Maths Learning Service: Revision ***Mathematics IA***

Quadratics

Mathematics IMA

Intro. to Fin. Maths I

A quadratic is an expression of the form

$$ax^2 + bx + c$$

where a, b, c are numbers ($a \neq 0$) and x is a pronumeral.

Examples:

(i)	$x^2 + 10x - 1$	$a = 1$	$b = 10$	$c = -1$
(ii)	$x^2 - 16$	$a = 1$	$b = 0$	$c = -16$
(iii)	$t^2 - t$	$a = 1$	$b = -1$	$c = 0$
(iv)	$x^2 + 4x + 4$	$a = 1$	$b = 4$	$c = 4.$

A *partial* factorization of this quadratic is obviously $x(x + b) + c$ but this not very useful. However, it is often possible to fully factorize in the following form

$$ax^2 + bx + c \;=\; (x + A)(x + B).$$

For example, it can be shown that

$$x^2 + 5x + 6 \;=\; (x + 2)(x + 3).$$

Check: $(x + 2)(x + 3) = x(x + 3) + 2(x + 3) = x^2 + 3x + 2x + 6 = x^2 + (3 + 2)x + 6$. This expansion illustrates the relationship between A, B and b, c when a is 1. That is $A \times B = c$ and $A + B = b$. Using this "trial and error" technique we can factorize many quadratics.

Example: Factorize $x^2 + 8x + 12$.

Solution: The possible factorizations of 12 are 1×12, 2×6 and 3×4. Only the second option involves numbers which add up to 8 so the factorization is $(x + 2)(x + 6)$.

Example: Factorize $x^2 + x - 12$.

Solution: In this case the value of c is negative, so one of its two factors is negative and the other positive. To produce a value of 1 for b we should use $4 \times (-3) = -12$ since $4 - 3 = 1$. The factorization is then $(x + 4)(x - 3)$.

Example: Factorize $x^2 - 13x + 12$.

Solution: The value of c is positive but b is negative. For this to happen, both factors of 12 must be negative. Since $(-1) \times (-12) = 12$ and $-1 - 12 = -13$, the factorization is $(x - 1)(x - 12)$.

Example: Factorize $x^2 - 25$.

Solution: Here the value of b is zero and c is negative. If we can find two identical factors, their difference will be zero. In this case $5 \times 5 = 25$ so the factorization is $(x - 5)(x + 5)$. This is known as the **difference of two squares** since the original quadratic is $x^2 - 5^2$.

Example: Factorize $x^2 + 8x + 16$.

Solution: The factorization here turns out to be $(x + 4)(x + 4) = (x + 4)^2$. This is known as a **perfect square** since both factors are identical.

Exercise

(1) Factorize:

(a)	$x^2 + 4x + 3$	(b)	$x^2 - 9x + 20$	(c)	$y^2 - 5y - 24$
(d)	$y^2 - 36$	(e)	$x^2 + 12x + 36$	(f)	$x^2 - 4x - 32$
(g)	$y^2 - 10y + 25$	(h)	$3x^3 + 18x^2 + 27x$	(Hint: Look for common factors first)	
(i)	$s^2 + 3s$	(j)	$8x^2 + 32x + 24$	(k)	$x^4 + 6x^2 - 16$

Quadratic Equations

Factorization is useful for solving quadratic equations of the form

$$ax^2 + bx + c = 0.$$

Unlike linear equations, there are up to *two* values of x which satisfy this equation.

Example: Solve $x^2 + 8x + 12 = (x + 2)(x + 6) = 0$.

Solution: We know that any number multiplied by zero will give an answer of zero, so if either bracket in the factorized version is zero, the answer will be zero. Hence the solutions are

$$\begin{array}{rrcl} & (x+2) & = & 0 \\ \text{or} & (x+6) & = & 0 \end{array}$$

Therefore, $x = -2$ or -6. Check to see that these two values satisfy $x^2 + 8x + 12 = 0$.

Example: Solve $x^2 + 8x = 0$.

Solution: In the absence of c we can simply factorize this as $x(x + 8) = 0$ and say that either $x = 0$ or -8.

Exercise

(2) Find the solution(s) to:

(a)	$x^2 - 4x + 3 = 0$	(b)	$x^2 - 9x = 10$	(c)	$y^2 = 4y$
(d)	$y^2 = 49$	(e)	$x^2 - 8x + 36 = 4x$	(f)	$y^2 - 26y + 25 = 0$
(g)	$10x^2 + 20x + 10 = 0$	(h)	$\theta^2 = -1$		

The Quadratic Formula

When a quadratic is difficult to factorize, the solutions to the equation $ax^2+bx+c=0$ $(a \neq 0)$ can be found by using

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Example: Solve $2x^2 - 4x - 3 = 0$.

$a = 2, b = -4, c = -3$ so

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 2 \times -3}}{2 \times 2} \\ &= \frac{4 \pm \sqrt{40}}{4} \\ &= \frac{4 \pm 2\sqrt{10}}{4} \\ &= \frac{2 \pm \sqrt{10}}{2}. \end{aligned}$$

Hence there are two solutions, $x = \dfrac{2-\sqrt{10}}{2} = -0.5811\ldots$ or $\dfrac{2+\sqrt{10}}{2} = 2.5811\ldots$.

Note: If $b^2 - 4ac < 0$, there are no real solutions (the solutions are complex numbers).

Exercise

(3) Solve the following equations:

(a) $2x^2 + 7x - 2 = 0$ (b) $z^2 + z + 1 = 0$ (c) $3y^2 - 4y - 5 = 0$

(d) $3y^2 - 4y = -5$

Quadratic Graphs (parabolas)

When sketching graphs of quadratic functions $y = ax^2 + bx + c$ you need to identify the following:

- the $y-$intercept: $(0, c)$,
- the $x-$intercept(s), if they exist, by solving $ax^2 + bx + c = 0$ and
- the co-ordinates of the vertex (turning point) of the parabola, which occurs at $x = -\dfrac{b}{2a}$.

Exercise

(4) Sketch the graphs of the following:

(a) $y = x^2 + 2x + 1$ (b) $y = x^2 - 1$ (c) $y = 2x - x^2$

(d) $y = -(x-2)^2 - 1$ (e) $y = 4 - (x-2)^2$ (f) $y = x^2 + 3x - 4$

Answers to Exercises

(1) (a) $(x+1)(x+3)$ (b) $(x-4)(x-5)$ (c) $(y+3)(x-8)$ (d) $(y+6)(y-6)$

(e) $(x+6)^2$ (f) $(x-8)(x+4)$ (g) $(x-5)^2$ (h) $3x(x+3)^2$

(i) $s(s+3)$ (j) $8(x+1)(x+3)$ (k) $(x^2+8)(x^2-2)$

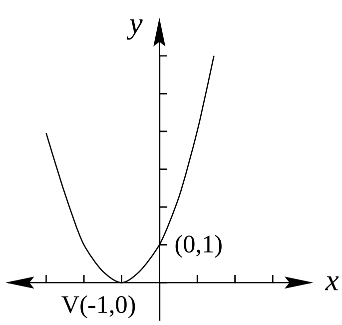
(2) (a) $x = 1$ or 3 (b) $x = -1$ or 10 (c) $y = 0$ or 4 (d) $y = -7$ or 7

(e) $x = 6$ (f) $y = 1$ or 25 (g) $x = -1$

(h) There are no solutions to this one, $\sqrt{-1}$ is not a real number.

(3) (a) $x = \dfrac{-7 \pm \sqrt{65}}{4}$ (b) no real solutions (c) $x = \dfrac{2 \pm \sqrt{19}}{3}$ (d) no real solutions

(4) (a)



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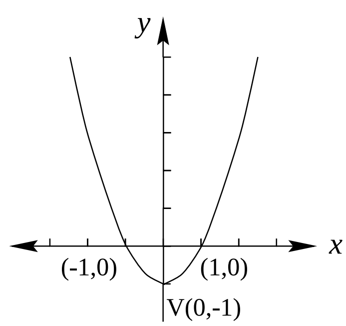

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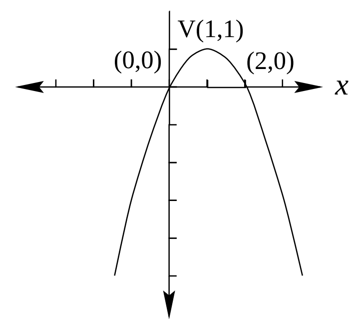

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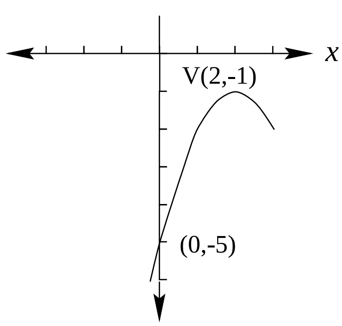

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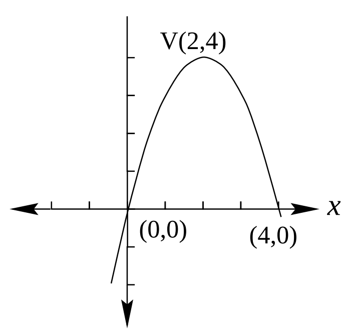

(f)