Mathematics IA
Worked Examples
ALGEBRA: OPTIMISATION AND
CONVEX SETS

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The University of Adelaide.
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The questions on this page have worked solutions and links to videos on
the following pages. Click on the link with each question to go straight to
the relevant page.

Questions

1. See Page 3 for worked solutions.
By drawing sketches, decide if the following sets in $\mathbb{R}^2$ are convex or not.
Also decide whether each set is bounded or unbounded.

(a) $\{(x, y) \in \mathbb{R}^2 | y \leq x^2\}$
(b) $\{(x, y) \in \mathbb{R}^2 | y \geq x^2\}$
(c) $\{(x, y) \in \mathbb{R}^2 | x + y \leq 1\}$
(d) $\{(x, y) \in \mathbb{R}^2 | x + y = 1\}$
(e) $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$
(f) $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \geq 1\}$
(g) $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 0\}$
(h) $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 0\}$

2. See Page 6 for worked solutions.
Sketch the following convex sets and write down any vertices they have.

(a) $\{(x, y) \in \mathbb{R}^2 | x + y \leq 3, y + 2x \geq 2, y \geq 0\}$
(b) $\{(x, y) \in \mathbb{R}^2 | x + y \geq 3, y + 2x \geq 2, y \geq 0\}$
(c) $\{(x, y) \in \mathbb{R}^2 | x + y \geq 3, y + 2x \geq 2, y \leq 0\}$
(d) $\{(x, y) \in \mathbb{R}^2 | x + y \geq 3, 2 \leq y + 2x \leq 4, y \geq 0\}$
(e) $\{(x, y) \in \mathbb{R}^2 | 2 \leq y + 2x \leq 4\}$

3. See Page 9 for worked solutions.
Let $\mathbf{v} = (3, 2)$. Show that the set $\{\mathbf{u} \in \mathbb{R}^2 | \mathbf{u} \cdot \mathbf{v} \leq 9\}$ is a convex set.

4. See Page 10 for worked solutions.
Solve the following problem graphically: Maximise $3x_1 + 5x_2$ subject to
$x_1 + 2x_2 \leq 16, 2x_1 + x_2 \leq 12, x_1 + 2x_2 \geq 2, x_1 \geq 0, x_2 \geq 0$. 
5. **See Page 11 for worked solutions.**

Consider the convex region \( C \) in \( \mathbb{R}^4 \) defined by \( x_1 + x_2 + x_3 \leq 6 \), \( x_2 + x_3 + x_4 \geq 2 \) and \( x_1, x_2, x_3, x_4 \geq 0 \). Use slack variables to find the vertices of \( C \).

6. **See Page 13 for worked solutions.**

A small company manufactures and sells two models of lamps FUNKY and SNAZZY, the profit per lamp being $15 and $10 respectively. They have two workers who make the lamps. Brian assembles the lamps and takes 20 minutes to assemble each FUNKY lamp, and 30 minutes to assemble each SNAZZY lamp. Bruce paints the lamps and takes 20 minutes to paint each FUNKY lamp and 10 minutes to paint each SNAZZY lamp. Brian and Bruce can work at most 100 hours and 80 hours per month respectively. How many of each lamp should be made each month, assuming that every lamp that is made can be sold?

7. **See Page 16 for worked solutions.**

Maximise \( F = 4x - y + 5z \) subject to the constraints \( x + y + z = 7 \), \( 3x + 4y + 6z \leq 36 \), \( 6x - y - 15z \leq 0 \) and \( x, y, z \geq 0 \).

8. **See Page 18 for worked solutions.**

Formulate the following problem as a linear program:

A company producing canned mixed fruit has a stock of 10 000 kg of pears, 12 000 kg of peaches and 8000 kg of cherries. The company produces three fruit mixtures, which it sells in 1-kg cans. The first mixture is half pears and half peaches and sells for $1.20. The second mixture has equal amounts of the three fruits and sells for $1.70. The third mixture is three-quarters peaches and one-quarter cherries and sells for $2.10. How many cans of each mixture should be produced to maximise the return on the current stock?
1. By drawing sketches, decide if the following sets in $\mathbb{R}^2$ are convex or not. Also decide whether each set is bounded or unbounded.

(a) $\{(x, y) \in \mathbb{R}^2 \mid y \leq x^2\}$  
(b) $\{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$

(c) $\{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1\}$  
(d) $\{(x, y) \in \mathbb{R}^2 \mid x + y = 1\}$

(e) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$  
(f) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1\}$

(g) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0\}$  
(h) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 0\}$

A set $C$ is convex if the line segment joining any two points of $C$ is contained entirely in $C$.

A set $C$ is bounded if it is contained in a finite region.
(a) \( \forall (x, y) \in \mathbb{R}^2 \mid y \leq x^2 \frac{3}{2} \) 
\[\text{Not convex. Unbounded}\]

(b) \( \exists (x, y) \in \mathbb{R}^2 \mid y \geq x^2 \frac{3}{2} \) 
\[\text{Convex. Unbounded}\]

(c) \( \forall (x, y) \in \mathbb{R}^2 \mid x + y \leq 1 \frac{3}{2} \) 
\[\text{Convex. Unbounded}\]

(d) \( \exists (x, y) \in \mathbb{R}^2 \mid 2x + y = 1 \frac{3}{2} \) 
\[\text{Convex. Unbounded}\]
(e) \( \exists (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \frac{3}{3} \) convex. bounded.

(f) \( \exists (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1 \frac{3}{3} \) Not convex unbounded.

(g) \( \exists (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0 \frac{3}{3} \) convex. bounded.

(h) \( \exists (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 0 \frac{3}{3} \) EMPTY convex. bounded.
2. Click here to go to question list.
Sketch the following convex sets and write down any vertices they have.

(a) \( \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 3, y + 2x \geq 2, y \geq 0\} \)
(b) \( \{(x, y) \in \mathbb{R}^2 \mid x + y \geq 3, y + 2x \geq 2, y \geq 0\} \)
(c) \( \{(x, y) \in \mathbb{R}^2 \mid x + y \geq 3, y + 2x \geq 2, y \leq 0\} \)
(d) \( \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 3, 2 \leq y + 2x \leq 4, y \geq 0\} \)
(e) \( \{(x, y) \in \mathbb{R}^2 \mid 2 \leq y + 2x \leq 4\} \)

Link to video on YouTube
(b) \{ (x, y) \in \mathbb{R}^2 \mid x + y \geq 3, y + 2x \geq 2, y \geq 0 \}\n
Vertices:
(-1, 4)
(3, 0)

(c) \{ (x, y) \in \mathbb{R}^2 \mid x + y \geq 3, y + 2x \geq 2, y \leq 0 \}\n
Vertex:
(3, 0)
(d) \( \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 3, 2 \leq y + 2x \leq 4, y \geq 0\} \)

Intersection of \( x + y = 3 \)
\& \( y + 2x = 4 \)

\( x = 1 \)
\( 1 + y = 3 \)
\( y = 2 \)

\[ \therefore \text{Vertices:} \]
\[ (1, 2) \]
\[ (-1, 4) \]
\[ (1, 0) \]
\[ (3, 0) \]

(e) \( \{(x, y) \in \mathbb{R}^2 \mid 2 \leq y + 2x \leq 4\} \)

No vertices.
3. Let \( \mathbf{v} = (3, 2) \). Show that the set \( \{ \mathbf{u} \in \mathbb{R}^2 \mid \mathbf{u} \cdot \mathbf{v} \leq 9 \} \) is a convex set.

Let \( \mathbf{v} = (3, 2) \). Show that the set \( \{ \mathbf{u} \in \mathbb{R}^2 \mid \mathbf{u} \cdot \mathbf{v} \leq 9 \} \) is a convex set.

Let \( C = \{ \mathbf{u} \in \mathbb{R}^2 \mid \mathbf{u} \cdot \mathbf{v} \leq 9 \} \).

The set \( C \) is convex if for any \( \mathbf{u}, \mathbf{w} \in C \), the point \( t \mathbf{u} + (1-t) \mathbf{w} \) lies in \( C \) for all \( t \in [0,1] \).

Let \( \mathbf{u}, \mathbf{w} \in C \) and \( t \in [0,1] \).

\[
(t \mathbf{u} + (1-t) \mathbf{w}) \cdot \mathbf{v} = (t \mathbf{u} \cdot \mathbf{v} + (1-t) \mathbf{w} \cdot \mathbf{v}) \leq t \cdot 9 + (1-t) \cdot 9 = 9
\]

\[\therefore (t \mathbf{u} + (1-t) \mathbf{w}) \cdot \mathbf{v} \leq 9\]

\[\therefore C \text{ is convex.}\]
Solve the following problem graphically: Maximise $3x_1 + 5x_2$ subject to $x_1 + 2x_2 \leq 16$, $2x_1 + x_2 \leq 12$, $x_1 + 2x_2 \geq 2$, $x_1 \geq 0$, $x_2 \geq 0$.

Link to video on YouTube
Consider the convex region $C$ in $\mathbb{R}^4$ defined by $x_1 + x_2 + x_3 \leq 6$, $x_2 + x_3 + x_4 \geq 2$ and $x_1, x_2, x_3, x_4 \geq 0$. Use slack variables to find the vertices of $C$.

Add slack variables $s_1, s_2 \geq 0$ so that

\[
\begin{align*}
    x_1 + x_2 + x_3 & \leq 6 \\
    -x_2 - x_3 - x_4 & \leq -2
\end{align*}
\]

\[
\begin{align*}
    x_1 + x_2 + x_3 + s_1 & = 6 \\
    -x_2 - x_3 - x_4 + s_2 & = -2
\end{align*}
\]

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$s_1$</th>
<th>$s_2$</th>
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<tbody>
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<tr>
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Basic solutions are given below:

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$s_1$</th>
<th>$s_2$</th>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>-2</td>
</tr>
</tbody>
</table>

• indicates a feasible solution

The vertices of $C$ are:

(4, 2, 0, 0)
(4, 0, 2, 0)
(6, 0, 0, 2)
(0, 2, 0, 0)
(0, 6, 0, 0)
(0, 0, 2, 0)
(0, 0, 6, 0)
(0, 0, 0, 2)
A small company manufactures and sells two models of lamps FUNKY and SNAZZY, the profit per lamp being $15 and $10 respectively. They have two workers who make the lamps. Brian assembles the lamps and takes 20 minutes to assemble each FUNKY lamp, and 30 minutes to assemble each SNAZZY lamp. Bruce paints the lamps and takes 20 minutes to paint each FUNKY lamp and 10 minutes to paint each SNAZZY lamp. Brian and Bruce can work at most 100 hours and 80 hours per month respectively. How many of each lamp should be made each month, assuming that every lamp that is made can be sold?

NOTE: There is an error in the solution here! I said that 80 hours times 60 minutes was 4000 minutes, when in fact it’s 4800 minutes! Oops! Consequently the answer given is wrong. Sorry! Hopefully you can still learn how the process works until I fix the actual answer.

Let $F$ be the number of FUNKY lamps made per month, and let $S$ be the number of SNAZZY lamps made per month.
<table>
<thead>
<tr>
<th>Lamp</th>
<th>FUNKY</th>
<th>SNAZZY</th>
<th></th>
<th>Total assembly</th>
</tr>
</thead>
<tbody>
<tr>
<td># per month</td>
<td>$15</td>
<td>$10</td>
<td></td>
<td>$\leq 100$ hrs</td>
</tr>
<tr>
<td>Profit per lamp</td>
<td>20 min</td>
<td>30 mins</td>
<td></td>
<td>$20F + 30S$ $\leq 6000$</td>
</tr>
<tr>
<td>Time to assemble per lamp</td>
<td>20 mins</td>
<td>10 mins</td>
<td></td>
<td>$2F + 3S$ $\leq 600$</td>
</tr>
<tr>
<td>Time to paint per lamp</td>
<td>20 F</td>
<td>30 S</td>
<td></td>
<td>$\leq 80$ hrs</td>
</tr>
<tr>
<td>Total assembly time</td>
<td>20 F</td>
<td>10 S</td>
<td></td>
<td>$20F + 10S$ $\leq 4000$</td>
</tr>
<tr>
<td>Total profit</td>
<td>15 F</td>
<td>10 S</td>
<td></td>
<td>$2F + S$ $\leq 400$</td>
</tr>
</tbody>
</table>
MAXIMISE \[ P = 15F + 10S \]
Subject to \[ 2F + 3S \leq 600 \]
\[ 2F + S \leq 400 \]
and \[ S, F \geq 0 \]

Find vertices:
\[ 2F + 3S = 600 \quad (1) \]
\[ 2F + S = 400 \quad (2) \]
\[ 2S = 200 \quad S = 100 \]
\[ 2F + 100 = 400 \quad 2F = 300 \quad F = 150 \]

<table>
<thead>
<tr>
<th>VERTEX</th>
<th>( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>15( \times 0 ) + 10( \times 0 ) = 0</td>
</tr>
<tr>
<td>((0, 200))</td>
<td>15( \times 0 ) + 10( \times 200 ) = 2000</td>
</tr>
<tr>
<td>((200, 0))</td>
<td>15( \times 200 ) + 10( \times 0 ) = 3000</td>
</tr>
<tr>
<td>((150, 100))</td>
<td>15( \times 150 ) + 10( \times 100 ) = 3250</td>
</tr>
</tbody>
</table>

Max profit of \$3250\ occurs when \( F = 150 \) and \( S = 100 \).
So 150 FUNKY lamps and 100 SNAZZY lamps should be made each month.
7. Click here to go to question list.
Maximise $F = 4x - y + 5z$ subject to the constraints $x + y + z = 7$, $3x + 4y + 6z \leq 36$, $6x - y - 15z \leq 0$ and $x, y, z \geq 0$.

Link to video on YouTube

Maximise $F = 4x - y + 5z$ subject to the constraints $x + y + z = 7$, $3x + 4y + 6z \leq 36$, $6x - y - 15z \leq 0$ and $x, y, z \geq 0$.

Add slack variables $s_1, s_2 \geq 0$ so that:

- $x + y + z = 7$
- $3x + 4y + 6z + s_1 = 36$
- $6x - y - 15z + s_2 = 0$

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & | & 7 \\
3 & 4 & 6 & 1 & 0 & | & 36 \\
6 & -1 & -15 & 0 & 1 & | & 0
\end{bmatrix}
\]
### Combinations

\[
\text{combinations} = \binom{5}{3} = \frac{5 \times 4 \times 3}{1 \times 2 \times 1} = 10
\]

### Feasible Solutions

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s1</th>
<th>s2</th>
</tr>
</thead>
<tbody>
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<td>9</td>
<td>0</td>
</tr>
<tr>
<td>-8</td>
<td>15</td>
<td>0</td>
<td>0</td>
<td>63</td>
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<tr>
<td>5</td>
<td>0</td>
<td>2</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
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<td>5</td>
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</tr>
<tr>
<td>7</td>
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<td>0</td>
<td>15</td>
<td>-92</td>
</tr>
<tr>
<td>0</td>
<td>7.5</td>
<td>0</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>63</td>
</tr>
<tr>
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<td>7</td>
<td>0</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>7</td>
<td>-6</td>
<td>105</td>
</tr>
</tbody>
</table>

Feasible solutions marked with *.

### Vertices and Objective Function

- \((1, 6, 0)\) \(F = 4 \times 1 - 6 + 5 \times 0 = -2\)
- \((5, 0, 2)\) \(F = 4 \times 5 - 0 + 5 \times 2 = 30\)
- \((2, 0, 5)\) \(F = 4 \times 2 - 0 + 5 \times 5 = 33\)
- \((7, 0, 0)\) \(F = 4 \times 7 - 0 + 5 \times 0 = 28\)
- \((0, 3, 4)\) \(F = 4 \times 0 - 3 + 5 \times 4 = 17\)
- \((10, 7, 0)\) \(F = 4 \times 0 - 7 + 5 \times 0 = -7\)

Maximum value of \(F\) is 33

which occurs when \(x = 2, y = 0, z = 5\).
Formulate the following problem as a linear program:
A company producing canned mixed fruit has a stock of 10 000 kg of pears, 12 000 kg of peaches and 8000 kg of cherries. The company produces three fruit mixtures, which it sells in 1-kg cans. The first mixture is half pears and half peaches and sells for $1.20. The second mixture has equal amounts of the three fruits and sells for $1.70. The third mixture is three-quarters peaches and one-quarter cherries and sells for $2.10. How many cans of each mixture should be produced to maximise the return on the current stock?

Let \( x_{pp} \) be the number of cans produced with mixture 1.

Let \( x_{ppc} \) be the number of cans produced with mixture 2.

Let \( x_{pc} \) be the number of cans produced with mixture 3.

1 can of mixture 1 has \( \frac{1}{2} \) kg pears and \( \frac{1}{2} \) kg peaches.

\[ x_{pp} \] cans of mixture 1 has \( \frac{1}{2} x_{pp} \) kg pears and \( \frac{1}{2} x_{pp} \) kg of peaches.

1 can of mixture 2 has \( \frac{1}{3} \) kg pears, \( \frac{1}{3} \) kg peaches and \( \frac{1}{3} \) kg cherries.

\[ x_{ppc} \] cans of mixture 2 have \( \frac{1}{3} x_{ppc} \) kg pears, \( \frac{1}{3} x_{ppc} \) kg peaches and \( \frac{1}{3} x_{ppc} \) kg cherries.

1 can of mixture 3 has \( \frac{3}{4} \) kg peaches and \( \frac{1}{4} \) kg cherries.

\[ x_{pc} \] cans of mixture 3 have \( \frac{3}{4} x_{pc} \) kg peaches and \( \frac{1}{4} x_{pc} \) kg cherries.
Total kg pears = \( \frac{1}{2} x_{pp} + \frac{1}{3} x_{ppc} \leq 10 000 \)
Total kg peaches = \( \frac{1}{2} x_{pp} + \frac{1}{3} x_{ppc} + \frac{3}{4} x_{pc} \leq 12 000 \)
Total kg cherries = \( \frac{1}{3} x_{ppc} + \frac{1}{4} x_{pc} \leq 8 000 \)

Total return = 1.20 \( x_{pp} \)
\[ + 1.70 x_{ppc} + 2.10 x_{pc} \]

The linear program is:

Maximise \( R = 1.20 x_{pp} + 1.70 x_{ppc} + 2.10 x_{pc} \)
Subject to
\( \frac{1}{2} x_{pp} + \frac{1}{3} x_{ppc} \leq 10 000 \)
\( \frac{1}{2} x_{pp} + \frac{1}{3} x_{ppc} + \frac{3}{4} x_{pc} \leq 12 000 \)
\( \frac{1}{3} x_{ppc} + \frac{1}{4} x_{pc} \leq 8 000 \)
and \( x_{pp}, x_{ppc}, x_{pc} \geq 0. \)