Mathematics IA
Worked Examples
ALGEBRA: THE VECTOR SPACE $\mathbb{R}^n$

Produced by the Maths Learning Centre,
The University of Adelaide.

May 1, 2013

The questions on this page have worked solutions and links to videos on the following pages. Click on the link with each question to go straight to the relevant page.

Questions

1. See Page 3 for worked solutions.
   Write $(0, -26, -9)$ as a linear combination of $(5, 3, 7)$ and $(2, -4, 1)$. Show that $(1, 3, 5)$ cannot be written as a linear combination of these two vectors.

2. See Page 5 for worked solutions.
   Determine which of the following lists of vectors are linearly independent:
   (a) $\{(1, 2, 0, -1, 5), (0, 0, 0, 0, 0), (15, 6, 2, -17, 0)\}$
   (b) $\{(5, 7)\}$
   (c) $\{(3, 1, 4), (-2, 2, 5), (3, 0, 4), (2, -1, -2)\}$
   (d) $\{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)\}$
   (e) $\{(1, 2, 3), (3, 2, 1), (2, 1, 3)\}$

3. See Page 6 for worked solutions.
   For what value(s) of $a$ are the following vectors linearly independent: $(1, 5, -2), (0, 6, a)$ and $(3, 13, -3)$?

4. See Page 7 for worked solutions.
   Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{Z}\}$, $V = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2yz = 6\}$ and $W = \{(a + b, a^2, a - b) \mid a, b \in \mathbb{R}\}$. For each of the following vectors, decide which (if any) of $U, V$ or $W$ the vector is in:
   (a) $(0, 0, 0)$  (b) $(-2, 1, 3)$  (c) $(\frac{2}{3}, \frac{1}{3}, 0)$
   (d) $(6, 0, 0)$  (e) $(0, \sqrt{3}, \sqrt{3})$
5. Let \( U = \{ x \in \mathbb{R}^4 \mid \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \\ 5 & 0 & 10 & 0 \end{bmatrix} x = 0 \} \)
and \( V = \text{span}\{ (1, \frac{1}{2}, 0, -1), (-2, 0, 1, 0) \} \).
Give four vectors in each of \( U \) and \( V \).

6. Show that the set \( \{ (x, -3x) \mid x \in \mathbb{R} \} \) is a subspace of \( \mathbb{R}^2 \).

7. Let \( W = \{ (x, y, z) \in \mathbb{R}^3 \mid 3x = 2y \} \). Prove that \( W \) is a subspace of \( \mathbb{R}^3 \).

8. Determine which of the following sets are subspaces of \( \mathbb{R}^3 \):
(a) \( \{ (x, y, z) \in \mathbb{R}^3 \mid x + 2y = 3z \} \)
(b) \( \{ (a, b, c) \in \mathbb{R}^3 \mid a, b, c \geq 0 \} \)
(c) \( \{ (0, \alpha, \alpha + 1) \mid \alpha \in \mathbb{R} \} \)
(d) \( \{ a(1, 0, -2) + b(5, 5, 7) \mid a, b \in \mathbb{R} \} \)
(e) \( \{ (k, m, n) \in \mathbb{R}^3 \mid k^2 = n^2 \} \)

9. Find a basis for the subspace \( V = \{ x \in \mathbb{R}^3 \mid Ax = 0 \} \), where \( A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 1 & -\frac{1}{2} \end{bmatrix} \). Also, write down the dimension of \( V \).

10. Find a basis for the subspace \( U = \{ (x, y, t) \in \mathbb{R}^4 \mid 3x + y - 7t = 0 \} \) and write down the dimension of \( U \).

11. Find a basis for, and write down the dimension of, the subspace \( W = \text{span}\{ (1, \frac{2}{3}, 0, -5), (-3, -2, 0, 15), (3, 0, -1, \frac{1}{2}), (\frac{7}{2}, \frac{1}{3}, -1, -2) \} \).

12. The subspace \( V \) of \( \mathbb{R}^3 \) contains the vectors \((0, 2, -1)\) and \((\sqrt{3}, 0, 0)\), but does not contain the vector \((0, -1, -\frac{1}{2})\). Find the dimension of and a basis for \( V \).

13. If \( \{ u, v \} \) is a basis for the subspace \( U \), show that \( \{ u + 2v, -3v \} \) is also a basis for \( U \).
1. Write \((0, -26, -9)\) as a linear combination of \((5, 3, 7)\) and \((2, -4, 1)\). Show that \((1, 3, 5)\) cannot be written as a linear combination of these two vectors.

\[
\begin{align*}
\text{Want to write} & \quad (0, -26, -9) = a (5, 3, 7) + b (2, -4, 1) \\
& = (5a, 3a, 7a) + (2b, -4b, 6b) \\
& = (5a + 2b, 3a - 4b, 7a + 6b) \\
5a + 2b &= 0 \\
3a - 4b &= -26 \\
7a + 6b &= -9
\end{align*}
\]

\[
\begin{bmatrix}
5 & 2 & 0 \\
3 & -4 & -26 \\
7 & 1 & -9
\end{bmatrix}
\]

\[
\begin{bmatrix}
5 & 2 & 0 \\
3 & -4 & -26 \\
-9 & 0 & 18 \\
7 & 1 & -9 \\
31 & 0 & -62 \\
1 & 0 & -2
\end{bmatrix}
\]

\[
\begin{align*}
\text{new R2, R3} &= R3, R2 \\
\text{new R1} &= R1 - 2R2 \\
\text{new R1} &= -\frac{1}{4} R1 \\
\text{new R2} &= R2 - 7R1 \\
\text{new R3} &= R3 - 31R1
\end{align*}
\]

\[
(0, -26, -9) = -2 (5, 3, 7) + 5 (2, -4, 1).
\]
Want to write $(1,3,5) = 9(5,3,7) + 6(2,-4,1)$

\[
\begin{array}{ccc|c}
5 & 2 & 1 \\
3 & -4 & 3 \\
7 & 1 & 5 \\
\hline
5 & 2 & 1 \\
7 & 1 & 5 \\
3 & -4 & 3 \\
\hline
-9 & 0 & -9 \\
7 & 1 & 5 \\
31 & 0 & 23 \\
\hline
1 & 0 & 1 \\
7 & 1 & 5 \\
31 & 0 & 23 \\
\hline
1 & 0 & 1 \\
0 & 1 & -2 \\
0 & 0 & -8 \\
\end{array}
\]

New $R_2, R_3 = R_3, R_2$

New $R_1 = R_1 - 2R_2$

New $R_3 = R_3 + 9R_2$

New $R_1 = -\frac{1}{9} R_1$

New $R_2 = R_2 - 7R_1$

New $R_3 = R_3 - 31R_1$

No solutions for $a$ and $b$.

\[\text{\therefore } (1,3,5) \text{ is not a lin. comb. of } (5,3,7) \text{ and } (2,-4,1).\]
2. **Click here to go to question list.**

Determine which of the following lists of vectors are linearly independent:

(a) \( \{(1, 2, 0, -1, 5), (0, 0, 0, 0), (15, 6, 2, -17, 0)\} \)
(b) \( \{(5, 7)\} \)
(c) \( \{(3, 1, 4), (-2, 2, 5), (3, 0, 4), (2, -1, -2)\} \)
(d) \( \{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)\} \)
(e) \( \{(1, 2, 3), (3, 2, 1), (2, 1, 3)\} \)

**Link to video on YouTube**

---

The vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) are linearly independent if \( a_1 \mathbf{v}_1 + \cdots + a_r \mathbf{v}_r = \mathbf{0} \) has only the solution \( a_i = \cdots = a_r = 0 \).

(a) \textbf{Lin. DEPENDENT} since one is the zero vector.

(b) \textbf{Lin. INDEPENDENT} since single nonzero vector.

(c) \textbf{Lin. DEPENDENT} since more than 3 vectors in \( \mathbb{R}^3 \).

(d) \textbf{Lin. INDEPENDENT} since none is a lin. comb. of those before it in the list.

(e) Solve:

\[
\begin{array}{ccc|c|c}
1 & 3 & 2 & 0 & \text{new } R2 = R2 - 2R1 \\
2 & 2 & 1 & 0 & \text{new } R3 = R3 - 3R1 \\
3 & 1 & 3 & 0 & \\
\hline
1 & -3 & 2 & 0 & \text{new } R2 = -R2 \\
0 & -1 & -3 & 0 & \\
0 & -8 & -3 & 0 & \\
\hline
1 & 3 & 2 & 0 & \text{new } R3 = R3 + 8R1 \\
0 & 1 & 3 & 0 & \\
0 & 0 & 21 & 0 & \\
\end{array}
\]

No free variables. So \textbf{lin. INDEPENDENT}. 

---

5
3. Click here to go to question list.
For what value(s) of \( a \) are the following vectors linearly independent:
\((1, 5, -2), (0, 6, a)\) and \((3, 13, -3)\)?

Link to video on YouTube

\[
\begin{align*}
a_1 (1, 5, -2) + a_2 (0, 6, a) + a_3 (3, 13, -3) &= 0 \\
\begin{bmatrix}
1 & 0 & 3 \\
5 & 6 & 13 \\
-2 & a & -3
\end{bmatrix} & \quad \text{new} R_2 = R_2 - 5R_1 \\
\begin{bmatrix}
1 & 0 & 3 \\
0 & 6 & -2 \\
0 & a & 3
\end{bmatrix} & \quad \text{new} R_3 = R_3 + 2R_1 \\
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & -\frac{1}{3} \\
0 & a & 3
\end{bmatrix} & \quad \text{new} R_2 = \frac{1}{6} R_2 \\
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & -\frac{1}{3} \\
0 & 0 & 3 + \frac{a}{3}
\end{bmatrix} & \quad \text{new} R_3 = R_3 - aR_2
\end{align*}
\]

Free variable when \( 3 + \frac{a}{3} = 0 \).

\[
3 + \frac{a}{3} = 0 \\
\frac{a}{3} = -3 \\
a = -9
\]

\[\therefore \text{Lin dependent when } a = -9.\]
\[\text{Lin. indep. when } a \neq -9.\]
4. Let \( U = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{Z}\} \), \( V = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2yz = 6\} \) and \( W = \{(a+b, a^2, a-b) \mid a, b \in \mathbb{R}\} \). For each of the following vectors, decide which (if any) of \( U \), \( V \) or \( W \) the vector is in:

- (a) \((0, 0, 0)\)
- (b) \((-2, 1, 3)\)
- (c) \((\frac{2}{3}, \frac{1}{3}, 0)\)
- (d) \((6, 0, 0)\)
- (e) \((0, \sqrt{3}, \sqrt{3})\)

<table>
<thead>
<tr>
<th>Vector</th>
<th>In U?</th>
<th>In V?</th>
<th>In W?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ((0, 0, 0))</td>
<td>((0, 0, 0) \in U ) since (0 \in \mathbb{Z}).</td>
<td>((0, 0, 0) \notin U ) since (0 + 2 \times 0 \neq 6).</td>
<td>((0, 0, 0) \in W ) since ((0, 0, 0) = (a+b, a^2, a-b)) for (a=0, b=0).</td>
</tr>
</tbody>
</table>
| (b) \((-2, 1, 3)\) | \((-2, 1, 3) \in U \) since \(-2, 1, 3 \in \mathbb{Z}\). | \((-2, 1, 3) \notin U \) since \(-2 + 2 \times 1 \times 3 = 4 \neq 6\). | \((-2, 1, 3) \notin W \) since \((-2, 1, 3) = (a+b, a^2, a-b)\):
  | \begin{align*}
  a+b &= -2 \\
  a^2 &= 1 \\
  a-b &= 3
  \end{align*}
  \Rightarrow \ a = \pm 1, a = \frac{1}{2}
  \text{No solution for } a. |
| (c) \((\frac{2}{3}, \frac{1}{3}, 0)\) |  |  |  |
| (d) \((6, 0, 0)\) |  |  |  |
| (e) \((0, \sqrt{3}, \sqrt{3})\) |  |  |  |

\( \therefore (-2, 1, 3) \notin W. \)
(c) \((\frac{2}{3}, \frac{1}{9}, 0) \notin \mathbb{W} \) since \(\frac{2}{3} \notin \mathbb{Z}\).

\((\frac{2}{3}, \frac{1}{9}, 0) \notin \mathbb{V} \) since \(\frac{2}{3} + 2 \cdot \frac{1}{9} \cdot 0 = \frac{2}{3} \notin \mathbb{N}\).

If \((\frac{2}{3}, \frac{1}{9}, 0) = (a+b, a^2, a-b)\),

Then \(a^2 = \frac{1}{9}\) and \(a+b = 2\).

\(a = \frac{1}{3}\) \(a-b = 0\)

\(\Rightarrow a = \frac{2}{3}\)

\(b = \frac{1}{3}\)

\((\frac{2}{3}, \frac{1}{9}, 0) \in \mathbb{W} \) since

\((\frac{2}{3}, \frac{1}{9}, 0) = (a+b, a^2, a-b)\)

for \(a = \frac{1}{3}, b = \frac{1}{3}\).

(d) \((6, 0, 0) \notin \mathbb{W} \) since \(6, 0 \notin \mathbb{Z}\).

\((6, 0, 0) \in \mathbb{V} \) since \(6 + 2 \cdot 0 \cdot 0 = 6\).

If \((6, 0, 0) = (a+b, a^2, a-b)\),

Then \(a^2 = 6\) and \(a+b = 0\).

\(a = 0\) \(a-b = 0\)

\(\Rightarrow 2a = 6\)

\(a = 3\)

No solution for \(a\).

\((6, 0, 0) \notin \mathbb{W} \).

(e) \((0, \sqrt{3}, \sqrt{3}) \notin \mathbb{W} \) since \(\sqrt{3} \notin \mathbb{Z}\).

\((0, \sqrt{3}, \sqrt{3}) \in \mathbb{V} \) since \(0 + 2 \cdot \sqrt{3} \cdot \sqrt{3} = 6\).

If \((0, \sqrt{3}, \sqrt{3}) = (a+b, a^2, a-b)\),

Then \(a^2 = \sqrt{3}\) and \(a+b = 0\).

\(a = \pm \sqrt{3}\)

\(a-b = \sqrt{3}\)

\(\Rightarrow 2a = 2\sqrt{3}\)

\(a = \sqrt{3}\)

No solution for \(a\).

\((0, \sqrt{3}, \sqrt{3}) \notin \mathbb{W} \).
Let $U = \{ x \in \mathbb{R}^4 \mid \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \\ 5 & 0 & 10 & 0 \end{bmatrix} x = 0 \}$

and $V = \text{span}\{(1,\frac{1}{2},0,-1),(-2,0,1,0)\}$.

Give four vectors in each of $U$ and $V$.

$V$ is the set of linear combinations of
$(1,\frac{1}{2},0,-1)$ and $(-2,0,1,0)$.

$(0,0,0,0) \in V \quad (-2,0,1,0) \in V$

$(1,\frac{1}{2},0,-1) \in V \quad (-1,\frac{1}{2},1,-1) \in V$

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 0 & 0 & -1 \\
5 & 0 & 10 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
so
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\in U.
\]
Show that the set \( \{(x, -3x) \mid x \in \mathbb{R}\} \) is a subspace of \( \mathbb{R}^2 \).

The set \( V \) is a subspace when:

1. \( 0 \in V \)
2. If \( u, v \in V \) then \( u + v \in V \)
3. If \( u \in V, \ c \in \mathbb{R} \) then \( cu \in V \)

Let \( V = \{(x, -3x) \mid x \in \mathbb{R}\} \)

1. \( (0, 0) = (0, -3 \cdot 0) \) so \( (0, 0) \in V \).
2. Let \( u, v \in V \)
   
   So \( u = (x, -3x) \) and \( v = (y, -3y) \) for some \( x, y \in \mathbb{R} \).
   
   \[ u + v = (x + y, -3x - 3y) \]
   
   \[ = (x + y, -3(x + y)) \]
   
   \[ \therefore u + v \in V \]

3. Let \( u \in V \) and \( c \in \mathbb{R} \)
   
   So \( u = (x, -3x) \) for some \( x \in \mathbb{R} \).
   
   \[ cu = (cx, c(-3x)) \]
   
   \[ = (cx, -3(cx)) \]
   
   \[ \therefore cu \in V. \]

Therefore, \( V \) is a subspace of \( \mathbb{R}^2 \).
Let \( W = \{(x, y, z) \in \mathbb{R}^3 \mid 3x = 2y\} \). Prove that \( W \) is a subspace of \( \mathbb{R}^3 \).

The set \( W \) is a subspace when:

1. \( \mathbf{0} \in W \)
2. If \( \mathbf{u}, \mathbf{v} \in W \) then \( \mathbf{u} + \mathbf{v} \in W \)
3. If \( \mathbf{u} \in W \), \( c \in \mathbb{R} \) then \( c\mathbf{u} \in W \).

1. \( 3\times0 = 2\times0 \) so \((0, 0, 0) \in W \)
2. Let \( \mathbf{u}, \mathbf{v} \in W \)

Write \( \mathbf{u} = (a, b, d) \) and \( \mathbf{v} = (e, f, g) \).

Then \( 3a = 2b \) and \( 3e = 2f \).

Now \( \mathbf{u} + \mathbf{v} = (a+e, b+f, d+g) \)

\[
3(a+e) = 3a + 3e \\
= 2b + 2f \\
= 2(b+f) \\
\therefore \mathbf{u} + \mathbf{v} \in W.
\]

3. Let \( \mathbf{u} \in W \) and \( c \in \mathbb{R} \).

Write \( \mathbf{u} = (a, b, d) \)

Then \( 3a = 2b \).

Now \( c\mathbf{u} = (ca, cb, cd) \)

From equation above,

\[
c(3a) = c(2b) \\
3(c a) = 2(c b) \\
\therefore c\mathbf{u} \in W
\]

Therefore, \( W \) is a subspace of \( \mathbb{R}^3 \).
8. Click here to go to question list.
Determine which of the following sets are subspaces of $\mathbb{R}^3$: 

(a) $\{(x, y, z) \in \mathbb{R}^3 \mid x + 2y = 3z\}$
(b) $\{(a, b, c) \in \mathbb{R}^3 \mid a, b, c \geq 0\}$
(c) $\{(0, \alpha, \alpha + 1) \mid \alpha \in \mathbb{R}\}$
(d) $\{a(1, 0, -2) + b(5, 5, 7) \mid a, b \in \mathbb{R}\}$
(e) $\{(k, m, n) \in \mathbb{R}^3 \mid k^2 = n^2\}$

**A set $V$ is a subspace of $\mathbb{R}^n$ when:**
1. $0 \in V$.
2. If $u, v \in V$ then $u + v \in V$.
3. If $u \in V$ and $c \in \mathbb{R}$ then $cu \in V$.

If $V$ is the set of solutions to homogeneous linear equations, then $V$ is a subspace.
If $V$ is the span of some vectors, then $V$ is a subspace.

**(a) $\exists (x, y, z) \in \mathbb{R}^3 \mid x + 2y = 3z$**

$= \exists (x, y, z) \in \mathbb{R}^3 \mid x + 2y - 3z = 0$ 

This is a subspace since it is the set of solutions to a homogeneous linear equation.

**(b) $\exists (a, b, c) \in \mathbb{R}^3 \mid a, b, c \geq 0$**

The vector $(1, 1, 1)$ is in the set but $(-1)(1, 1, 1) = (-1, -1, -1)$ is not. So the set is not a subspace since it is not closed under scalar multiplication.
(c) \( \exists (0, \alpha, \alpha+1) \mid \alpha \in \mathbb{R}^3 \)

If \((0,0,0) = (0, \alpha, \alpha+1)\)

then \(\alpha = 0\) and \(\alpha + 1 = 0\).

No solution for \(\alpha\), so \((0,0,0)\) is not in the set.

\(\therefore\) The set is not a subspace.

(d) \( \exists a(1,0,-2) + b(5,5,7) \mid a, b \in \mathbb{R}^3 \)

= \(\text{Span} \{ (1,0,-2), (5,5,7) \} \)

\(\therefore\) The set is a subspace.

(e) \( \exists (k,m,n) \in \mathbb{R}^3 \mid k^2 = n^2 \)

\((1,0,1)\) is in the set since \(1^2 = 1^2\)

\((1,0,-1)\) is in the set since \(1^2 = (-1)^2\)

but \((1,0,1) + (1,0,-1) = (2,0,0)\)

is not in the set since \(2^2 \neq 0^2\).

So the set is not a subspace since it is not closed under vector addition.
9. Find a basis for the subspace \( V = \{x \in \mathbb{R}^3 \mid Ax = 0\} \), where 
\[
A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 1 & -\frac{1}{2} \end{bmatrix}.
\]
Also, write down the dimension of \( V \).

**A basis for \( V \) is a set of vectors in \( V \) which**

1. **Span \( V \)**
2. are lin. indep.

**Solve \( Ax = 0 \):**

\[
\begin{array}{c|c}
2 & 0 & 5 \\
1 & 1 & -\frac{1}{2} \\
\hline
1 & 0 & \frac{1}{2} \\
1 & 1 & -\frac{1}{2} \\
\hline
1 & 0 & \frac{1}{2} \\
0 & 1 & -3
\end{array}
\]

new \( R1 = \frac{1}{2} R1 \)

new \( R2 = R2 - R1 \)

**Let \( x_3 = t \) for \( t \in \mathbb{R} \).**

\[
x_1 = -\frac{5}{2} t \\
x_2 = 3 t
\]

So \( x = \begin{pmatrix} -\frac{5}{2} t \\ 3 t \\ t \end{pmatrix} = t \begin{pmatrix} -\frac{5}{2} \\ 3 \\ 1 \end{pmatrix} \) for \( t \in \mathbb{R} \).

\[
\begin{pmatrix} -\frac{5}{2} \\ 3 \\ 1 \end{pmatrix}
\]

**is a basis for \( V \).**

The dim. of \( V \) is the number of basis vectors.

\[
\therefore \text{dim of } V = 1.
\]
10. Find a basis for the subspace \( U = \{ (x, y, z, t) \in \mathbb{R}^4 \mid 3x + y - 7t = 0 \} \) and write down the dimension of \( U \).

A basis for \( U \) is a set of vectors in \( U \) which:
1. Span \( U \)
2. are lin indep.

The dim. of \( U \) is the number of vectors in a basis for \( U \).

Solve \( 3x + y - 7t = 0 \)
\[ y = -3x + 7t \]
So \( (x, y, z, t) = (x, -3x + 7t, z, t) \)
\[ = x(1, -3, 0, 0) + z(0, 0, 1, 0) + t(0, 7, 0, 1) \]

So a basis for \( U \) is \( \{ (1, -3, 0, 0), (0, 0, 1, 0), (0, 7, 0, 1) \} \)
\[ \therefore \text{ dim of } U \text{ is } 3. \]
Find a basis for, and write down the dimension of, the subspace
\( W = \text{span}\{(1, \frac{2}{3}, 0, -5), (-3, -2, 0, 15), (3, 0, -1, \frac{1}{2}), (\frac{7}{2}, \frac{1}{3}, -1, -2)\} \).

A basis for \( W \) is a set of vectors in \( W \) which:
1. span \( W \)
2. are lin. indep

The dim. of \( W \) is the number of vectors in a basis for \( W \).

\[
\begin{bmatrix}
1 & -3 & 3 & \frac{7}{2} \\
\frac{2}{3} & -2 & 0 & \frac{1}{3} \\
0 & 0 & -1 & -1 \\
-\frac{5}{2} & 15 & \frac{1}{2} & -2 \\
\end{bmatrix}
\]

\[
\begin{align*}
\text{new } R_1 &= 2R_1 \\
\text{new } R_2 &= 3R_2 \\
\text{new } R_3 &= -R_3 \\
\text{new } R_4 &= 2R_4 \\
\text{new } R_2 &= R_2 - R_1 \\
\text{new } R_4 &= R_4 + 5R_1 \\
\text{new } R_2, R_3 &= R_3, R_2 \\
\text{new } R_1 &= R_1 - 6R_2 \\
\text{new } R_3 &= R_3 + 6R_2 \\
\text{new } R_4 &= R_4 - 3R_2 \\
\end{align*}
\]

Columns that become pivot columns are the basis

So a basis for \( W \) is \( \left\{ (1, \frac{2}{3}, 0, -5), (3, 0, -1, \frac{1}{2}) \right\} \)

\( \therefore \) dim of \( W \) is 2.
The subspace $V$ of $\mathbb{R}^3$ contains the vectors $(0, 2, -1)$ and $(\sqrt{3}, 0, 0)$, but does not contain the vector $(0, -1, -\frac{1}{2})$. Find the dimension of and a basis for $V$. 

\[
\begin{align*}
&V \text{ contains two lin. indep. vectors.} \\
&\text{So dim. of } V \geq 2. \\
&V \text{ does not contain all vectors in } \mathbb{R}^3. \\
&\text{So dim of } V \neq 3. \\
&\therefore \text{ dim of } V = 2. \\
&\text{The vectors } (0, 2, -1) \text{ and } (\sqrt{3}, 0, 0) \text{ are lin. indep. and span a 2 dim subspace.} \\
&\text{But } V \text{ is a 2-dim. subspace so they span } V. \\
&\therefore \text{ A basis for } V \text{ is } \{(0, 2, -1), (\sqrt{3}, 0, 0)\}.
\end{align*}
\]
13. If \{\mathbf{u}, \mathbf{v}\} is a basis for the subspace \(U\), show that \{\mathbf{u} + 2\mathbf{v}, -3\mathbf{v}\} is also a basis for \(U\).

The set \{\mathbf{u}, \mathbf{v}\} is a basis for \(U\), so \(\mathbf{u}, \mathbf{v}\) span \(U\) and \(\mathbf{u}, \mathbf{v}\) are lin. indep.

Since 2 vectors in a basis, \dim(U) = 2.

Want to show \(\mathbf{u} + 2\mathbf{v}\) and \(-3\mathbf{v}\) span \(U\).

Let \(\mathbf{w} \in U\). Want to show \(\mathbf{w}\) is a lin. comb. of \(\mathbf{u} + 2\mathbf{v}\) and \(-3\mathbf{v}\).

Write \(\mathbf{w} = a\mathbf{u} + b\mathbf{v}\) for \(a, b \in \mathbb{R}\).

Now \(\mathbf{v} = -\frac{1}{3}(-3\mathbf{v})\)
\(\mathbf{u} = \mathbf{u} + 2\mathbf{v} + \frac{2}{3}(-3\mathbf{v})\)

So \(\mathbf{w} = a\left((\mathbf{u} + 2\mathbf{v}) + \frac{2}{3}(-3\mathbf{v})\right) + b\left(-\frac{1}{3}(-3\mathbf{v})\right)\)
which is a lin. comb. of \(\mathbf{u} + 2\mathbf{v}\) and \(-3\mathbf{v}\).

\(\mathbf{u} + 2\mathbf{v}\) and \(-3\mathbf{v}\) span \(U\).

Want to show \(\mathbf{u} + 2\mathbf{v}\) and \(-3\mathbf{v}\) are lin. indep.

Solve \(a(\mathbf{u} + 2\mathbf{v}) + b(-3\mathbf{v}) = \mathbf{0}\)
\(a\mathbf{u} + 2a\mathbf{v} - 3b\mathbf{v} = \mathbf{0}\)
\(a\mathbf{u} + (2a - 3b)\mathbf{v} = \mathbf{0}\)

\(a = 0\) and \(2a - 3b = 0\)
\(-3b = 0\)
\(b = 0\)

So \(a = 0, b = 0\).
\(\mathbf{u} + 2\mathbf{v}\) and \(-3\mathbf{v}\) are lin. indep.

Therefore \{\mathbf{u} + 2\mathbf{v}, -3\mathbf{v}\} is a basis for \(U\).