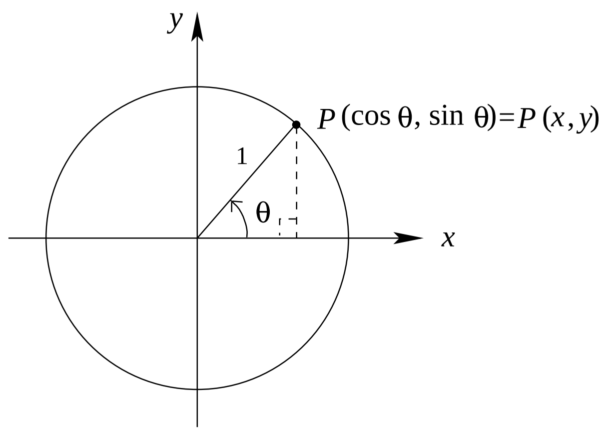
More Trigonometry

Extending the Trigonometric Ratios to angles $> \frac{\pi}{2}$ $(90°)$ or < 0



In the first quadrant of the unit circle above, the co-ordinates of the point P on the circle are, by definition,

$$x = \cos\theta \quad \text{and} \quad y = \sin\theta$$

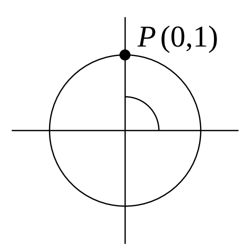
where θ is the angle measured anti-clockwise around from the positive $x-$axis.

Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, we can also say that

$$\frac{y}{x} = \tan\theta.$$

This relationship holds for *any* angle and point on the circle.

Example: What are $\sin\frac{\pi}{2}$ and $\cos\frac{\pi}{2}$?



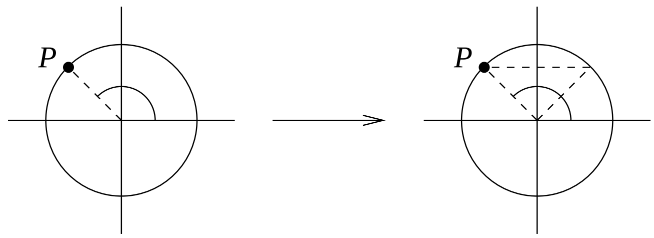
From the unit circle,

$$\sin\frac{\pi}{2} = y - \text{co-ordinate of } P = 1.$$

$$\cos\frac{\pi}{2} = x - \text{co-ordinate of } P = 0.$$

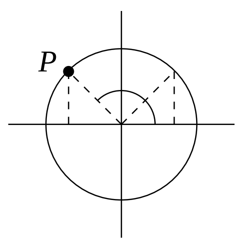
Check these results on a calculator.

Example: What are $\sin\frac{3\pi}{4}$, $\cos\frac{3\pi}{4}$ and $\tan\frac{3\pi}{4}$?



By symmetry, the $y-$co-ordinate of P is the same as for the first quadrant angle $\frac{\pi}{4}$. Hence

$$\sin\frac{3\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$



The $x-$co-ordinate of P is the negative of that for the first quadrant angle $\frac{\pi}{4}$. Hence

$$\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$$

Finally,

$$\tan\frac{3\pi}{4} = \frac{1/\sqrt{2}}{-1/\sqrt{2}} = -1.$$

Exercises

1. Find the values of

   (a) $\sin 0$, $\sin\frac{\pi}{2}$, $\sin\left(-\frac{\pi}{2}\right)$, $\sin\pi$, $\sin\frac{3\pi}{2}$, $\sin 2\pi$.

   (b) $\cos 0$, $\cos\frac{\pi}{2}$, $\cos\left(-\frac{\pi}{2}\right)$, $\cos\pi$, $\cos\frac{3\pi}{2}$, $\cos 2\pi$.

   (c) $\tan 0$, $\tan\pi$, $\tan\frac{3\pi}{4}$, $\tan\frac{7\pi}{4}$, $\tan\left(-\frac{\pi}{4}\right)$.

   (d) $\sin\frac{7\pi}{6}$, $\cos\frac{7\pi}{6}$, $\tan\frac{7\pi}{6}$.

(e) $\sin\frac{11\pi}{6}$, $\cos\frac{11\pi}{6}$, $\tan\frac{11\pi}{6}$.

2. Use the unit circle to show that

   (a) $\sin(\pi - \theta) = \sin\theta$ (b) $\cos(\pi - \theta) = -\cos\theta$ (c) $\tan(\pi + \theta) = \tan\theta$

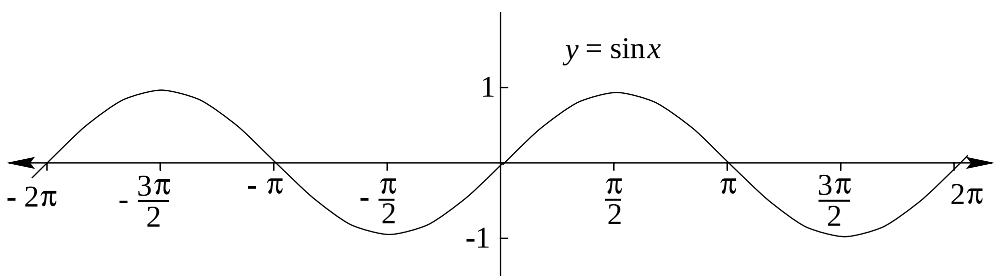
   (d) $\cos(2\pi - \theta) = \cos\theta$

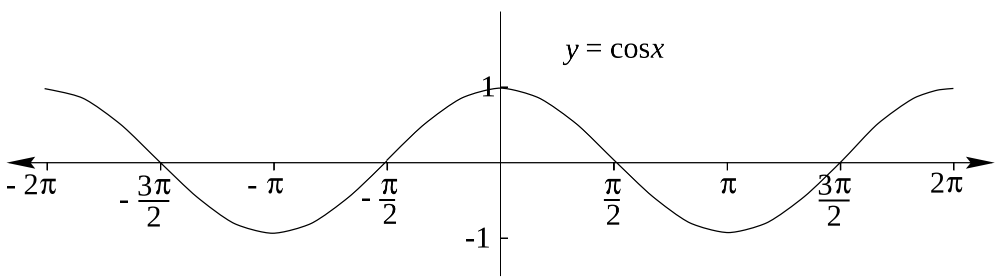
Circular Functions

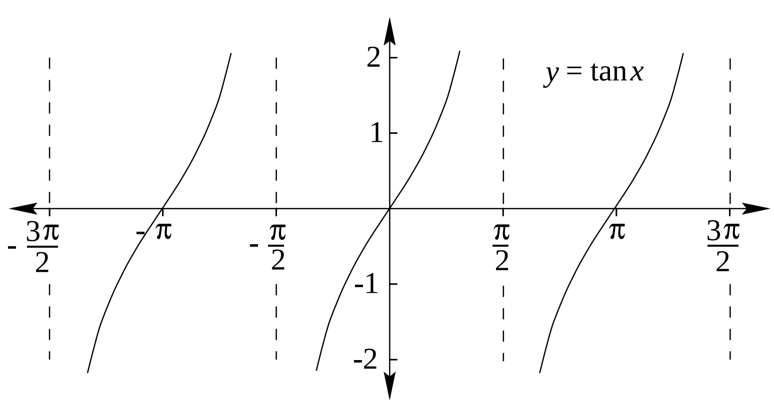
Many situations are affected by circular motion (eg. day length in temperate areas such as Adelaide) and can be modelled using functions of the form

$$y = \sin x, \quad y = \cos x \quad \text{or} \quad y = \tan x.$$

Plotting points from the unit circle produces these distinctive graphs (where x is measured in radians):





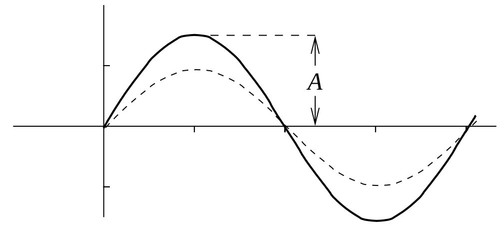


Note: The shape of the cosine graph is the same as for sine, but shifted backwards $\frac{\pi}{2}$ units along the $x-$axis.

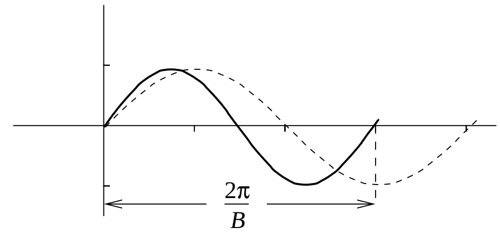
These basic sine waves can be modified using the parameters A, B, C and D as follows:

$$y = A\sin(B(x+C)) + D$$

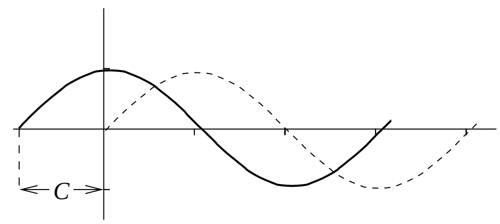
A : changes the amplitude of the sine wave from 1 to A.



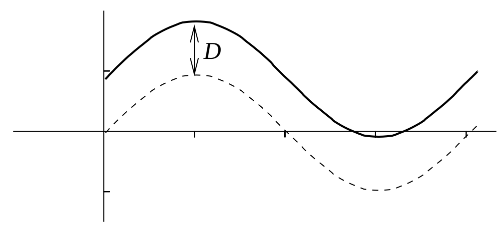
B : changes the period of the sine wave from 2π to $\frac{2\pi}{B}$.



C : changes the horizontal position of the sine wave by C in the *opposite direction to its sign* (ie. backwards if C is positive).



D : changes the vertical position of the sine wave by D units (upwards if D is positive).



Exercises

3. Sketch the graphs of the following functions:

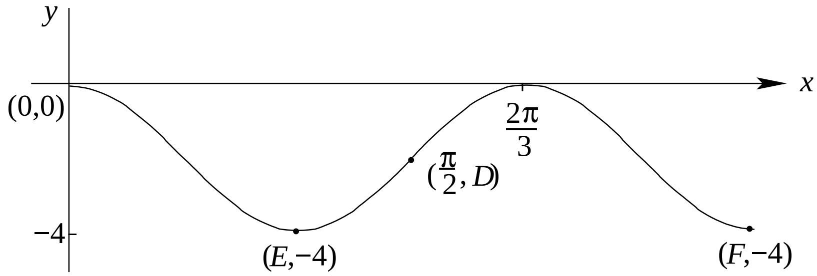
   (a) $y = 1 + \sin x$ (b) $y = 2\cos x$ (c) $y = \sin\left(x + \frac{\pi}{4}\right)$ (d) $y = \cos 2x$

4. *[No answers given]* Sketch the graphs of the following functions:

   (a) $y = -\sin x$ (b) $y = 2 + \cos x$ (c) $y = \cos\left(x - \frac{\pi}{6}\right)$ (d) $y = -\frac{1}{2}\cos x$

   (e) $y = 2\sin 3x$

5. Find the values of A, B, C, D, E and F for $y = A\cos Bx + C$ given that it has the following graph.



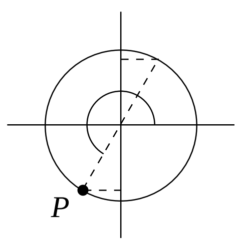
Trigonometric Equations

Example: Find all solutions to $2\sin x + \sqrt{3} = 0$.

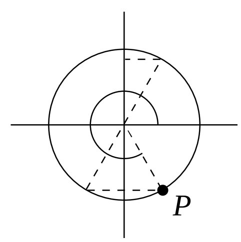
Re-arranging this equation gives

$$\sin x = -\frac{\sqrt{3}}{2}.$$

We know that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ (first quadrant of unit circle) so, by symmetry



or



$$\text{ie. } \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2} \quad \text{or} \quad \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}.$$

Since there is no restriction on x, $2\pi + \frac{4\pi}{3}$ is also a solution since the cycle repeats once we travel all the way around the circle. In fact, the complete solution is

$$x = 2k\pi + \frac{4\pi}{3} \quad \text{or} \quad 2k\pi + \frac{5\pi}{3}$$

where k is any integer.

Exercises

6. Find all solutions of the following equations (you'll need a calculator for (a), (b) and (d)):

   (a) $\cos x = -0.7$ (b) $\sin 2x = 0.4$ (c) $\cos x = \sin x$ (d) $\tan 3x = 2$

7. Sketch the graph of $y = 4\cos 2x$ and use it to find the number of solutions, in the domain $0 \le x \le 2\pi$, to the simultaneous equations

$$y = 4\cos 2x \quad \text{and} \quad 4y = x$$

Trigonometric Identities

Since the point $P(\cos\theta,\ \sin\theta)$ was defined to lie on the unit circle, it follows that (by Pythagorus) that

$$\cos^2\theta + \sin^2\theta = 1$$

From this simple identity we can easily derive others. For example, if we divide through by $\cos^2\theta$ we get

$$1 + \tan^2\theta = \sec^2\theta$$

and, if we divide through by $\sin^2\theta$ we get

$$\cot^2\theta + 1 = \mathrm{cosec}^2\theta.$$

Another class of trigonometric identities are the *addition formulae* which are listed here without proof:

$$\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B \\ \cos(A+B) &= \cos A\cos B - \sin A\sin B \end{aligned}$$

From these may be derived a number of other formulae:

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &\text{or}\quad 2\cos^2\theta - 1 \quad \text{using } \cos^2\theta + \sin^2\theta = 1 \end{aligned}$$

Exercises

8. Rewrite $\sin\theta\cos\theta$ in terms of $\sin 2\theta$.
9. Rewrite $\sin^2\theta$ in terms of $\cos 2\theta$.
10. Use the unit circle to show that

$$\sin(-B) = -\sin(B) \quad \text{and} \quad \cos(-B) = \cos(B).$$

Hence, use the addition formulae to derive the subtraction formulae

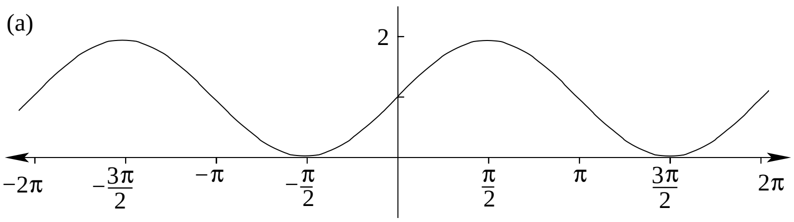
$$\sin(A-B) \quad \text{and} \quad \cos(A-B).$$

ANSWERS

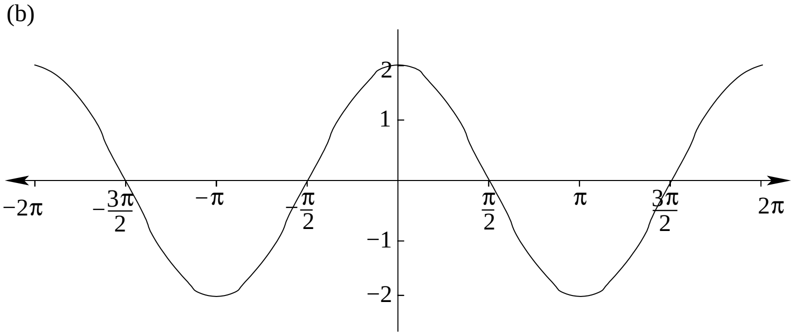
1(a) $0,\ 1,\ -1,\ 0,\ -1,\ 0$ (b) $1,\ 0,\ 0,\ -1,\ 0,\ 1$ (c) $0,\ 0,\ -1,\ -1,\ -1$

(d) $-\frac{1}{2},\ -\frac{\sqrt{3}}{2},\ \frac{1}{\sqrt{3}}$ (e) $-\frac{1}{2},\ \frac{\sqrt{3}}{2},\ -\frac{1}{\sqrt{3}}$

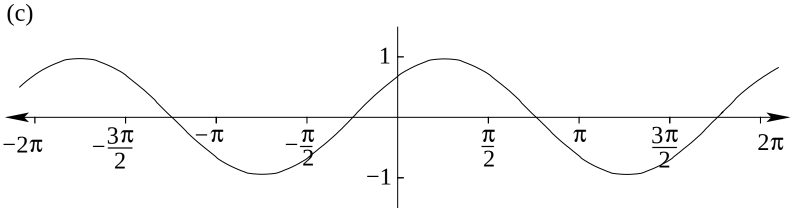
3 (a)



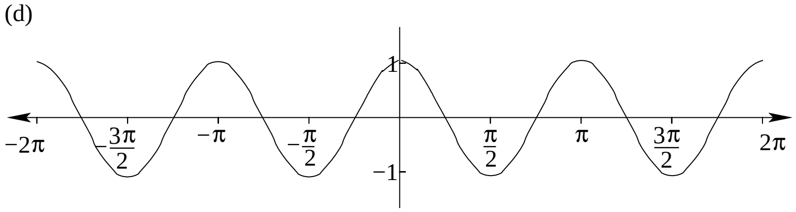
(b)



(c)



(d)



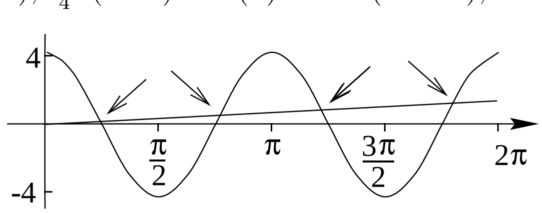
5. $A = 2,\ B = 3,\ C = -2,\ D = -2,\ E = \frac{\pi}{3},\ F = \pi$

6. All answers in radians (with degrees in brackets) and $\pm 2\pi k$ where k is an integer

(a) 2.346 (134.43°), 3.967 (225.57°) (b) 0.206 (11.79°), 1.365 (78.21°)

(c) $\frac{\pi}{4}$ (45°), $\frac{5\pi}{4}$ (225°) (d) 0.369 (21.14°), 1.416 (81.14°)

7. four



8. $\sin\theta\cos\theta = \dfrac{\sin 2\theta}{2}$ 9. $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

10. $\sin(A-B) = \sin A\cos B - \cos A\sin B$ $\cos(A-B) = \cos A\cos B + \sin A\sin B$