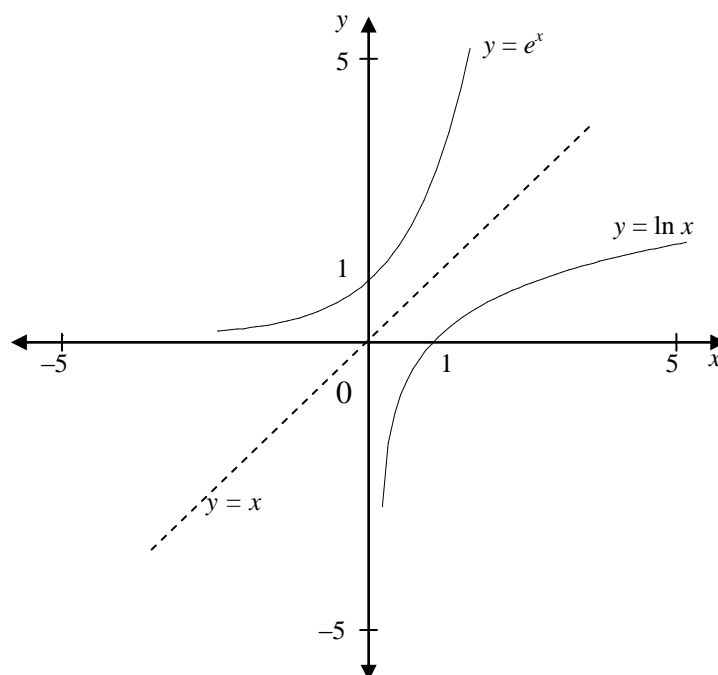


Topic 8

Logarithms



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This Topic...

This topic introduces logarithms and exponential equations. Logarithms are used to solve exponential equations, and so are used along with exponential functions when modelling growth and decay. The logarithmic function is an important mathematical function and you will meet it again if you study calculus. It is used in many areas of advanced applicable mathematics and in statistics.

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— Prerequisites —

You will need to have a scientific calculator.

We also assume you have read Topic 7: Exponential Functions

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— Contents —

Chapter 1 Exponential Equations and Logarithms.

Chapter 2 Logarithm Functions.

Chapter 3 Growth & Decay II.

Appendices

A. Answers

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1

Exponential Equations & Logarithms

1.1 Exponential Equations

An *exponential equation* is an equation like $2^x = 16$ or $10^x = 3.267$. The first equation has answer $x = 4$, but the second equation is much harder to solve. An exponential equation has the general form $a^x = b$, where the base a and the number b are known and we wish to find the unknown index x .

These type of equations arise frequently in growth and decay problems.

Example

If the population of a town is initially 1000 and is growing at a constant rate of 2% per year, then its population $P(t)$ after t years is given by

$$P(t) = 1000e^{0.02t}.$$

To find how long it takes for the population to reach 2000, we need to solve the equation

$$1000e^{0.02t} = 2000 \quad \text{or} \quad e^{0.02t} = 2.$$

This equation can be solved once we know that $e^x = 2$ has solution $x = 0.6931$ (check this), because then we would have $0.02t = 0.6931 \Rightarrow t = 34.66$ years.

A *logarithm* is just an index. We use *log* as an abbreviation for the word logarithm.

To find the value of a logarithm we need to solve an exponential equation.

Example

(a) The solution of $2^x = 8$ is $x = 3$.

We can write this in logarithm notation as $\log_2 8 = 3$ 'log of 8 to base 2 is 3'

(b) $x = 5$ is the solution of $2^x = 32$.

We can write this using logarithms as $\log_2 32 = 5$ 'log of 32 to base 2 is 5'

(c) $10^2 = 100$.

We can write this as $\log_{10} 100 = 2$ 'log of 100 to base 10 is 2'

Problems 1.1

1. Rewrite the following in logarithm notation:

- (a) $2^4 = 16$ (b) $2^{10} = 1024$ (c) $2^{-1} = 0.5$ (d) $2^0 = 1$
 (e) $3^4 = 81$ (f) $4^5 = 1024$ (g) $4^{-0.5} = 0.5$ (h) $10^0 = 1$

2. Find the values of the following logarithms:

- (a) $\log_2 4$ (b) $\log_2 16$ (c) $\log_2 1$ (d) $\log_2 0.5$
 (e) $\log_4 4$ (f) $\log_4 16$ (g) $\log_3 1$ (h) $\log_{10} 0.01$

1.2 Logarithms

We use can logarithms to solve exponential equations:

The solution of $a^x = b$ is $x = \log_a b$

For example, the solution of $e^x = 2$ is $x = \log_e 2$. To find the value of this logarithm, we need to use a calculator: $\log_e 2 = 0.6931$.

Note

Logarithms were invented and used for solving exponential equations by the Scottish baron John Napier (1550 – 1617). In those days, before electronic calculators, all logarithms to bases 10 and e were listed in tables. As you can imagine, it was a herculean task constructing these tables of numbers, but the task was made easier because of some properties of logarithms that you'll see later.

Logarithms to the base 10 are called *common logarithms*. Over their long history, two notations developed: $\log b$ (read as 'log b') and $\log_{10} b$. These both represent the logarithm of b to the base 10.

Logarithms to the base e are used in research, and are called *natural logarithms*. Once again two notations developed over a long period of time: $\ln b$ and $\log b$. These both represent the logarithm of b to the base e .

In most modern texts, including this one, $\log b$ refers to the common logarithm of b (base 10), and $\ln b$ refers to the natural logarithm of b (base e).

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Example

- To find the value of $\ln 2$ on a calculator: use $\boxed{\ln} \ 2 \ \boxed{=}$ $\Rightarrow 0.6931$
- To find the value of $\log 2$ on a calculator: use $\boxed{\log} \ 2 \ \boxed{=}$ $\Rightarrow 0.3010$

Example

Solve the equation $e^x = 3$

Answer

$$\begin{aligned}e^x &= 3 \\x &= \ln 3 \\&= 1.099\end{aligned}$$

Check by
calculating $e^{1.099}$

Example

Solve the equation $10^x = 3$

Answer

$$\begin{aligned}10^x &= 3 \\x &= \log 3 \\&= 0.4771\end{aligned}$$

Check by
calculating $10^{0.4771}$

Example

Solve the equation $e^{2x} = 3$

Answer

$$\begin{aligned}e^{2x} &= 3 \\2x &= \ln 3 \\x &= \frac{\ln 3}{2} \\&= 0.5493\end{aligned}$$

First find the index
using logs, then find x .

Example

Solve the equation $e^{-2x} = 3$

Answer

$$\begin{aligned}e^{-2x} &= 3 \\-2x &= \ln 3 \\x &= \frac{\ln 3}{(-2)} \\&= -0.5493\end{aligned}$$

Find the index first, then x .

Example

Solve the equation $2e^{4x} = 3$

Answer

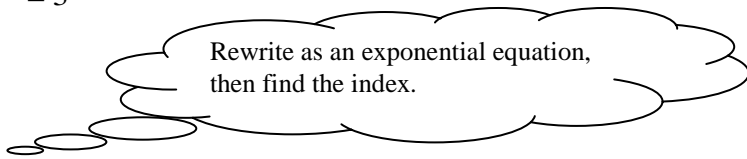
$$2e^{4x} = 3$$

$$e^{4x} = 1.5$$

$$4x = \ln 1.5$$

$$x = \frac{\ln 1.5}{4}$$

$$= 0.1013$$



Rewrite as an exponential equation,
then find the index.

Example

Solve the equation $(e^x)^2 = 3$

Answer 1

Take square roots of both sides:

$$(e^x)^2 = 3$$

$$e^x = \sqrt{3}$$

$$x = \ln(\sqrt{3})$$

$$= 0.5493$$

Answer 2

Use index rule 3 from *Topic 7*: $(a^n)^m = a^{nm}$

$$(e^x)^2 = 3$$

$$e^{2x} = 3$$

$$2x = \ln 3$$

$$x = \frac{\ln 3}{2}$$

$$= 0.5493$$

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Problems 1.2

1. Solve the following equations:

(a) $e^x = 1$ (b) $e^x = 2$ (c) $e^x = 3$ (d) $e^x = 2^2$ (e) $e^x = 3^2$
(f) $10^x = 1$ (g) $10^x = 2$ (h) $10^x = 2^2$ (i) $e^x = 2^{-2}$ (j) $e^x = \sqrt{3}$

2. Solve the following equations:

(a) $e^x - 2 = 0$ (b) $2e^x - 5 = 0$ (c) $3e^x = 2 + 2e^x$ (d) $\frac{1}{1+e^x} = 0.75$
(e) $10^{2x} = 1$ (f) $10^{2x} = 2$ (g) $10^{2x} = 2^2$ (h) $\frac{1}{1+10^{2x}} = 0.75$

3. Use the index rules to solve the following equations.

(a) $e^x \cdot e^{2x} \cdot e^{3x} = 30$ (b) $(e^{2x})^3 = 10$ (c) $\sqrt{e^{5x}} = 2.7$ (d) $(e^{3x})^4 = 10e^{2x}$

4. If the population of the earth was 6.5 billion in 2000, and was increasing at 2% per year,

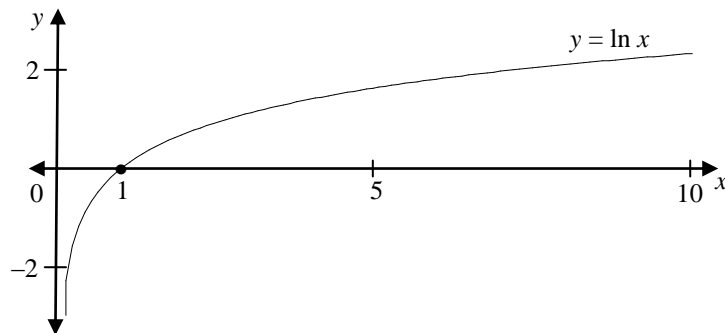
- (a) what would the population be in 20 years time?
(b) when would the population reach 15 billion?

2

Logarithm Functions

2.1 The Natural Logarithm Function and its Graph

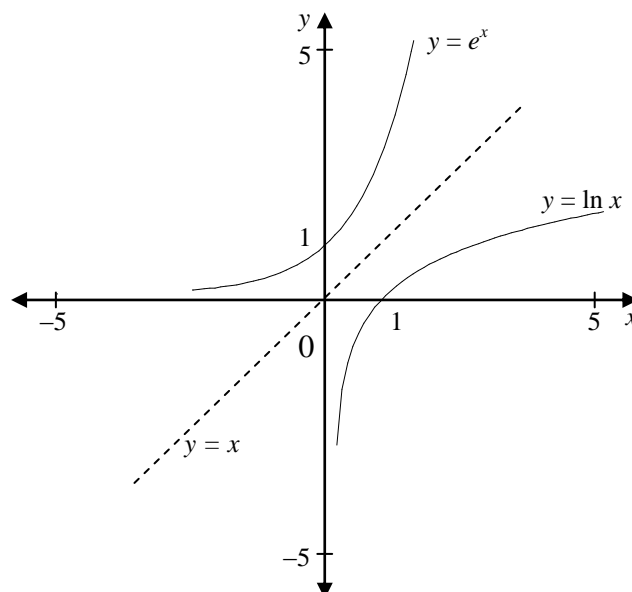
The equation $e^y = x$ has a solution $y = \ln x$ for every positive value of x , so the natural domain of $\ln x$ is $\{x : x > 0\}$. The graph is show below.



We won't be using this graph, but it shows some of the properties of the logarithm function.

- the x -intercept is $(1, 0)$ because $e^0 = 1 \Rightarrow \ln 1 = 0$.
- the graph is above the x -axis when $x > 1$
- the graph is below the x -axis when $0 < x < 1$.

It's not all obvious but the graphs of $y = e^x$ and $y = \ln x$ are related. The diagram below shows that one is the reflection of the other across the line $y = x$.



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The reason for this is that $e^r = s \Leftrightarrow r = \ln s$.

However, $e^r = s$ means that the point (r, s) is on the curve $y = e^x$, and $r = \ln s$ means that the point (s, r) is on the curve $y = \ln x$. This means that (r, s) is on the curve $y = e^x$ whenever (s, r) is on the curve $y = \ln x$. For example, $(0, 1)$ is on $y = e^x$ and $(1, 0)$ is on $y = \ln x$. This is another way of saying that the curves $y = e^x$ and $y = \ln x$ are symmetric about the line $y = x$. Check this with a few points!

2.2 Properties of the natural logarithm

The natural logarithm has three special properties:

If u and v are any positive numbers, and n is any index, then

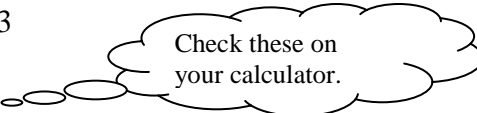
- $\ln uv = \ln u + \ln v$
- $\ln\left(\frac{u}{v}\right) = \ln u - \ln v$
- $\ln u^n = n \ln u$

Example

(a) $\ln 6 = \ln (2 \times 3) = \ln 2 + \ln 3$

(b) $\ln (6/3) = \ln 3 - \ln 2$

(c) $\ln 2^3 = 3 \ln 2$



Check these on your calculator.

The properties above are very useful for solving general exponential equations.

Example

Solve $2^x = 3.99$

Answer

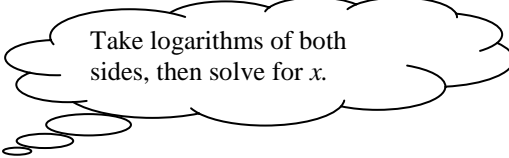
$$2^x = 3.99$$

$$\ln(2^x) = \ln 3.99$$

$$x \ln 2 = \ln 3.99$$

$$x = \frac{\ln 3.99}{\ln 2}$$

$$= 1.996$$



Take logarithms of both sides, then solve for x .

Problems 2.2

1. Express each of the following as a single logarithm.

- (a) $\ln 6 + \ln 3$ (b) $\ln 56 - \ln 7$ (c) $2 \ln 5$ (d) $\ln 5 + 2 \ln 2$
 (e) $2 \ln 10 - 2 \ln 2$ (f) $1 - \ln(2e)$ (g) $2 + \ln 3$ (h) 0

2. Solve the following equations

- (a) $\ln(x-1) - \ln x = \ln 0.5$ (b) $\ln(x-1) + \ln x = \ln 6$

3. Solve the following exponential equations

- (a) $2^x = 4.1$ (b) $3^x = 9.1$ (c) $2 \times 3^x = 53$ (d) $41 - 10 \times 3^x = 23$

2.3 Properties of the common logarithm

The graph of the common logarithm function $y = \log x$ is similar to the graph of the natural logarithm $y = \ln x$. It is the reflection of the graph of the graph of $y = 10^x$ across the line $y = x$.

The common logarithm has similar properties to the natural logarithm:

If u and v are any positive numbers, and n is any index, then

- $\log uv = \log u + \log v$
- $\log\left(\frac{u}{v}\right) = \log u - \log v$
- $\log u^n = n \log u$

The common logarithm is not used very much in mathematics, mainly because you can solve exponential functions using \ln , even if the base is 10.

3

Growth & Decay II

3.1 Growth and Decay

We can solve any exponential equation using logarithms.

A population that is growing at a constant rate will have

$$P(t) = P(0) e^{rt}$$

members after time t , where

- $P(0)$ is the initial population and
- r is the constant growth rate per unit time.

Example

The population of China was 850,000,000 in 1990 and was growing at the rate of 4% per year. When did the population reach 1,000,000,000?

Answer

The initial population (in 1990) is $P(0) = 850,000,000$.

The growth rate is 0.04 per year.

The model is $P(t) = P(0)e^{0.04t}$, we need to find when $P(t) = 1,000,000,000$

Put $850,000,000e^{0.04t} = 1,000,000,000$

$$e^{0.04t} = \frac{1,000,000,000}{850,000,000}$$

$$= \frac{100}{85}$$

$$0.04t = \ln\left(\frac{100}{85}\right)$$

$$t = \frac{\ln\left(\frac{100}{85}\right)}{0.04}$$

$$= 4.1 \text{ (2 sf)}$$

Rewrite 4% as a decimal number.

Rewrite as an exponential equation.

Take logarithms of both sides.

The population reached 1,000,000,000 in 1994.

Example

The population of China was 850,000,000 in 1990 and reached 1,000,000,000 in 1994. If it grew at a constant growth rate, what is this growth rate?

Answer

The initial population (in 1990) is $P(0) = 850,000,000$.

The population in 1994 is $P(4) = 1,000,000,000$

The growth model is $P(t) = P(0)e^{rt}$, we need to find the rate r .

Put $850,000,000e^{r \times 4} = 1,000,000,000$

$$e^{4r} = \frac{1,000,000,000}{850,000,000}$$

$$= \frac{100}{85}$$

$$4r = \ln\left(\frac{100}{85}\right)$$

$$r = \frac{\ln\left(\frac{100}{85}\right)}{4}$$

$$= 0.041 \text{ (2 sf)}$$

The population grew at a constant rate of 4.1% per year.

Problems 3.1

1. The population of the earth is now 4 billion, and is increasing at a constant rate of 2% per year. If it continues to grow at this rate, when will the population reach 5 billion?
2. The population in Britain in 1600 is believed to have been about 5 million. Three hundred fifty years later the population had increased to 50 million. What was the average percentage growth during that period? (Assume that the growth is constant.)
3. Radioactive radium decays at a rate of 0.044% per year. How many years does it take 10 gm of radium to decay so that only 8 gm of radium remains? How long will it take for a further 2 gm of radium to decay?

4. The ratio of radioactive isotope C^{14} to the regular isotope C^{12} of carbon is fixed in the atmosphere. Living matter breathes in air, and this same ratio of C^{14} to C^{12} is found in all its cells. When it dies and can no take breath in air, the amount of C^{14} begins to decay at a constant rate of 1.24×10^{-4} . This is the principle of carbon dating.

If the amount $Q(t)$ of radioactive carbon C^{14} in a human bone is measured to be 58% of the amount $Q(0)$ found in the atmosphere, how old is the bone?

3.2 Doubling Time and Half-life.

The growth rate of a population is usually quite small, and its hard to imagine how fast a polulation is actually growing. Because of this the *doubling time* is often quoted instead. The doubling time of a population is the time it takes to double.

Example

If a town had an initial population of 1000 and a doubling time of 30 years, then the population would be 2000 after 30 years, 4000 after another 30 years, 8000 after a further 30 years (ie. after a total of 90 years from the beginning).

A population growing at a constant growth rate r will double in size every $\frac{\ln 2}{r}$ units of time

Reason

The population growth model is $P(t) = P(0)e^{rt}$, where $P(0)$ is the starting time.

If the population doubles, then $P(t) = 2P(0)$ and we have the equation,

$$2P(0) = P(0)e^{rt}$$

$$e^{rt} = 2$$

$$rt = \ln 2$$

$$t = \frac{\ln 2}{r}$$

so the population doubles after $\frac{\ln 2}{r}$ units of time.

Example

If a town had an initial population of 1000 and grew at a constant rate, if the population doubled every 30 years, what is the growth rate?

Answer

$$\text{Put } \frac{\ln 2}{r} = 30,$$

$$\text{then } \ln 2 = 30r$$

$$r = \frac{\ln 2}{30} = 0.023$$

Similarly, the decay rate of a quantity is usually quite small, and its hard to imagine how fast it decays. Because of this the *half-life* is often quoted instead. The *half-life* of a quantity is the time it takes to halve.

A quantity which is decaying at a constant rate will have the amount

$$Q(t) = Q(0) e^{-rt}$$

left after time t , where

- $Q(0)$ is the initial amount and
- r is the constant *decay* rate per unit time.

Example

One kilogram of a radioactive isotope of iodine has a half life of 7.967 years. After this period of time only 500 gm will remain. After a further 7.967 years only 250 gm (ie. half of 500gm) will remain.

A population decaying at a constant decay rate r will be reduced by half every $\frac{\ln 2}{r}$ units of time

Reason

The decay model is $Q(t) = Q(0)e^{-rt}$, where $Q(0)$ is the initial amount.

If the quantity halves, then $Q(t) = 0.5Q(0)$ and we have:

$$0.5Q(0) = Q(0)e^{-rt}$$

$$0.5 = e^{-rt}$$

$$e^{rt} = 2$$

$$rt = \ln 2$$

$$t = \frac{\ln 2}{r}$$

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Example

One kilogram of a radioactive isotope of iodine decays at a rate of 8.7% per day. What is its half life

Answer

$$t = \frac{\ln 2}{r},$$
$$t = \frac{\ln 2}{0.087}$$
$$= 7.967 \text{ years}$$

The half life is 7.967 years.

Problems 3.2

1. A culture of bacteria doubles in weight every 24 hours. If it originally weighed 10 g, what would be its weight after 18 hours?
2. The half-life of radium is 1590 years. If 10 g of radium is left after 1000 years, how much was there originally?

A

Appendix: Answers

Section 1.1

- 1(a) $\log_2 16 = 4$ (b) $\log_2 1024 = 10$ (c) $\log_2 0.5 = -1$ (d) $\log_2 1 = 0$
(e) $\log_3 81 = 4$ (f) $\log_4 1024 = 5$ (g) $\log_4 0.5 = -0.5$ (h) $\log_{10} 1 = 0$
- 2(a) 2 (b) 4 (c) 0 (d) -1
(e) 1 (f) 2 (g) 0 (h) -2

Section 1.2

- 1 (a) 0 (b) 0.6931 (c) 1.099 (d) 1.386 (e) 2.197
(f) 0 (g) 0.301 (h) 0.6021 (i) -1.386 (j) 0.5493
- 2 (a) 0.6931 (b) 0.9163 (c) 0.6931 (d) -1.099
(e) 0 (f) 0.1505 (g) 0.3010 (h) -0.2386
- 3 (a) 0.5669 (b) 0.3838 (c) 0.3973 (d) 0.2303
- 4 (a) 9.70 billion (b) 41.81 years

Section 2.2

- 1 (a) $\ln 18$ (b) $\ln 8$ (c) $\ln 25$ (d) $\ln 20$
(e) $\ln 25$ (f) $\ln 0.5$ (g) $\ln 3e^2$ (h) $\ln 1$
- 2(a) 2 (b) 3 Note: $x = -2$ is not a solution as only positive numbers are in the domain of $\ln x$.
- 3(a) 2.036 (b) 2.010 (c) 2.983 (d) 0.5350

Section 3.1

- 11.2 years
- 0.6579%
- 507.1 years, 653.8 years
- 4392.7 years

Section 3.2

- 16.82 g
- 15.47 g