PARAMETRIC CURVES

What ARE curves?

PICTURE:

Follow a point as it moves.

Can have a parameter $t$ (or $x$ or $\theta$) to tell you where the point is up to.
Also

Any set of points \((x, y)\) ought to have an equation to tell if point is in the set.

By equation \(x^2 + y^2 = 1\)

Is \((1, 3)\) in this set?

\[1^2 + 3^2 = 10 \neq 1 \quad \text{No}\]

Is \((1, 0)\) in this set?

\[1^2 + 0^2 = 1 \quad \text{Yes}\]
CONVERTING BETWEEN FORMS

**Example Curve:** \((t+1, t^2) \quad t \in \mathbb{R}\)

\[
\begin{align*}
x &= t + 1 \\
y &= t^2
\end{align*}
\]

Check:
- Sub in \((1, 0)\): \((1-1)^2 = 0^2 = 0\) ✓
- Sub in \((2, 1)\): \((2-1)^2 = 1^2 = 1\) ✓
- Sub in \((0, 0)\): \((0-1)^2 = 1 
eq 0\) ✗

Cartesian equation is \(y = (x-1)^2\)
\[ y = (2 \cos \theta, 3 \sin \theta), \quad \theta \in [0, 2\pi] \]

\[ \theta = 0: \quad (2 \cos 0, 3 \sin 0) = (2, 0) \]

\[ \theta = \frac{\pi}{2}: \quad (2 \cos \frac{\pi}{2}, 3 \sin \frac{\pi}{2}) = (0, 3) \]

\[(x, y)\text{ is on curve,}\]

\[ \text{then } x = 2 \cos \theta \Rightarrow \cos \theta = \frac{x}{2} \]
\[ y = 3 \sin \theta \Rightarrow \sin \theta = \frac{y}{3} \]

\[ \text{know } (\cos \theta)^2 + (\sin \theta)^2 = 1 \]

\[ \text{sub in:} \]
\[ \left( \frac{x}{2} \right)^2 + \left( \frac{y}{3} \right)^2 = 1 \]

\[ \text{check: } (2, 0): \quad \left( \frac{2}{2} \right)^2 + \left( \frac{0}{3} \right)^2 = 1 \]
Convert from cartesian to parametric.

\[ y = x^3 \]

Points on curve are

\[ (x, x^3) \quad \text{for } x \in \mathbb{R} \]

\[ 2x + y - 2 = 0 \]

**Option 1:**

\[ 2x + y - 2 = 0 \]

\[ y = 2 - 2x \]

Points are: \( (x, 2 - 2x), x \in \mathbb{R} \)

**Option 2:**

\[ 2x + y - 2 = 0 \]

\[ 2x = 2 - y \]

Let \( t = 2x \) and \( t = 2 - y \)

\[ t = x \quad y = 2 - t \]

\( (\frac{t}{2}, 2 - t) \) for \( t \in \mathbb{R} \)
SPECIFIC KIND OF CURVE: BÉZIER CURVE

Beginning point: \( P_0 \)
Ending point: \( P_n \)

Other points it doesn't go through but used to define curve:
\( P_1, \ldots, P_{n-1} \)

Points on curve:

(Two points) \( (1-t)P_0 + tP_1 \)

(Three points) \( (1-t)^2P_0 + 2(1-t)tP_1 + t^2P_2 \)

(Four points) \( (1-t)^3P_0 + 3(1-t)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3 \)

\[
\begin{array}{ccccccc}
& & & 1 \\
& & 1 & 1 \\
& 1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]
A \left( \frac{1}{3}, t+2 \right) \text{ for } t \in (0, 3]

\begin{align*}
t = 0: & \text{ Not defined} \\
t = 3: & \left( \frac{1}{3}, 5 \right) \text{ endpoint.}
\end{align*}

**Tangents**

Slope of \( t + 1 \) = Slope of curve
\[ \Delta \text{Point} = (\Delta x, \Delta y) \]
\[ = \left( \left( \frac{dx}{dt} \right) \Delta t, \left( \frac{dy}{dt} \right) \Delta t \right) \]

Slope of \( y = f(t) \) = \[ \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \]

\text{how quickly y-coordinate moves}

\text{how quickly x-coordinate moves}

\[ (t^2, t^3+1) \text{ for } t \in [0,5] \]

Slope of curve when \( t = 2 \).

Point on curve: \((x, y) = (4, 9)\)

\[ x = t^2 \quad y = t^3 + 1 \]
\[ \frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 3t^2 \]

\[ \text{slope} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2}{2t} = \frac{3}{2}t \]

At point where \( t = 2 \):

\[ \text{slope} = \frac{3}{2} \times 2 = 3 \]

Equation of tangent:

\[ y = mx + c \]

for some \( m, c \)
Slope is 3 so

\[ y = 3x + C \]

passes thru \((4, q)\)

so \[ q = 3 \cdot 4 + C \]

\[ q = 12 + C \]

\[-3 = C \]

So eqn of tgt is \[ y = 3x - 3. \]

Specific tangents

Vertical tgts happen when \[ \frac{dx}{dt} = 0 \]

Horizontal tgts happen when \[ \frac{dy}{dt} = 0 \]