Vectors

\[ (5, -1, 2/3) \in \mathbb{R}^3 \]

Represented as:
* point
* arrow
* column matrix

\[
\begin{bmatrix}
5 \\
-1 \\
2/3
\end{bmatrix}
\]

Adding vectors

\[
\begin{array}{c}
\text{Start point} \\
(1, 5)
\end{array} +
\begin{array}{c}
\text{Direction} \\
(3, 2)
\end{array} =
\begin{array}{c}
\text{New point} \\
(4, 7)
\end{array}
\]

\[
(x, y) = (1, 5) + t(3, 2)
\]

(new point) (start point) (direction)
SETS

RULE for being in/out

4 WAYS TO MAKE SETS

1. List all of the vectors
   RULE: In the list ⇒ In the set
   Not in list ⇒ Not in the set

   \[ S = \{ (1,0,3), (0,0,1), (1,1,2) \} \]

   • \((1,0,3) \in S\) since it is in the list
   • \((1,1,1) \notin S\) since not in the list.

2. Show how to make the vectors
   RULE: If you can make it this way ⇒ In the set
   If you cannot ⇒ Not in the set

   \[ S = \{ \frac{1}{2} \cdot (1,2,3) + \epsilon (1,0,1) \mid \epsilon \in \mathbb{R} \} \]

   • \((2,2,4) \in S\) because \((2,2,4) = (1,2,3) + 1\cdot(1,0,1)\)
1. Is \((2, 2, 3) \in S\)?

   If it was, then
   \[
   (2, 2, 3) = (1, 2, 5) + 6(1, 0, 1)
   \]

   1st coord: \(2 = 1 + 6 \Rightarrow \varepsilon = 1\)
   2nd coord: \(2 = 2 + 0 \varepsilon\)
   3rd coord: \(3 = 3 + \varepsilon \Rightarrow \varepsilon = 0\)

   No solution for \(\varepsilon\)

   \[\therefore \quad (2, 2, 3) \notin S\]

3. Have an equation to satisfy

   \[\text{eg} \quad S = \{ (x, y, z) \in \mathbb{R}^3 \mid x + 2y - z = 15 \}\]

   **RULE**
   If point satisfies equation \(\Rightarrow\) in the set
   If point doesn't satisfy \(\Rightarrow\) not in the set.

   • Is \((1, 1, 3) \in S\)?

     \[1 + 2 \times 1 - 3 = 1 + 2 - 3 = 0 \checkmark \quad \text{NO}\]

   • Is \((1, 1, 2) \in S\)?

     \[1 + 2 \times 1 - 2 = 1 \checkmark \quad \text{YES}\]
• Find a point in $S$.
  
  Pick $x = 0$
  
  $y = 0$

  Then $0 + 2x0 - z = 1$
  
  $z = -1$

  $\therefore (0, 0, -1) \in S$.

  Pick $x = 1$
  
  $y = 2$

  Then $1 + 2x2 - z = 1$
  
  $5 - z = 1$
  
  $z = 4$

  $\therefore (1, 2, 4) \in S$

4. Construct from other sets

$S$ is a set, $T$ is a set

New set $S \cup T$ "intersection"

Rule: if in both $S \cup T \Rightarrow \in$ set

not in both $\Rightarrow \not\in$ set
New set $S U T$  "union"

Rule: $\begin{align*}
    & \text{If in } S \text{ or } T \text{ or both} \Rightarrow \text{in set} \\
    & \text{If in neither} \Rightarrow \text{not in set}.
\end{align*}$

- $S U T$

New set $S \setminus T$ "without"

Rule: $\begin{align*}
    & \text{If in } S \text{ but not in } T \Rightarrow \text{in new set} \\
    & \text{If not in } S \text{ or in } T \Rightarrow \text{not in set}.
\end{align*}$
With vectors I can make
linear combinations

if \( \mathbf{y} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \ldots + a_k \mathbf{v}_k \)

then \( \mathbf{y} \) is a lin. comb. of \( \mathbf{v}_1, \ldots, \mathbf{v}_k \).

About some vectors I can say
if they are linearly independent

**DEFINITION** \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) are lin. indep.

when \( a_1 \mathbf{v}_1 + \ldots + a_k \mathbf{v}_k = \mathbf{0} \)

is only possible if \( \begin{cases} 
\text{or has only trivial soln} \end{cases} \)

\( a_1 = 0, \ldots, a_k = 0 \).

Things that tell you they are lin. indep.

- None is a lin. comb. of any others
- Matrix with vectors as cols \( [\mathbf{v}_1 \ldots \mathbf{v}_k] \)
  row reduces to get a pivot in every column.
Things that tell you they are NOT lin indep

- It's possible to write
  
  \[ a_1v_1 + \ldots + a_kv_k = 0 \]

  with at least one \( a_i \neq 0 \).

- Any one vector is a lin. comb. of any others.

- matrix with vectors as cols \([v_1 \ldots v_k]\)
  row reduces to get a col with no pivot

- one of the vectors is \( \mathbf{0} \)

- More vectors than \# coords
**Subspaces**

**Definition:**
A subspace of \( \mathbb{R}^n \) is a set of vectors \( S \subseteq \mathbb{R}^n \) where:
1. \( \mathbf{0} \in S \)
2. If \( \mathbf{v} \in S \) and \( k \in \mathbb{R} \) then \( k\mathbf{v} \in S \)
3. If \( \mathbf{u}, \mathbf{v} \in S \) then \( \mathbf{u} + \mathbf{v} \in S \)

**Things that are:**
- \( \mathbb{R}^n \)
- Line thru origin
- Plane thru origin
- \( \mathbb{R}^n \)

**Things that aren't:**
- Things with gaps

- \( \mathbb{R}^n \)
- \( \mathbb{R}^n \) with \( (1,2,3) \) and \( (4,0,1) \)
- Plane
- Span \( \mathbb{R} \) \( (1,2,3) \)
- Span \( \mathbb{R} \) \( (1,2,3), (4,0,1) \)
THINGS THAT ARE

- Span of any vectors is a subspace.
- \( \exists \) vectors \( \text{linear eqns } = 0 \) is a subspace.
- Solutions to \( Ax = 0 \)
  
  \[ \begin{align*}
  & \exists (x, y) \in \mathbb{R}^2 | x + y = 0 \\
  & \exists (x, y) \in \mathbb{R}^2 | 2x + y = 0, x = 0
  \end{align*} \]

  \[ \begin{align*}
  & \exists \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 | \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
  \end{align*} \]

  \[ \exists (x, y) \in \mathbb{R}^2 | x^2 + y = 0 \]
  
  because only solution is \( (x, y) = (0, 0) \)

THINGS THAT AREN'T

- Lines not thru origin
- Plane not thru origin

- \( \exists \) \((1, 2, 3) + (1, 1, 0) \) \( \not\in \mathbb{R}^3 \)
  - Not all combos

- \( \exists \) \((1, 2) + 5(1, 1) \) \( \not\in \mathbb{R}^2 \)
  - Not all combos

- \( \exists \) \((x, y) | x^2 + y = 0 \)
  - Not linear

- \( \exists \) \((x, y) | ay + y = 0 \)
  - Not linear

- \( \exists \) \((x, y) | x + 3y = 1 \)
  - Not \( = 0 \).
\[ E \cap \{(r, r+5, t-35) \mid r, s \in \mathbb{R}^3 \} = \exists \ r(1,1,0) \]
\[ + s(0,1,-3) \]
\[ + t(0,0,1) \mid r,s,t \in \mathbb{R}^3 \]
\[ = \pm \exists (1,1,0), (0,1,-3), (0,0,1) \mathbb{R}^3 \]
So it's a subspace of \( \mathbb{R}^3 \)

\[ f(\mathbb{R}^3) \mid \varphi(x-y) = 0 \]
\[ 5^{-5} = 5 \]
\[ (0,1) \in S \]
\[ \text{since } 0 \times (0-1) = 0 \checkmark \]
\[ (1,1) \in S \]
\[ \text{since } 1 \times (1-1) = 0 \checkmark \]
but \( (0,1) + (1,1) = (1,2) \notin S \)
\[ \text{since } 1 \times (1-2) = -1 \neq 0 \]
\[ \Rightarrow S \text{ not a subspace.} \]
\[ x^2 + y = 0 \]

For \( x, y \in \mathbb{R} \)

\[ (1,-1) \in S \]

since \( 1^2 + (-1) = 0 \)

but

\[ (2, -1/2) \notin S \]

since \( 2^2 + (-1/2) = 2 \neq 0 \)

\[ \therefore \text{S not a subspace.} \]

\( \exists (x, y, z) = (1, 1, 1) \in \mathbb{R}^3 \)

\[ (1, 1, 1) \in S \]

since \( 1, 1, 1 \in \mathbb{Z} \)

but

\[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \notin S \]

\[ \therefore \text{S not a subspace} \]

"(Come back to proofs later)"
Basis

- Info about a subspace.
- Idea is to write subspace as span. (most efficiently)
  
  If S is subspace
  write $S = \text{span } \{ \text{vectors } \}$

- Coord. Axes
**Definition:** Given a subspace \( S \), a basis for \( S \) is a list of vectors in \( S \) which
1. span \( S \)
2. are lin. indep.

**Def of Dimension**
3. Vectors in basis = dimension of \( S \)

Any two of 1, 2, 3 make it a basis

**Advice:** To find basis, first rewrite as span.

eg \( S = \text{span} \{ (1, 1, 2), (0, 1, 3), (1, 2, 5) \} \)

1st method
\[(1, 2, 5) = (1, 1, 2) + (0, 1, 3)\]
so lin. comb. of others
so \( S = \text{span} \{ (1, 1, 2), (0, 1, 3) \} \)


End method

Find if vectors lin. indep

\[
\begin{array}{cc}
1 & 0 & 1 \\
1 & 1 & 2 \\
2 & 3 & 5 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 3 & 3 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{array}
\]

\[\begin{align*}
R_2 &= R_2 - R_1 \\
R_3 &= R_3 - 2R_1 \\
R_3 &= R_3 - 3R_2
\end{align*}\]

No pivot in 3rd col. 
Choose cols with pivots

so \( E(1,1,2) \), \((0,1,3)\) is a basis for \( S \).

\[
E(S) = \{(x,y,z,w) | x + z = 0, 2z - w = 0\}
\]

Find all points in \( S \).
So solve \[
\begin{align*}
x + z &= 0 \\
2z - w &= 0
\end{align*}\]
\[ x 
 y 
 z \\
1 
0 
1 
0 \\
0 
0 
2 
-1 \\
0 
0 
-2 
1 \\
\hline
1 
0 
1 
0 \\
0 
0 
-1 
1 \]

\[ \mathbf{y}, \mathbf{z} \text{ are free.} \]

so let \[ y = s \]
\[ z = t \]

1st row: \[ x = -t \]
2nd row: \[ w = 2t \]

\[
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} =
\begin{pmatrix}
-t \\
s \\
t \\
2t
\end{pmatrix} =
t \begin{pmatrix}
-1 \\
0 \\
0 \\
0
\end{pmatrix} +
s \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}
\]

\[ S = \mathbb{E} \begin{pmatrix}
(-1, 0, 1, 2) \\
(0, 1, 0, 0)
\end{pmatrix} \mathbf{t}, \mathbf{s} \in \mathbb{R}^2 \]

\[ = \mathbb{E} \begin{pmatrix}
(-1, 0, 1, 2) \\
(0, 1, 0, 0)
\end{pmatrix} \]

\[ \therefore \mathbb{E} \begin{pmatrix}
(-1, 0, 1, 2) \\
(0, 1, 0, 0)
\end{pmatrix} \text{ is a basis for } S. \]
\[ e \in S \implies (x, y, z) = (x, y, 2) \]
\[ = 2(1, 0, 1) + y(0, 1, 0) \]
\[ \therefore S = \text{Span } \{ (1, 0, 1), (0, 1, 0) \} \]
\[ \{ (1, 0, 1), (0, 1, 0) \} \text{ is a basis for } S. \]

**Approach 1:**

Find all points in $S$.

Solve equation.

$x - e = 0$

$x = e$

$(x, y, z) = (e, y, 2)$

$= e(1, 0, 1) + y(0, 1, 0)$

$\therefore S = \text{Span } \{ (1, 0, 1), (0, 1, 0) \}$

$\{ (1, 0, 1), (0, 1, 0) \}$ is a basis for $S$.

**Approach 2:**

Find all points in $S$.

Solve equation.

$x - e = 0$

$\begin{bmatrix}
2 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & -1
\end{bmatrix}$

$y, z$ are free

$y = 5$

$z = 6 \implies x = 6$
\[
\begin{pmatrix}
2 \\
3 \\
4
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix} + s \begin{pmatrix}
1 \\
2 \\
0
\end{pmatrix}
\]

\[
\therefore \exists (1,0,1), (0,1,0) \text{ is a basis for } S.
\]

**Approach 3:**

S is a plane (1 linear eqn in \( \mathbb{R}^3 \)).

\[\therefore S \text{ has dim 2.}\]

\[(1,0,1) \in S \text{ since } 1-1=0\]

\[(1,1,1) \in S \text{ since } 1-1=0\]

These vectors are lin indep and \# vectors = dim(S).

\[\therefore \exists (1,0,1), (1,1,1) \text{ is a basis for } S.\]
PROVING WITH SUBSPACES

Def of subspace:

1. \( 0 \in S \)
2. If \( x \in S \) then \( x + \mathbb{R} \in S \)
3. If \( y, z \in S \) then \( y + z \in S \)

Ex: Prove that \( S = \{ (x, y, z) \mid x + y = 0, x - 2y + 3z = 0 \} \) is a subspace of \( \mathbb{R}^3 \).

Rule for being in \( S \)
is \((x, y, z)\) must satisfy

\[ x + y = 0 \quad \text{and} \quad x - 2y + 3z = 0 \]

1. \( 0 + 0 = 0 \) \( \checkmark \)
2. \( 0 - 2 \cdot 0 + 3 \cdot 0 = 0 \) \( \checkmark \)

\( \therefore 0 \in S \)
2. Suppose \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \)

Let \( \mathbf{v} = (a, b, c) \).

Know that \( a + b = 0 \) & \( a - 2b + 3c = 0 \).

Now \( \mathbf{k v} = (ka, kb, kc) \)

\[
(ka) + (kb) = k(a+b) \quad \frac{(ka) - (kb) - 3(kc)}{kx0} = 0 \\
= kx0 \\
= kx0 \\
= 0
\]

\( \therefore k \mathbf{v} \in \mathbb{S} \)

3. Suppose \( \mathbf{u}, \mathbf{v} \in \mathbb{S} \)

Let \( \mathbf{u} = (a, b, c) \) & \( \mathbf{v} = (d, e, f) \)

\( a + b = 0, \quad a - 2b + 3c = 0 \) & \( d + e = 0, \quad d - 2e + 3f = 0 \)

Now \( \mathbf{u} + \mathbf{v} = (a + d, b + e, c + f) \)

\[
(a+d) + (b+e) \quad \frac{(a+d) - 2(b+e) + 3(c+f)}{0 + 0} = 0 \\
= a + b + d + e \quad = a - 2b + 3c + d - 2e + 3f \\
= 0 + 0 \quad = 0 + 0 \\
= 0 \\
= 0
\]

\( \therefore \mathbf{u} + \mathbf{v} \in \mathbb{S} \).

Since all conditions satisfied,

\( \mathbb{S} \) is a subspace \( \mathbb{R}^3 \).
Prove $S = \left\{ (r+s, -r, s+r) \mid s, r \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$

$S = \left\{ r(1, -1, 1) + s(1, 0, 1) \mid s, r \in \mathbb{R} \right\}$

1. $0 = 0(1, -1, 1) + 0(1, 0, 1)$

$i.e. 0 \in S$

2. Suppose $v \in S$, $k \in \mathbb{R}$

So $v = a(1, -1, 1) + b(1, 0, 1)$ for some $a, b \in \mathbb{R}$

$k v = k(a(1, -1, 1) + b(1, 0, 1))$

$= (ka)(1, -1, 1) + (kb)(1, 0, 1)$

$i.e. k v \in S$

3. Suppose $u, v \in S$

So $u = a(1, -1, 1) + b(1, 0, 1)$

$v = c(1, -1, 1) + d(1, 0, 1)$

$u + v = (a+c)(1, -1, 1) + (b+d)(1, 0, 1)$

$i.e. u + v \in S$
Let $A$ be an $m \times n$ matrix and let $S = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$.

Prove $S$ is a subspace of $\mathbb{R}^n$.

1. $A0 = 0 \checkmark$
   
   $\therefore 0 \in S$

2. Suppose $v \in S$, $k \in \mathbb{R}$
   
   So $Av = 0$
   
   $A(kv) = kAv = k0 = 0$
   
   $\therefore kv \in S$

3. Suppose $u, v \in S$
   
   So $Au = 0$ and $Av = 0$
   
   $A(u + v) = Au + Av$
   
   $= 0 + 0 = 0$
   
   $\therefore u + v \in S$

So $S$ is a subspace of $\mathbb{R}^n$. 