**BASIS**

**Questions**
- "Proof" way to find basis
- Are there "better" bases

Basis is information about a subspace.

Says how to construct the whole subspace out of linear combinations efficiently.

**Technical details**

Basis for subspace $W$ is a set of vectors that
1. spans $W$
2. is lin. indep.

That is, everything in $W$ is a lin. comb. of vectors.
To find a basis:

1. Rewrite as a span
2. Make sure linearly independent.

Example: Let $W = \{ (x, y, z, t) \mid x - 3y = t, \quad 2z + 5x = 0 \}$

Find a basis for $W$.

Find all vectors in $W$.
So find all vectors that satisfy

$$x - 3y = t \Rightarrow x - 3y - t = 0$$
$$2z + 5x = 0$$

$$\begin{bmatrix}
1 & -3 & 0 & -1 & 0 \\
5 & 0 & 1 & 0 & 0 \\
-1 & 3 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Row echelon form:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Free \quad Points

$r_1 = -r_1$
Let \( x = 5 \)
\( y = r \)
then \(-5 + 3r + t = 0\)
\( t = 5 - 3r \)
and \( 5s + t = 0 \)
\( t = -5s \)

\((x, y, z, t) = (5, r, -5s, 5 - 3r)\)

\[ = 5(1, 0, -5, 1) + r(0, 1, 0, -3) \]

so \( W = \text{span}\{ (1, 0, -5, 1), (0, 1, 0, -3) \} \)

these are lin. indep

\( \therefore \{ (1, 0, -5, 1), (0, 1, 0, -3) \} \)

is a basis for \( W \).

OR

\[
\begin{bmatrix}
1 & -3 & 0 & -1 & | & 0 \\
5 & 0 & 10 & | & 0 \\
1 & -3 & 0 & -1 & | & 0 \\
0 & 15 & 15 & | & 0 \\
\end{bmatrix}
\]

\( R_2 = R_2 - 5R_1 \)

\( R_2 = \frac{1}{5} R_2 \)
\[
\begin{bmatrix}
1 & -3 & 0 & -1 & 10 \\
0 & 1 & 15 & 5 & 10 \\
1 & 0 & 15 & 1 & 10 \\
0 & 0 & 15 & 2 & 10 \\
\end{bmatrix}
\]

Pivots free

Let \( z = 5 \)

\( t = r \)

\[ x + \frac{1}{5} s = 0 \Rightarrow x = -\frac{1}{5} s \]

\[ y + \frac{1}{15} s + \frac{1}{5} r = 0 \Rightarrow y = -\frac{1}{15} s - \frac{1}{5} r \]

\[
\begin{pmatrix}
x \\ y \\ z \\ t \\
\end{pmatrix}
= \begin{pmatrix}
-\frac{1}{5} s \\ -\frac{1}{15} s - \frac{1}{5} r \\ 5 \\ r \\
\end{pmatrix}
= s \begin{pmatrix}
-\frac{1}{5} \\ -\frac{1}{15} \\ 5 \\ 0 \\
\end{pmatrix}
+ r \begin{pmatrix}
0 \\ -\frac{1}{5} \\ 0 \\ 1 \\
\end{pmatrix}
\]

Basis is \[ \frac{1}{2} \left( -\frac{1}{5}, -\frac{1}{15}, 5, 0 \right), \left( 0, -\frac{1}{5}, 0, 1 \right) \]
Example: \( W = \text{span}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}) \mid x - 3y = 0 \)

Find all vectors.

Solve

\[ \begin{align*} x - 3y &= t \\ x - 2y - t &= 0 \end{align*} \]

\[ \begin{bmatrix} 1 & -3 & 0 & -1 & 0 \end{bmatrix} \]

pivot free

Let \( y = s \)

\( z = r \)

\( t = q \)

\( x = 3s + q \)

\[ \begin{pmatrix} x \\ y \\ z \\ t \\ e \end{pmatrix} = \begin{pmatrix} 3s + q \\ s \\ r \\ q \\ e \end{pmatrix} = s \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + q \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + e \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]
\[ \text{OR} \quad \text{know that } x - 3y = t \]

\[
\begin{pmatrix}
  y \\
  z \\
  t
\end{pmatrix} = \begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} - \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}
\]

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  1 \\
  0 \\
  -3
\end{pmatrix} + y \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}
\]

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}
\]

\[
\therefore \quad W = \text{span} \begin{pmatrix}
  \frac{1}{3} \\
  0 \\
  \frac{1}{3}
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}
\]

A Basis is \( \begin{pmatrix}
  \frac{1}{3} \\
  0 \\
  \frac{1}{3}
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix} \)

\[
\begin{pmatrix}
  1 \\
  2 \\
  3
\end{pmatrix} + 2 \begin{pmatrix}
  -1 \\
  -2 \\
  -1
\end{pmatrix} = \begin{pmatrix}
  -1 \\
  0 \\
  -2
\end{pmatrix}
\]

\[
W = \text{span} \begin{pmatrix}
  \frac{1}{3} \\
  0 \\
  \frac{1}{3}
\end{pmatrix}, \begin{pmatrix}
  -1 \\
  -2 \\
  -1
\end{pmatrix}
\]

These are lin indep

So a basis is \( \begin{pmatrix}
  \frac{1}{3} \\
  0 \\
  \frac{1}{3}
\end{pmatrix}, \begin{pmatrix}
  -1 \\
  -2 \\
  -1
\end{pmatrix} \)
put cols as in a matrix

\[
\begin{bmatrix}
1 & -1 & -1 & 0 \\
0 & 2 & 4 & 0 \\
3 & \frac{1}{2} & 4 & 0 \\
\end{bmatrix}
\]

\[R_3 = R_3 - 3R_1\]

\[
\begin{bmatrix}
1 & -1 & -1 & 0 \\
0 & 2 & 4 & 0 \\
0 & \frac{1}{2} & 7 & 0 \\
\end{bmatrix}
\]

\[R_2 = R_2 / 2\]

\[
\begin{bmatrix}
1 & -1 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & \frac{1}{2} & 1 & 0 \\
\end{bmatrix}
\]

\[R_3 = R_3 - \frac{1}{2}R_2\]

\[
\begin{bmatrix}
1 & -1 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

no pivot in this col, \Rightarrow \text{in clp}

\[\text{basis is } \begin{pmatrix} \frac{1}{3} \\ -1/2 \\ \frac{1}{3} \end{pmatrix} \]