Therefore, the critical jamming ratio $j_0 > 1$.

For irregular shapes of pores or particles, the critical jamming ratio j_0 can either be less or more than unity (continuous lines in Fig. 5). Very small particles do not retain by straining, so probability p is equal p_a for small j-values in case of size exclusion.

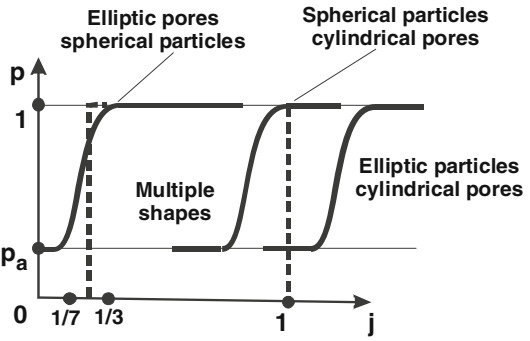

Fig. 5 Dependence of the particle capture probability p on the jamming ratio $j = r_s/r_p$ for different particle and pore shapes

If there is a variation of irregular pore and particle shapes, capture probability is an S-shaped function which is equal p_a for small j-values and to unity for large j.

The continuous curve in Fig. 5 corresponds to "one-third one-seventh" rule of the filtration theory (van Oort et al. 1993). The "rule of thumb" criterion in filtration theory says that particles smaller than one-seventh of the pore size flow in the filter without being captured by straining, particles larger than one-third of the pore size are captured at the filter inlet and do not enter the filter, and the intermediate size particles perform deep bed filtration with non-zero probability to be captured. The bulk of the probability change, from p_a to unity, occurs between jamming ratios 1/7 and 1/3.

The continuous curve corresponding to the "one-third one-seventh" rule can be approximated by step-function, where the shock occurs at some intermediate value of jamming ratio j_0 (the case $j_0 = 1$ corresponding to spherical particles and cylindrical pores is shown in Fig. 5). In this case, j_0 becomes a phenomenological parameter, which can be used for the model tuning.

3 System of Governing Equations

In this section, the population balance equations accounting for pore space accessibility and particle flux reduction, if compared with carrier water flux, are derived. The effect of suspension storage in plugged accessible pores is introduced in the model. It results in loss of the conservation law form for the particle balance equation.

3.1 Conservation Law for Particles

Let us derive mass balance equation for particles accounting for the effect of dead end pore plugging and the particles which remain in cut-off pore space and do not participate in the flow after plugging (Fig. 6).

The number of particles in a unitary volume, is equal to the total of suspended particles in accessible pore volume $\phi_a C$, of suspended particles in cut off accessible volume $(\phi_{a0} - \phi_a)C_t$ and of retained particles $\Sigma(r_s, x, t)$. The r_s-particles are transported by accessible flux.

The diffusive (dispersive) fluxes that are proportional to concentration gradients are ignored in large scale approximation, since they are negligibly smaller than the advective particle fluxes. The resulting model has discontinuous solutions with shock concentration fronts. The shock fronts can be "smoothened" by introduction of diffusion (dispersion) near to front and by matching the asymptotic expansions (Barenblatt et al. 1991; Bedrikovetsky 1993).

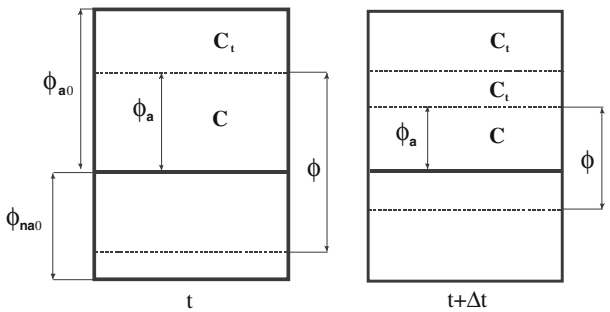

Fig. 6 Schema of particle storage in accessible pores, cut off from the suspension stream by plugged particles

Accounting for local concentration gradients is a separate complex problem of suspension flow in rocks (see Kechagia et al. 2002; Altoe et al. 2006) and is outside the scope of the current article.

The mass balance equation can be written for size distribution functions of suspended and retained particles as (Sharma and Yortsos 1987a,b,c)

$$\frac{\partial}{\partial t}\left[\phi_a C\left(r_s,x,t\right)+\left(\phi_{a0}-\phi_a\right)C_t\left(r_s,x,t\right)\right]+\frac{\partial}{\partial x}\left[C\left(r_s,x,t\right)U_a\right]=-\frac{\partial\Sigma\left(r_s,x,t\right)}{\partial t} \quad (34)$$

Now let us derive the kinetics of particle exchange between the accessible and cut-off accessible volumes due to pore plugging. The dead-end pore with entrance plugged at some moment is an example of cut-off volume—the particles "left" in the pore must be accounted for in mass balance equation. Discuss the accessible and cut-off accessible pore volumes for moments t and $t+\Delta t$ (Fig. 6). During the particle retention process, the retained concentration increases, cut-off pore volume increases; the total porosity decreases along with its accessible fraction. The accessible pore volume that was cut off during the period Δt is

$$\begin{aligned}&\left\{\phi_a\left[H\left(p_p,x,0\right),r_s\right]-\phi_a\left[H\left(p_p,x,t+\Delta t\right),r_s\right]\right\}-\left\{\phi_a\left[H\left(p_p,x,0\right),r_s\right]\right.\\&\left.-\phi_a\left[H\left(p_p,x,t\right),r_s\right]\right\}=\phi_a\left[H\left(p_p,x,t+\Delta t\right),r_s\right]-\phi_a\left[H\left(p_p,x,t\right),r_s\right]\end{aligned}$$

The particles, suspended at the moment t in the fraction of the accessible volume that was cut-off during the period Δt, become cut off the conductive pore cluster at the moment $t+\Delta t$

$$\begin{aligned}&C\left(r_s,x,t\right)\left\{\left(\phi_{a0}-\phi_a\left(t+\Delta t\right)\right)-\left(\phi_{a0}-\phi_a\left(t\right)\right)\right\}\\&=\left(\phi_{a0}-\phi_a\left(t+\Delta t\right)\right)C_t\left(r_s,x,t+\Delta t\right)-\left(\phi_{a0}-\phi_a\left(t\right)\right)C_t\left(r_s,x,t\right)\end{aligned}$$

The kinetics of particle capture due to cutting off the accessible pore volume can be obtained by tending Δt to zero in the previous equation, that is

$$\frac{\partial}{\partial t}\left[\left(\phi_{a0}-\phi_a\right)C_t\left(r_s,x,t\right)\right]=C\left(r_s,x,t\right)\frac{\partial\left(\phi_{a0}-\phi_a\right)}{\partial t} \quad (35)$$

The mass exchange (35) is caused by change of pore space geometry. This term is analogous to that for kinetics of oil ganglia saturation during miscible WAG flooding (Bedrikovetsky 2003), and for kinetics of polymer adsorbed during polymer solution injection in pores filled by oil, during the production stage with huff-n-puff conformance control process (Holleben et al. 1997).

The term of suspended concentration distribution in front of the accumulation term in right hand side of (35) means that particles "enter" into cut-off volume from suspension with its own concentration distribution $C(r_s,x,t)$ in the moving suspension. In the case of retained particle release, the cut-off particles enter the suspension from the cut-off volume with concentration distribution $C_t(r_s,x,t)$ in cut-off volume; this term appears in right hand side of (35) in front of the accumulation term. So, the upscaled model contains the hysteretic term.

The geometry-change-induced kinetics (35) does not have the conservation law form. Therefore, the resulting mass balance equation does not have a conservation law form either.

Substitution of (35) into (34) results in exclusion of unknown concentration distribution C_t from the system of governing equations, that is

$$\frac{\partial}{\partial t}\left[\phi_a C\left(r_s,x,t\right)\right]+\frac{\partial}{\partial x}\left[C\left(r_s,x,t\right)U_a\right]=-\frac{\partial\Sigma\left(r_s,x,t\right)}{\partial t}+C\left(r_s,x,t\right)\frac{\partial\phi_a}{\partial t}$$

Performing derivation in accumulation term in the left hand side of mass balance equation results in disappearing of the particle storage term in right hand side

$$\phi_a \frac{\partial}{\partial t}\left[C\left(r_s, x, t\right)\right] + \frac{\partial}{\partial x}\left[C\left(r_s, x, t\right) U_a\right] = -\frac{\partial \Sigma\left(r_s, x, t\right)}{\partial t} \tag{36}$$

The difference between mass balance equation with and without the cut-off-volume effect is placing the porosity outside and inside the accumulation term, respectively. Usually, accounting for extra physics phenomenon in mathematical model results in introduction of a new coefficient (function). In the case of accounting for cut-off volume, it results in the change of term placement without introduction of new coefficients.

3.2 Kinetics for Particle Capture and Pore Plugging

The derivations of the kinetic equations for particle capture and pore plugging are based on the above-mentioned geometric porous medium model of parallel capillaries alternated by mixing chambers (Figs. 1–4). The complete mixing in chambers corresponds to the assumption, that the particles coming to the pores at the chamber outlet are being distributed among them, independently of relations between the particle sizes and the pore sizes, and the prehistory of the pore filling (Fig. 3). Before "collision" between the particles and the pores, their behaviour is totally uncorrelated. The assumption is similar to the Boltzmann's assumption about "molecular chaos" (Landau and Lifshitz 1980, 1999).

On this assumption, the number of r_s-particles arriving to pores with size r_p per unit of rock volume during time Δt is equal to

$$q\left(r_p\right) H^v\left(r_p\right) C\left(r_s, x, t\right) \Delta t$$

So, p-th fraction of these particles is captured by the rock

$$\frac{\partial \Sigma\left(r_s, r_p, x, t\right)}{\partial t} = \left[1 - \nu\left(r_s/r_p\right) + \nu p_a\right] q\left(r_p\right) H^v\left(r_p\right) C\left(r_s, x, t\right) \tag{37}$$

The plugging of one pore by one particle can be interpreted as chemical reaction with unitary stoichiometric coefficients. So, kinetics Eq. 37 can be interpreted as active mass law (Yortsos 1990; Fogler 1998).

The retention rate for particles with size r_s is obtained by integration of (37) in r_p

$$\begin{aligned}\frac{\partial \Sigma\left(r_s, x, t\right)}{\partial t} &= \int_0^\infty \left[1 - \nu\left(r_s/r_p\right) + \nu p_a\right] q\left(r_p\right) H^v\left(r_p\right) C\left(r_s, x, t\right) \mathrm{d}r_p \\ &= \int_0^\infty \left[1 - \nu\left(r_s/r_p\right) + \nu p_a\right] \frac{k_1\left(r_p\right)}{k} U H^v\left(r_p\right) C\left(r_s, x, t\right) \mathrm{d}r_p \end{aligned} \tag{38}$$

Finally, the expression for retention rate of r_s-particles is

$$\frac{\partial \Sigma\left(r_s, x, t\right)}{\partial t} = \frac{UC\left(r_s, x, t\right)}{lk} \int_0^\infty \left[1 - \nu\left(r_s/r_p\right) + \nu p_a\right] k_1\left(r_p\right) H\left(r_p\right) \mathrm{d}r_p \tag{39}$$

One captured particle plugs one pore. It allows deriving the kinetics of pore plugging

$$\frac{\partial H\left(r_p,x,t\right)}{\partial t}=l\frac{\partial H^V\left(r_p,x,t\right)}{\partial t}=-l\int\limits_0^\infty\frac{\partial\sum\left(r_s,r_p,x,t\right)}{\partial t}\mathrm{d}r_s$$
$$=-\int\limits_0^\infty\left[1-\nu\left(r_s/r_p\right)+\nu p_a\right]q\left(r_p\right)H\left(r_p\right)C\left(r_s,x,t\right)\mathrm{d}r_s$$

Finally, the expression for r_p-pore plugging rate is

$$\frac{\partial H\left(r_p,x,t\right)}{\partial t}=-\frac{k_1\left(r_p\right)}{k}UH\left(r_p\right)\int\limits_0^\infty\left[1-\nu\left(r_s/r_p\right)+\nu p_a\right]C\left(r_s,x,t\right)\mathrm{d}r_s \tag{40}$$

For pure straining case $p_a=0$, and the kinetics rate (39) for retained particles becomes

$$\frac{\partial\Sigma\left(r_s,x,t\right)}{\partial t}=\frac{UC\left(r_s,x,t\right)}{lk}\int\limits_0^\infty\left[1-\nu\left(r_s/r_p\right)\right]k_1\left(r_p\right)H\left(r_p\right)\mathrm{d}r_p=\frac{U_{na}C\left(r_s,x,t\right)}{l} \tag{41}$$

The retention rate (41) is proportional to inaccessible flux.

3.3 Closed System of Two Equations

Substitution of the capture kinetics (39) into particle balance (36) excludes the retained concentration distribution from the model

$$\phi_a\left[H,r_s\right]\frac{\partial}{\partial t}\left[C\left(r_s,x,t\right)\right]+U\frac{\partial}{\partial x}\left[C\left(r_s,x,t\right)f\left[H,r_s\right]\right]$$
$$=-\frac{UC\left(r_s,x,t\right)}{lk}\int\limits_0^\infty\left[1-\nu\left(r_s/r_p\right)+\nu p_a\right]k_1\left(r_p\right)H\left(r_p\right)\mathrm{d}r_p \tag{42}$$

Finally, the population balance model consists of two equations of particle balance (42) and of pore plugging kinetics (40). The unknowns in the system of integro-differential equations (40, 42) are suspended particle and vacant pore concentration distributions, C and H, respectively.

The system of Eqs. 40, 42 describes suspension transport in porous media modelled by system of parallel capillary alternated by mixing chambers (Fig. 1). The empirical functionals ϕ_a and f are given by formulae (7) and (28), respectively. For any arbitrary rock geometry, the functionals of accessible porosity and fractional flow must be determined from laboratory tests or from micro modelling data.

For the case of cylindrical pores and spherical particles, particle balance Eq. 42 becomes

$$\left[\int_{r_s}^{\infty} H\left(r_p,x,t\right)\pi r_p^2\left(1-\frac{r_s}{r_p}\right)^2 \mathrm{d}r_p\right]\frac{\partial C\left(r_s,x,t\right)}{\partial t} + U\frac{\partial}{\partial x}\left[C\left(r_s,x,t\right)\frac{\int_{r_s}^{\infty}\nu\left(r_s/r_p\right)r_p^4 H\left(r_p,h\right)\mathrm{d}r_p}{\int_0^{\infty} r_p^4 H\left(r_p,h\right)\mathrm{d}r_p}\right] = -\frac{UC\left(r_s,x,t\right)}{l}\frac{\int_0^{r_s} r_p^4 H\left(r_p\right)\mathrm{d}r_p + \int_{r_s}^{\infty}\left[1-\left(1-p_a\right)\nu\left(r_s/r_p\right)\right]r_p^4 H\left(r_p\right)\mathrm{d}r_p}{\int_0^{\infty} r_p^4 H\left(r_p,h\right)\mathrm{d}r_p} \quad (43)$$

where accessibility factor ν is given by formula (22).

Equation (43) differs from that presented by Santos and Bedrikovetsky (2006), by the accessible porosity term, which is outside the time derivative of the accumulation term in left hand side of equation, expressing cutting the suspended particles in the accessible pores plugged off the bulk suspension. The model (43) also captures the effect of varying porosity; it contains the term ϕ_a in left hand side, while the model by Santos and Bedrikovetsky (2006) assumes constant porosity and contains the fraction ϕ_a/ϕ in the accumulation term of the particle balance equation.

The capture rate term in right hand side contains two different terms showing that all particle that enter smaller pores are captured, while just the fraction of particles in larger pores is captured.

Another important difference between the models is: introduction of particle capture probability p_a allows description of both, size exclusion and attachment mechanisms, for particle capture, for the case of "infinite" rock capacity to attach particles.

The problem of suspension injection into a clean bed corresponds to initial and boundary conditions for system (42, 40)

$$t=0: C\left(r_s,x,0\right)=0,\ H\left(r_p,x,0\right)=H_0\left(r_p,x\right) \quad (44)$$

$$x=0: C\left(r_s,0,t\right)=C^0\left(r_s,t\right) \quad (45)$$

where H_0 is an initial pore size distribution and C^0 is a particle size distribution in the injected suspension.

The initial-boundary value problem (44, 45) has a Goursat type: while both unknowns are fixed at $t=0$, only $C=C^0$ is given at $x=0$; unknown H is found from ordinary-differential-integral equation

$$\frac{\mathrm{d}H\left(r_p,0,t\right)}{\mathrm{d}t} = -\frac{k_1\left(r_p\right)}{k}UH\left(r_p,0,t\right)\int_0^{\infty}\left[1-\nu\left(r_s/r_p\right)+\nu p_a\right]C^0\left(r_s,t\right)\mathrm{d}r_s$$

which is obtained from (40) by substitution of $C=C^0$ from boundary condition (45).

3.4 Overall Conservation for Particles and Pores

Let us calculate the total retention rate. Accounting for (1, 37, 39) we obtain

$$\frac{\partial \sigma\,(x,t)}{\partial t} = \int_0^\infty \frac{\partial \Sigma\,(r_s,x,t)}{\partial t}\mathrm{d}r_s = \int_0^\infty \int_0^\infty \frac{\partial \underline{\Sigma}\left(r_s,r_p,x,t\right)}{\partial t}\mathrm{d}r_p\mathrm{d}r_s$$
$$= \int_0^\infty \left[\int_0^\infty \frac{\partial \underline{\Sigma}\left(r_s,r_p,x,t\right)}{\partial t}\mathrm{d}r_s\right]\mathrm{d}r_p = \int_0^\infty \left[-\frac{\partial H^v\left(r_p,x,t\right)}{\partial t}\right]\mathrm{d}r_p = -\frac{\partial h\,(x,t)}{\partial t}$$

The obtained conservation of the sum for concentrations of retained particles and vacant pores

$$\sigma\,(x,t) + h\,(x,t) = g\,(x)$$

follows from the model assumption that one pore is plugged by one retained particle, and vice versa. The constant in right hand side of the previous equality can be calculated from initial conditions

$$\sigma\,(x,t) + h\,(x,t) = h_0 + \sigma_0 \tag{46}$$

4 Transport of Mono Dispersed Suspensions

In this section we discuss the flow of single-size dispersed suspensions as a particular case of poly dispersed suspension transport, discussed in the previous section.

Consider transport of a single size particle suspension in porous media with pores distributed over size

$$C\,(r_s,x,t) = c\,(x,t)\,\delta\,(r_s - r_0) \tag{47}$$

where r_0 is a particle size.

Let us find the simplified form of the system of governing Eqs. 40, 42 for filtration of mono sized suspension.

Substituting the form of solution (47) into particle population balance Eq. 42 and integrating both parts in r_s yields:

$$\phi_a\,[H,r_0]\,\frac{\partial}{\partial t}\,[c\,(x,t)] + U\,\frac{\partial}{\partial x}\,[c\,(x,t)\,f\,[H,r_0]]$$
$$= -\frac{U c\,(x,t)}{lk\,[H]}\int_0^\infty \left[1 - \nu\left(r_s/r_p\right) + \nu p_a\right] k_1\left(r_p\right) H\left(r_p\right)\mathrm{d}r_p \tag{48}$$

Kinetics equation for single particle size suspension is obtained by substitution of (47) into (40)

$$\frac{\partial H\left(r_p,x,t\right)}{\partial t} = -\frac{k_1\left(r_p\right)}{lk}U H\left(r_p\right) c\,(x,t)\left[1 - \nu\left(r_0/r_p\right) + \nu p_a\left(r_0/r_p\right)\right] \tag{49}$$

The unknowns in system (48, 49) are total suspension concentration $c(x,t)$ and pore concentration distribution $H(r_p,x,t)$.

Lemma *For governing system* (48, 49) *subject to uniform initial conditions for pore size distribution*

$$t = 0 : h = h_0, H\left(r_p, x, t\right) = H_0\left(r_p\right) \tag{50}$$

the solution is given by the formula

$$H\left(r_p, x, t\right) = \tilde{H}\left(r_p, h\right) \tag{51}$$

where $\tilde{H}(r_\mathrm{p}, h)$ *is the solution of an ordinary-differential-integral equation.*

Proof Calculate kinetics of total vacant pore concentration by integration of kinetics Eq. 49 in r_p:

$$\frac{\partial h\left(x, t\right)}{\partial t} = \int_0^\infty \frac{\partial H\left(r_p, x, t\right)}{\partial t} \mathrm{d}r_p$$

$$= -\frac{Uc\left(x, t\right)}{lk} \int_0^\infty H\left(r_p\right)\left[1 - \nu\left(r_0/r_p\right) + \nu p_a\left(r_0/r_p\right)\right] k_1\left(r_p\right) \mathrm{d}r_p \tag{52}$$

Equations (49) and (52) does not contain x-derivatives. So, co-ordinate x can be considered in (49, 52) as a scalar parameter, and time partial derivative can be substituted by ordinary derivative

$$\frac{\mathrm{d}H\left(r_p, x, t\right)}{\mathrm{d}t} = -\frac{k_1\left(r_p\right)}{lk} U H\left(r_p\right) c\left(x, t\right)\left[1 - \nu\left(r_0/r_p\right) + \nu p_a\left(r_0/r_p\right)\right] \tag{53}$$

$$\frac{\mathrm{d}h\left(x, t\right)}{\mathrm{d}t} = -\frac{Uc\left(x, t\right)}{lk} \int_0^\infty H\left(r_p\right)\left[1 - \nu\left(r_0/r_p\right) + \nu p_a\left(r_0/r_p\right)\right] k_1\left(r_p\right) \mathrm{d}r_p \tag{54}$$

Let $h(x, t)$ be the solution of system (53, 54) subject to initial conditions (50). The total vacant pore concentration monotonically decreases with time, so the variable t can be expressed via h and x by the inverse function:

$$t = t\left(x, h\right) \tag{55}$$

Let us use h instead of t as a parameter along each h-characteristic $x = \mathrm{const}$. Substituting (55) into (53) accounting for (54) yields:

$$\frac{\mathrm{d}H\left(r_p, x, h\right)}{\mathrm{d}h} = \frac{k_1\left(r_p\right) H\left(r_p\right)\left[1 - \nu\left(r_0/r_p\right) + \nu p_a\left(r_0/r_p\right)\right]}{\int_0^\infty H\left(r_p\right)\left[1 - \nu\left(r_0/r_p\right) + \nu p_a\left(r_0/r_p\right)\right] k_1\left(r_p\right) \mathrm{d}r_p} \tag{56}$$

Initial condition (50) becomes:

$$h = h_0 : H\left(r_p, x, 0\right) = H_0\left(r_p\right) \tag{57}$$

The ordinary integro-differential equation (56) with initial data (57) allows for unique solution. Since right hand side of Eq. 56 and initial conditions (57) are independent of x, the solution also does not depend on x, and formula (51) follows.

Let us illustrate this statement by the example of porous media with discrete pore sizes

$$H\left(r_p, x, t\right) = \sum_{i=1}^{n} h_i\left(x, t\right) \delta\left(r_p - r_i\right)$$

Substitution of the solution form into Eq. 53 and initial conditions (54) and integration of the resulting expressions into r_p result into following Caushy problem for system of ordinary differential equations:

$$\frac{\mathrm{d}h_i\,(x,h)}{\mathrm{d}h} = \frac{k_1\,(r_i)\,[1-\nu\,(r_0/r_i)+\nu p_a\,(r_0/r_i)]\,h_i}{\sum\limits_{i=1}^{n} k_1\,(r_i)\,[1-\nu\,(r_0/r_i)+\nu p_a\,(r_0/r_i)]\,h_i} \tag{58}$$

$$h = \sum_{i=1}^{n} h_{i0} = h_0 : h_i = h_{i0} \tag{59}$$

The solution of Cauchy problem (58, 59)

$$h_i = h_i\,(h) \tag{60}$$

provides with solution (51) for this particular case. □

Proof that with n tending to infinity and simultaneously $r_i - r_{i-1}$ tending to zero, the solution (55) tends to solution of the problem (56, 57)

$$H\left(r_p, x, t\right) = \tilde{H}\left(r_p, h\right)$$

is outside the scope of this work.

Form of the function $\tilde{H}\left(r_p, h\right)$ depends on empirical model functions $k_1(r_p)$, $p(j)$, $\nu(j)$ and on initial pore size distribution $H_0(r_p)$.

Equation (56) can be integrated exactly allowing for implicit expression for solution (51). Let us find it.

Consider system of two integral-ordinary-differential equations

$$\frac{\mathrm{d}H}{\mathrm{d}y} = -k_1\left(r_p\right) H\left(r_p\right)\left[1-\nu\left(r_0/r_p\right)+\nu p_a\left(r_0/r_p\right)\right] \tag{61}$$

$$\frac{\mathrm{d}h}{\mathrm{d}y} = -\int_0^{\infty} H\left(r_p\right)\left[1-\nu\left(r_0/r_p\right)+\nu p_a\left(r_0/r_p\right)\right] k_1\left(r_p\right)\mathrm{d}r_p \tag{62}$$

subject to initial conditions

$$y = 0 : h = h_0,\ H = H_0\left(r_p\right) \tag{63}$$

Introduction of parameter h instead of y reduces system (61, 62) to Eq. (56).

Separation of variables in (61) accounting for initial conditions (63) leads to solution

$$H\left(y, r_p\right) = H_0\left(r_p\right) e^{-k_1(r_p)[1-\nu(r_0/r_p)+\nu p_a(r_0/r_p)]y} \tag{64}$$

Substituting solution (64) into Eq. (62)

$$\frac{\mathrm{d}h}{\mathrm{d}y} = -\int_0^{\infty} H\left(r_p\right)\left[1-\nu\left(r_0/r_p\right)+\nu p_a\left(r_0/r_p\right)\right] k_1\left(r_p\right)\mathrm{d}r_p$$

$$= -\int_0^{\infty} H_0\left(r_p\right) e^{-k_1(r_p)[1-\nu(r_0/r_p)+\nu p_a(r_0/r_p)]y}\left[1-\nu\left(r_0/r_p\right)+\nu p_a\left(r_0/r_p\right)\right] k_1\left(r_p\right)\mathrm{d}r_p$$

and then integrating the result in y yields:

$$h(y) = h_0 - \int_0^y \mathrm{d}y \left[\int_0^\infty H_0(r_p)\, e^{-k_1(r_p)[1-\nu(r_0/r_p)+\nu p_a(r_0/r_p)]y} [1-\nu+\nu p_a] k_1(r_p) \mathrm{d}r_p \right]$$
$$= h_0 - \int_0^\infty H_0(r_p)\, e^{-k_1(r_p)[1-\nu(r_0/r_p)+\nu p_a(r_0/r_p)]y} \mathrm{d}r_p$$

Finally, pore concentration distribution $H(r_p, h)$ is given by formula (64) where $y(h)$ is implicitly expressed from equation

$$h(y) = h_0 - \int_0^\infty H_0(r_p)\, e^{-k_1(r_p)[1-\nu(r_0/r_p)+\nu p_a(r_0/r_p)]y} \mathrm{d}r_p \tag{65}$$

as an inverse function.

Formulae (64), (65) form implicit analytical solution (51) for system (48, 49) of mono dispersed filtration in porous media.

Remark 1 If for initial conditions for the ordinary-differential-integral equation (56)

$$h = h_0 : H(r_p, x, t) = H_0(r_p)$$

takes place the equality

$$h_0 = \int_0^\infty H_0(r_p)\, \mathrm{d}r_p$$

it also takes place for the solution at any $h > h_0$:

$$h = \int_0^\infty H(r_p, h)\, \mathrm{d}r_p$$

The above equality is obtained by integration of both parts of Eq. 56 in h.

In other words, if a formal independent parameter h in (54) is an overall pore concentration at the moment $t = 0$, it remains the overall concentration for any $t > 0$.

Let us denote the solution (51) as

$$H(r_p, h) = \bar{H}\left[r_p, h, h_0, H_0(r_p)\right] \tag{66}$$

in order to show that the solution depends on initial conditions.

Remark 2 The solution of governing system (48, 49) with non-uniform initial data

$$t = 0 : h = h_0(x), \quad H(r_p, x, 0) = H_0(r_p, x)$$

can be expressed as a function of the total retention concentration

$$H(r_p, x, t) = \tilde{H}(r_p, h)$$

if and only if

$$H_0(r_p, x) = \bar{H}\left[r_p, h_0(x), h_0(x_0), H_0(r_p, x_0)\right] \tag{67}$$

Previous equation means that the initial distribution $H_0\left(r_p, x\right)$ is already a solution of ordinary-differential-integral equation (56) with initial data at some reference point x_0. The initial pore concentration distribution in point x, where the total retention concentration equals $h_0(x)$, is equal to solution (51) with initial data at point x_0, where the total retention concentration equals $h_0(x_0)$.

The solution of the problem with non-uniform initial data $h_0(x)$ is given by formula

$$H\left(r_p, x, t\right) = \tilde{H}\left[r_p, h + h\left(x_0\right), h\left(x_0\right), H_0\left(r_p, x_0\right)\right]$$

5 Derivation of Upscaled System for Mono Dispersed Suspension

The fact that the vacant pore concentration distribution $H(r_p, x, t)$ can be expressed as a function of the total vacant pore concentration $h(x, t)$, $H(r_p, x, t) = \tilde{H}(r_p, h)$, derived in the previous section, allows for excluding H from all terms in governing system (48, 49) and for exact averaging of the micro scale system with distributed concentrations and derivation of the closed system on the macro scale. This is the subject of the current section and the main purpose of the current article.

Substituting (51) into (3) and (7) we obtain that the total and accessible porosities are functions of h

$$\phi_a = \phi_a\left(h\right), \quad \phi = \phi\left(h\right)$$

Fractional flow function (28) is also h-dependent.

The conservation law (46) for the sum of particle and pore concentrations allows expressing the total pore concentration via the total retained concentration

$$h\left(x, t\right) = h_0 + \sigma_0 - \sigma\left(x, t\right) \tag{68}$$

and concluding that the fractional flow and porosities are functions of the total retained concentration

$$\phi_a = \phi_a\left(\sigma\right), \quad f = f\left(\sigma\right), \quad \phi = \phi\left(\sigma\right) \tag{69}$$

The above derivations were performed for mono dispersed particles, so the functions (69) depend also on particle size

$$\phi_a = \phi_a\left(r_0, \sigma\right), \quad \phi = \phi\left(r_0, \sigma\right), \quad f = f\left(r_0, \sigma\right)$$

The closed averaged system for suspension transport in porous media is obtained by substituting the functions (69) into Eqs. 48 and 52:

$$\phi_a\left(\sigma\right)\frac{\partial c}{\partial t} + U\frac{\partial}{\partial x}\left[cf\left(\sigma\right)\right] = -\frac{\partial \sigma}{\partial t} \tag{70}$$

$$\begin{aligned}
\frac{\partial \sigma}{\partial t} &= \frac{1}{l}\frac{k_c\left(\sigma\right)}{k\left(\sigma\right)}Uc \\
k_c\left(\sigma\right) &= \int_0^\infty \left[1 - \nu\left(r_0/r_p\right) + \nu p_a\left(r_0/r_p\right)\right] k_1\left(r_p\right) H\left(r_p\right) \mathrm{d}r_p \\
&= \int_0^\infty p\left(r_0/r_p\right) k_1\left(r_p\right) H\left(r_p\right) \mathrm{d}r_p
\end{aligned} \tag{71}$$

Equation for capture rate (71) has the same form as that in the classical filtration model—the rate is proportional to particle flux, and the filtration function is

$$\lambda'(\sigma) = \frac{p'}{l}, \quad p' = \frac{k_c(\sigma)}{k(\sigma)} \tag{72}$$

where p' is a probability for particle to be captured during its motion over the unit trajectory.

Let us consider the physics meaning of the filtration coefficient (72) for different particular cases.

Substituting (71) into (72) and accounting for expression (28) for fractional flow function, we obtain

$$\frac{k_c(\sigma)}{k(\sigma)} = 1 - f + \frac{\int_0^\infty \left[\nu p_a\left(r_0/r_p\right)\right] k_1\left(r_p\right) H\left(r_p\right) \mathrm{d}r_p}{\int_0^\infty k_1\left(r_p\right) H\left(r_p, x, t\right) \mathrm{d}r_p}$$

Therefore, capture kinetics Eq. 71 for straining capture where $p_a = 0$ takes the form:

$$\frac{\partial \sigma}{\partial t} = \frac{1}{l}(1 - f) Uc \tag{73}$$

So, for size exclusion case, the retention rate becomes proportional to flux via inaccessible pores.

In case of constant attachment probability, the retention rate expression takes the form

$$\frac{k_c(\sigma)}{k(\sigma)} = 1 - f(\sigma) + p_a f(\sigma) \tag{74}$$

and

$$\frac{\partial \sigma}{\partial t} = \frac{1 - f(\sigma) + p_a f(\sigma)}{l} Uc \tag{75}$$

The expression for capture probability follows from comparison between (75) and (72):

$$p' = 1 - f(\sigma) + p_a f(\sigma) \tag{76}$$

Fractional flow f can be interpreted as a probability for particle to get into accessible pore volume. Formula (76) becomes Bayes' theorem for particle in porous media: the capture probability is the probability to enter inaccessible volume plus the probability to enter accessible volume times conditional probability to be captured inside the pore.

So, the Bayes' theorem is valid for a pore scale (5) and is obtained for the rock scale (76) after averaging.

Comparing (73) with the retention rate expression proposed by Santos and Bedrikovetsky (2006)

$$\frac{\partial \sigma(x,t)}{\partial t} = \lambda_0 [1 - f(r_s, \sigma)] Uc(x,t) \tag{77}$$

we obtain the filtration coefficient for size exclusion capture

$$\lambda_0 = \frac{1}{l} \tag{78}$$

Introduction of the "suspension saturation"

$$s = \varphi_a / \varphi \tag{79}$$

transforms Eq. 67 to the fractional flow form

$$\phi(\sigma)\, s(\sigma)\, \frac{\partial c}{\partial t} + \frac{\partial \sigma}{\partial t} + U \frac{\partial}{\partial x}\left[c f(\sigma)\right] = 0 \tag{80}$$

The function $f = f(s)$ as obtained from two parametric dependencies $f(\sigma)$ and $s(\sigma)$ is a fractional flow function defined for two-phase flow in porous media.

Darcy's law (15, 16) for the overall flux completes the governing system for suspension transport in porous media.

Finally, the fractional flow model for mono dispersed suspension flow in porous media consists of equations for particle balance, for particle capture kinetics and Darcy's law accounting for permeability damage:

$$\phi(\sigma)\, s(\sigma)\, \frac{\partial c}{\partial t} + \frac{\partial \sigma}{\partial t} + U \frac{\partial}{\partial x}\left[c f(\sigma)\right] = 0 \tag{81}$$

$$\frac{\partial \sigma}{\partial t} = \lambda'(\sigma)\, U c \tag{82}$$

$$U = -\frac{k(\sigma)}{\mu} \frac{\partial P}{\partial x} \tag{83}$$

System (81–83) contains four empirical functions: accessible porosity $\phi_a(\sigma)$, fractional flow function $f(\sigma)$, filtration function $\lambda'(\sigma)$ nd absolute permeability $k(\sigma)$. For the case of constant attachment probability p_a, the filtration function is expressed via fractional flow function and two constants $-p_a$ and dispersivity l.

Since none of the mentioned empirical functions depend on pressure, Eqs. 81, 82 separate from Darcy's eqs. (83). System (81, 82) determines two unknowns—total suspension and retained concentrations, $c(x, t)$ and $\sigma(x, t)$, respectively. The pressure $P(x, t)$ is determined from (83) where concentrations $c(x, t)$ and $\sigma(x, t)$ are already known from solution of eqs. (81, 82).

Now discuss the limitations of the model (81–83) and where is can be applied.

It is important to emphasize that, according to the lemma, the model (81–83) is valid, strictly speaking, just for mono dispersed suspensions. In the case of distribution by size particles, the above described upscaling procedure is not valid.

The model (81–83) was derived for porous space of a parallel capillary bundle alternated by mixing chambers. Nevertheless, Eq. 81 is particle mass balance accounting for particles cut off the flux by retained particles which is valid for any pore space geometry.

Equations of particle capture kinetics (82) and of momentum conservation (83) are also valid for any rock. Therefore, system of governing Eqs. 81–83 for suspension transport can be applied for any rock, and four empirical functions $\phi_a(\sigma)$, $f(\sigma)$, $\lambda'(\sigma)$ and $k(\sigma)$ must be determined from laboratory tests with this rock. For particular case of the pore space geometry shown in Fig. 1, the functions can be calculated using micro scale formulae (7, 28, 16) with further upscaling using procedures (64, 65). Numerical network modelling can be used to calculate the empirical functions for more complex pore space geometries (Sahimi and Imdakm 1991; Siqueira et al. 2003). For network models with small pore size variation, the effective medium formulae can be applied (Chen et al. 2003). Percolation formulae can be applied for networks with high variation of pore sizes (Seljakov and Kadet 1996).

Another restriction is either uniform initial pore size distribution mentioned in the lemma, or the specific initial non-uniform pore size distribution derived in Remark 2. If the initial

values for (48, 49) provide different x-dependent initial conditions for system of ordinary-differential-integral equations (56), the solution H is also x-dependent, and the equality (51) is not valid any more. In the case where the conditions (66, 67) are violated, the particles may be displaced into finite clusters of accessible pore volume, i.e. they will be captured instead of being transported by the accessible flux, and the flow is not described by the model (81–83) anymore. It may result in non-unique determination of upscaled functions $\phi_a(\sigma)$, $f(\sigma)$, $\lambda'(\sigma)$ and $k(\sigma)$ and in appearance of hysteresis.

The model (81–83) is obtained for continuous injection corresponding to boundary conditions (44, 45). Sequential injection of suspension and production from the same well, which takes place during drilling and well clean-up, is an example where the model (81–83) is not valid.

For continuous injection of mono dispersed suspension in clean bed, the particles occupy the accessible part of porous space, i.e. infinite cluster of pores which are larger than the particles. The same takes place during primary displacement of wetting phase by non-wetting phase, where the non-wetting fluid enters the pores in order of increasing capillary pressure, i.e. in order of decreasing of the pore size (Lake 1989; Barenblatt et al. 1991; Dullien 1992). Transport properties (relative phase permeabilities, electrical conductivity and capillary pressure) for primary drainage are calculated using an infinite cluster of pores accessible for non-wetting phase (Larson et al. 1981; Seljakov and Kadet 1996). Therefore, the process of mono dispersed suspension injection is analogous to primary drainage for two-phase flow, and the fractional flow function for primary drainage can be used for preliminary qualitative analysis of the deep bed filtration model (81–83).

The micro model (40, 42) and, consequently, the upscaled macro model (81–83), were derived for diluted suspensions. So, it was assumed that the suspension concentration variation does not cause change of suspension density and viscosity. Therefore, the overall suspension flux is conserved, and the term U can be taken out of square brackets in the suspended particles flux term in left hand side of Eq. 42. In the case of highly concentrated suspensions, the Amagat law of the volume additivity with mixing can be violated, and the total suspension flux can be not conserved. The total suspension flux $U(x, t)$ becomes another unknown in the system of governing equations.

For medium concentrated suspensions, viscosity μ in Darcy's law (83) becomes function of the total suspended concentration c. The high concentration suspension may become non-Newtonian liquid resulting in non-linear dependency of suspension flux versus pressure gradient.

6 Dimensional Analysis

Different scales for time and for retention concentration result in different asymptotic models for suspension transport in natural rocks. These models are described in the current section.

Introduction of the following dimensionless variables and parameters

$$X = \frac{x}{L}, \quad C = \frac{c}{c^0}, \quad \phi = \frac{\phi'}{\phi_0}, \quad \lambda = \lambda' L, \quad \varepsilon = \frac{c^0}{\alpha} \tag{84}$$

$$T = \frac{Ut}{\phi_0 L} \tag{85}$$

$$\bar{S} = \frac{\sigma}{\alpha \phi_0} \tag{86}$$

into Eqs. 81, 82 results in dimensionless form of the governing system

$$\phi\left(\bar{S}\right) s\left(\bar{S}\right) \frac{\partial C}{\partial T} + \frac{\partial}{\partial X}\left[C f\left(\bar{S}\right)\right] = -\lambda\left(\bar{S}\right) C$$
$$\frac{\partial \bar{S}}{\partial T} = \varepsilon\lambda\left(\bar{S}\right) C \tag{87}$$

Here, α is the critical porosity fraction, showing which fraction of the porous space is filled by particles up to the end of particle motion; the vacant pores does not form an infinite cluster any more where the retained concentration is equal to α (Pang and Sharma 1994). Usually α varies between 0.03 and 0.4 (da Silva et al. 2004) while injected concentration varies between 0.1 and 1,000 ppm. So, the ratio $\varepsilon = c^0/\alpha$ is often significantly less than unity, $\varepsilon \ll 1$, and may be considered in (87) as a small parameter.

Small parameter ε was introduced by Marchesin et al. (2002), in the numerical model for deep bed filtration.

The small parameter ε appears in system (87) in the expression for retention rate. Neglecting the retention rate and accounting for zero initial condition (no retained particles in the rock before the injection) allows concluding, that the retention concentration remains zero. So, during the retention of a small amount of particles, all volumetric and transport coefficients in first Eq. 87 are constant; the accessible volume does not change and the storage term in right hand side of (87) disappears. Finally, system (87) degenerates into one linear hyperbolic equation for suspended concentration:

$$\phi(0) s(0) \frac{\partial C}{\partial T} + f(0) \frac{\partial C}{\partial X} = -\lambda(0)[1 - f(0)] C \tag{88}$$

Consider the slow time $\tau = \varepsilon T$. From (85) follows that

$$\tau = \varepsilon T = \frac{c^0 U t}{\alpha\varphi L}, \tag{89}$$

which is the volume of injected particles divided by critical pore volume that should be filled in order to stop the particle transport.

Introduction of slow time (89) into governing system (81, 82) instead of fast time (85) yields

$$\varepsilon\phi\left(\bar{S}\right) s\left(\bar{S}\right) \frac{\partial C}{\partial \tau} + \frac{\partial}{\partial X}\left[C f\left(\bar{S}\right)\right] = -\lambda\left(\bar{S}\right) C$$
$$\frac{\partial \bar{S}}{\partial \tau} = \lambda\left(\bar{S}\right) C \tag{90}$$

allowing omitting accumulation and storage terms in mass balance equation:

$$\frac{\partial}{\partial X}\left[C f\left(\bar{S}\right)\right] = -\lambda\left(\bar{S}\right) C$$
$$\frac{\partial \bar{S}}{\partial \tau} = \lambda\left(\bar{S}\right) C \tag{91}$$

Herzig et al. (1970) neglected the accumulation term based on observation of laboratory and numerical modelling data.

Consider dimensionless time T measured in pore volumes injected (85) and dimensionless retained concentration normalised by injected particle concentration per rock volume unit c^0 instead of (86):

$$S = \frac{\sigma}{c^0\phi_0} \tag{92}$$

The system (81, 82) yields:

$$\phi(S)\, s(S)\, \frac{\partial C}{\partial T} + \frac{\partial}{\partial X}\left[C f(S)\right] = -\frac{\partial S}{\partial T}$$
$$\frac{\partial S}{\partial T} = \lambda(S)\, C \tag{93}$$

The obtained averaged particle balance Eq. 93 generalises the model for deep bed filtration, which accounts for flux reduction and accessibility factors (Santos and Bedrikovetsky (2006)), by the effect of suspended particle storage in the cut-off fraction of accessible pore space, and also by the term of varying porosity $\phi(\sigma)$. Introduction of the suspended particle storage in the cut-off fraction of accessible pore space changes the form of mass balance equation—it is not a conservation law any more. The accounting for a new effect does not introduce any additional constitutive empirical function into the governing system (93), it still contains the same functions of accessible porosity $\phi_a(\sigma)$, fractional flow $f(\sigma)$ and filtration function $\lambda(S)$.

7 Structure of the Analytical Model for Clean Bed Suspension Injection

Assumptions of no accessibility and flux reduction factors, $s = f = 1$, and of constant porosity $\phi = \phi_0$ transforms the particle balance Eq. 93 into that for the classical deep bed filtration model (Herzig et al. 1970; Payatakes et al. 1974)

$$\frac{\partial (C+S)}{\partial T} + \frac{\partial C}{\partial X} = 0 \tag{94}$$

The retention rate is governed by any arbitrary filtration function in the classical deep bed filtration model

$$\frac{\partial S}{\partial T} = \lambda(S)\, C \tag{95}$$

It is worth mentioning that for the case of straining ($p_a = 0$) and complete accessibility ($f = 1$), as it follows from (73), no retention occurs for size exclusion particle capture. The corresponding physics schema is as follows: the particle cannot pass thin pore throat and cannot stick to the rock matrix due to short range repulsion forces; the particle remains until it will be taken out by small flux fluctuation and will join the suspension flux.

Nevertheless, the classical model (94, 95) is applied for straining capture. So, the classical deep bed filtration model (94, 95) is not a particular case for the averaged size exclusion model of suspension transport in porous media (81, 82).

For the general particle capture case, the retention rate expressions (82) and (95) coincide. The classical deep bed filtration model (84, 85) is a particular case of the averaged model (83) for the general case of particle capture.

Let us consider the initial-boundary value problem for suspension injection into a particle-free rock

$$T = 0 : C = S = 0 \tag{96}$$
$$X = 0 : C = 1 \tag{97}$$

and investigate the solution structure.

The Goursat problem (85, 87) allows deriving formula for retained concentration at the core inlet $X = 0$, without solving the whole problem (94–97) (see Tikhonov and Samarskii

1990). Accounting for the inlet boundary condition (97) in eq. (85) yields:

$$\frac{\mathrm{d}S\,(0,\,T)}{\mathrm{d}T} = \lambda\,(S) \tag{98}$$

Separation of variables in ordinary differential equation (98) leads to

$$T = \int\limits_{0}^{S(0,T)} \frac{\mathrm{d}y}{\lambda\,(y)} \tag{99}$$

which is an implicit expression for retained concentration at the core inlet $C\,(0,\,T)$.

Performing first differentiation of the advective term in left hand side of first Eq. 90 and then substituting the retention rate kinetics into the result yields:

$$\frac{\partial}{\partial T}\left[\phi\,(S)\,s\,(S)\,C\right] + \frac{\partial}{\partial X}\left[Cf\,(S)\right] = \varepsilon\lambda\left[1 - f\,(S)\right]C^2\left[\phi\,(S)\,s\,(S)\right]' - \lambda\left[1 - f\,(S)\right]C \tag{100}$$

Discuss characteristics of quasi-linear hyperbolic partial differential equation (100) which start at axes $T = 0$. The differential condition on characteristics for Eq. 100 is an ordinary differential equation. As it follows from zero-value initial condition (96), suspended concentration along each characteristic is also zero. From (95) follows that if $C = 0$ along all characteristics which start at axes $T = 0$, S is zero too. All transport and volumetric coefficients in (100) become constant, including dimensionless porosity ($\phi(0) = 1$), which simplifies Eq. 100:

$$\frac{\partial C}{\partial T} + \frac{f\,(0)}{s\,(0)}\frac{\partial C}{\partial X} = -\lambda\frac{\left[1 - f\,(0)\right]}{s\,(0)}C\left\{1 - \varepsilon\left[\phi\,(S)\,s\,(S)\right]'(0)\,C\right\} \tag{101}$$

Since, as it follows from (95), function $S(X, T)$ is continuous, the retained concentration is equal zero along the concentration front $X{=}vT$, $v{=}f(0)/s(0)$. The suspended concentration behind the front is given by an ordinary differential equation, which is the differential condition for Eq. 100 along the front:

$$\frac{\mathrm{d}C}{\mathrm{d}T} = \lambda\frac{\left[1 - f\,(0)\right]}{s\,(0)}C\left\{\varepsilon\left[\phi\,(S)\,s\,(S)\right]'(0)\,C - 1\right\},\quad C(0) = 1 \tag{102}$$

The solution of ordinary differential equation (102) is obtained by separation of variables

$$C\,(T) = \frac{N\exp\left(-MT\right)}{1 + N - \exp\left(-MT\right)},\quad M = \lambda\frac{\left[1 - f\,(0)\right]}{s\,(0)},\quad N = -\left[\varepsilon\left[\phi\,(S)\,s\,(S)\right]'(0)\right]^{-1}$$

This formula shows how suspended concentration behind the front decreases from unity on injector up to zero when time tends to infinity.

The structure of the flow pattern is shown in Fig. 7. The concentration front propagates with speed v. Both concentrations are zero ahead of the front.

8 Prediction of Breakthrough Time

The breakthrough moment is calculated from the concentration front speed (see (101)):

$$T_{br} = \frac{1}{v} = \frac{s\,(r_s,\,0)}{f\,(r_s,\,0)} \tag{103}$$

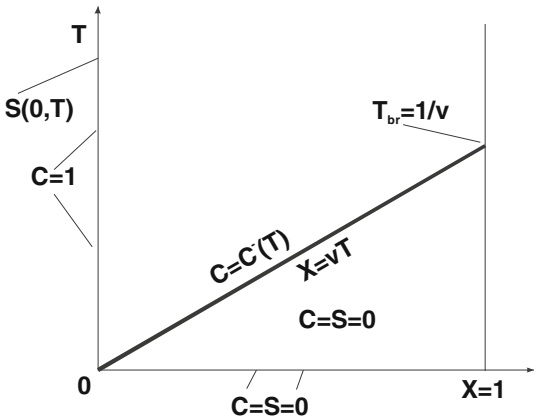

Fig. 7 Structure of the flow pattern during suspension injection into a particle-free rock

Since in system (94, 95) $s = f = 1$, the classical deep bed filtration model predicts that the breakthrough time is equal to unity.

Yet, several authors report different breakthrough times that vary from 0.3–0.5 up to 100 pore volumes injected, p.v.i. (Harvey and Garabedian 1991; Roque et al. 1995; Kretzschmar et al. 1997; Chauveteau et al. 1998; Camesano et al. 1999; Veerapen et al. 2001; Bolster et al. 1998,1999; Massei et al. 2002). It is important to notice that breakthrough times, presented in the literature, always exceed 0.3 p.v.i. and can reach whatever high value like 100 p.v.i. and even more. Roque et al. (1995) and Chauveteau et al. (1998) emphasise that these are large particles that exhibit high breakthrough times.

Corapcioglu and Choi (1996) introduced the porosity reduction due to particle deposition into classical model (94, 95). The obtained equations are equivalent to those (93) with accessibility factor ($s < 1$) but without flux reduction, $f = 1$. This only explains an early breakthrough of particles observed in laboratory tests—from (93) it follows that $v > 1$ and $T_{br} < 1$.

Introduction of both factors s and f explains the breakthrough times that are either less or exceed unity. Below we show that both cases can be explained by S-shape of fractional flow function $f(s)$.

As it was mentioned above, the model (93) is valid only for filtration of equal size particles, and the particle size r_0 is a model parameter. Consider the dependencies of saturation (fractional flow) on particle size, and express $f(s)$ parametrically.

The form of the fractional flow function (28) as calculated for the bundle of parallel capillaries accounting for inaccessibility of thin pores and for accessible volume inside large pores (15, 25, 28) is shown in Fig. 8. The fractional flow function is convex, $f'' < 0$. For a given porous space geometry, both saturation and fractional flow are determined by particle size r_s, so each size particle corresponds to a point on the fractional flow curve. The increase of the particle size from zero to minimum pore radius corresponds to "remaining" in point $s = f = 1$. The particle size increase from r_{pmin} up to r_{pmax} is shown in Fig. 8 as the point motion from the upper right point of the fractional flow curve up to point $f = s = 0$. Particles larger than r_{pmax} remain in point (0,0). The function $f(s)$ is concave, so the tangent f/s always exceeds unity. The larger is the particle, the higher is its velocity in each capillary, and the lower is the breakthrough time. The parallel capillary model always predicts an early breakthrough.

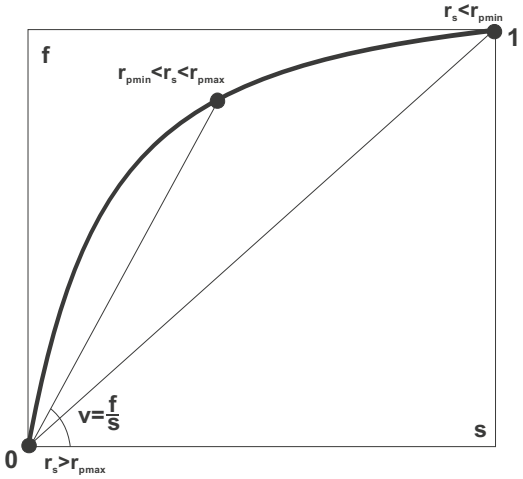

Fig. 8 Fractional flow function for a bundle of parallel capillaries alternated by mixing chambers

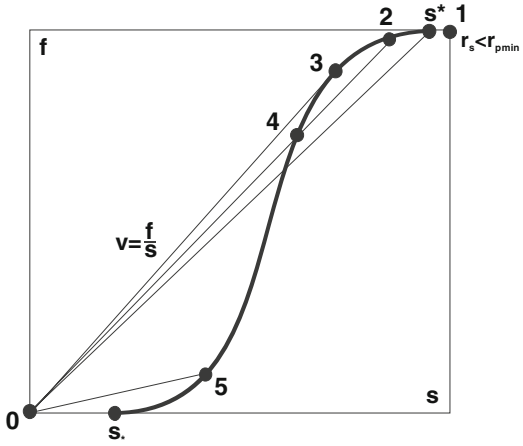

Fig. 9 Fractional flow function for network model of the porous space

Now consider more realistic network structure of the rock. Typically, percolation and effective medium models predict the S-shaped fractional flow functions for network models of porous media (Fig. 9) (Larson et al. 1981; Dullien 1992; Bedrikovetsky 1993; Seljakov and Kadet 1996). The typical form of fractional flow as determined from corefloods is also S-shaped (Lake 1989; Barenblatt et al. 1990). Strict definition of pore size distributions and network geometries resulting in S-shaped fractional flow functions is outside the scope of this work.

Consider any arbitrary S-shaped fractional flow curve (Fig. 9).

Zero-sized particles percolate via all pores, so $s = 1$, $f = 1$ and breakthrough time also equals unity. The same takes place when particle size increases up to the minimum pore size. Some further increase of the particle size results in creation of some inaccessible pore volume, i.e. s becomes less than unity. Nevertheless, flux via inaccessible pores remains zero until the saturation reaches the first threshold value where the inaccessible pore cluster

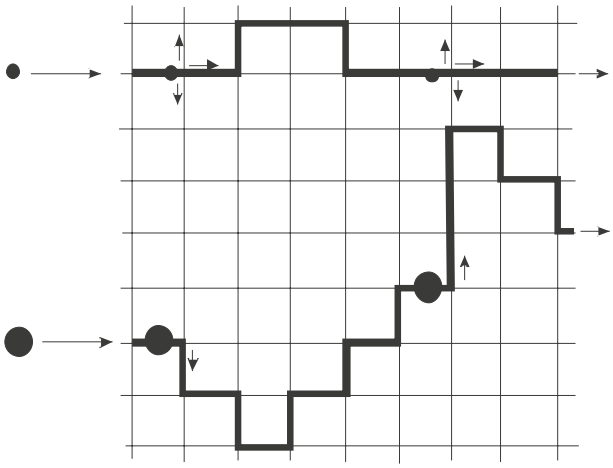

Fig. 10 Path tortuosity increase for large particles

becomes infinite. The corresponding pore radius and threshold saturation are denoted as r^* and s^*, respectively. For larger particles, the suspension moves via accessible pore space, and the particle-free water moves via inaccessible part of the pore space. At some large particle radius, the accessible pore cluster density reaches its threshold value, and clusters for larger particles become finite, i.e. there is no transport via accessible pores any more. It corresponds to $f = 0$ for some positive saturation $s < s_*$ that is called the second threshold.

Increase of the particle size from zero to second threshold value r_* corresponds to movement of the point on fractional flow curve from the position (1,1) to that $(s_*, 0)$. The breakthrough time decrease, while the increase of the particle size from zero to some value that exceeds the first threshold r^*, is explained by increase of particle velocity in each pore due to inaccessible volume near to pore walls (so called velocity enhancement factor, see Bartelds et al. 1997).

When the particle size increases, its probability to enter a pore decreases, the probability for particle to take a tortuous path increases, and the mean path becomes more tortuous (Fig. 10). Furthermore, when r_s approaches the second threshold r_*, the accessible cluster density tends to zero, and its tortuosity tends to infinity (Seljakov and Kadet 1996). Therefore, the breakthrough time tends to infinity too. This is what is observed at low values of fractional flow (Fig. 9): when s tends to s_*, the interstitial particle speed f/s tends to zero.

The intermediate part of the fractional flow curve is a result of competition between these two factors—increase of flow velocity in each pore and increase of path tortuosity, both with increase of the particle size. The tortuosity factor is equal to unity for parallel capillaries, so just the velocity enhancement factor is in play, which explains the convex shape of fractional flow function (Fig. 8).

Depending on the form for relative phase permeabilities, the frontal velocity for immiscible fluids with the same viscosity, as calculated from the tangent to Buckley–Leverett function, usually varies between 1.5 and 3 (Lake 1989; Bedrikovetsky 1993). It agrees with the fact that the lowest found in the literature breakthrough time is equal 0.3.

The concentration front speed f/s tends to zero for large particles, Fig. 9. It agrees with high values of breakthrough times, presented in the literature.

9 Discussion

The population balance model for suspension flow in porous media, which accounts for particle and pore size distributions and for particle storage in accessible pores cut off the flow by pore plugging, allows for exact upscaling in the case of a single-particle-size suspension filtration.

Introduction of the effect of particle storage in cut off accessible pores results in loss of the conservation form for averaged particle balance equation.

The obtained averaged equations generalise the classical model for suspension flow in porous media and its latter modifications. The generalised averaged model shows that in particular case of pure particle straining, the retention rate is proportional to inaccessible flux. So, the straining retention rate equals zero in the case of complete accessibility, while the filtration coefficient is not zero in the classical model, i.e. the averaged straining model is not a particular case of the classical deep bed filtration model.

The generalised model has a fractional flow form.

The breakthrough time in bundle of parallel capillaries is less than unity, which is explained by velocity enhancement and pore accessibility. The larger is the particle, the faster it appears at the core outlet.

For the porous medium with a network pore structure, the path for large particles is tortuous, and the larger is the particle, the more tortuous is the path and the higher is the breakthrough time. This effect competes with the velocity enhancement effect resulting in non-monotonous dependency of breakthrough time versus particle size in rocks with network pore structure.

The fractional flow model explains the acceleration and slowing down of the particles in porous media and the breakthrough times which, as it was observed in laboratory, can be either more or less than one pore volume injected. Experimentally observed breakthrough times, which are higher than 0.3 p.v.i. and can be as high as 100 p.v.i., agree with typical shapes of fractional flow function.

Nevertheless, widely laboratory observed deviation of retention profile from classical colloid filtration theory cannot be explained by the proposed model.

The classical model (94, 95) contains single empirical function—the filtration coefficient $\lambda(\sigma)$. As it was mentioned above, the filtration coefficient is constant for the case of low concentrated suspensions. The filtration coefficient λ can be calculated from the breakthrough concentration. Then the deposition profile $\sigma(x,t)$ can be calculated. The comparison between the predicted and measured deposition profiles allows validating the theory.

Numerous works report on significant deviation of the deposition profile as predicted by the breakthrough-concentration-tuned model and that measured by computer tomography.

The deviation was explained by heterogeneity of the rock surface charge in the case of electric attachment of particles (Tufenkji and Elimelech 2005). The deviation was not explained in case of size exclusion particle capture (Bradford et al. 2002, 2003, 2004; Bradford and Bettahar 2006).

The laboratory tests by Bradford have been performed with mono-sized particles. The majority of tests exhibit unity breakthrough time, so the fractional flow effects—accessibility and flux reduction—are negligible. As it is proven in the current article, transport of mono-sized particles in rock with varying pore sizes is described by large scale Eqs. 93 which degenerate into the classical model for the case of complete pore accessibility. Therefore, the modified model (93) does not explain deviation of modelling data from those obtained experimentally.

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