MathsTrack

(NOTE Feb 2013: This is the old version of MathsTrack. New books will be created during 2013 and 2014)

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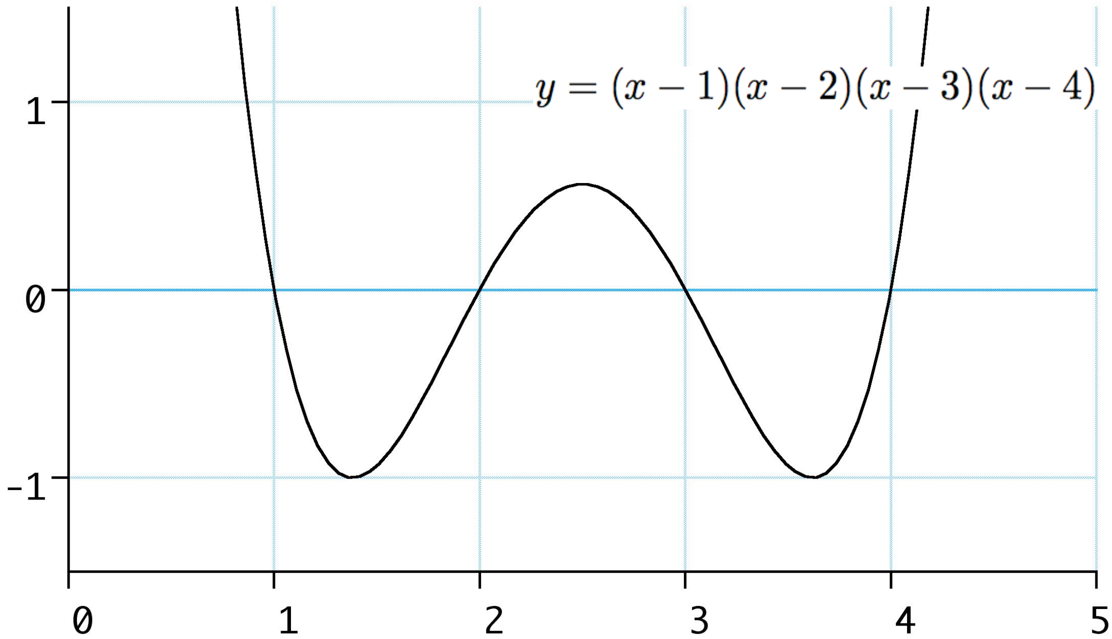
Topic 1

Polynomials



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This Topic ...

Many real situations can be represented by mathematical models that are built from three kinds of **elementary functions**. These functions are:

- algebraic functions
- exponential and logarithmic functions
- trigonometric functions

The most common type of algebraic function is the polynomial. This module revises and explores polynomial functions. Later, calculus will be used to investigate polynomials further.

The Topic has 2 chapters:

Chapter 1 begins by revising the algebra of polynomials. Polynomial division and the remainder theorem are introduced. The relationship between the zeros and factors of a polynomial is explored, and the special case of polynomials with integer coefficients is considered.

Chapter 2 explores the graphs of polynomial functions. It begins by examining how the leading term of the polynomial influences the global shape of its graph, and then explores how the factors of the polynomial influence the local shape of its graph.

Auhor: Dr Paul Andrew Printed: February 24, 2013

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Chapter 1

The Algebra of Polynomials

1.1 Introduction

The most common type of algebraic function is a **polynomial function**.[1] A polynomial in x is a sum of multiples of powers of x. If the highest power of x is n, then the polynomial is described as having **degree** n in x.

Definition
A general polynomial of degree n in the variable x is written as

$$a_n x^n + a_{n-1}x^{n-1} + \ \ldots \ + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$. The numbers a_i are called **coefficients***, with a_n the* **leading coefficient** *and a_0 the* **constant term**.[2]

It is common to use subscript notation a_i for the coefficients of general polynomial functions, but we use the simpler forms below for polynomials of low degree.

Zero degree:	$f(x) = a$	*Constant function*
First degree:	$f(x) = ax + b$	*Linear function*
Second degree:	$f(x) = ax^2 + bx + c$	*Quadratic function*
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	*Cubic function*

linear
quadratic
cubic
quartic
quintic
$\vdots$

Example

a. The linear function

$$2x + 3$$

has degree 1, leading coefficient 2 and constant term 3.

[1] *Poly-nomial* means 'many terms'.
[2] It is called the constant term as it doesn't change when the variable is given different values.

b. The quadratic function

$$x^2 - 2x$$

has leading coefficient 1 and no constant term. The coefficient of x is -2.

c. The cubic polynomial

$$-t^3 + 3t - 17$$

has leading coefficient -1 and constant term -17. The coefficient of t^2 is 0.

d. The polynomials

$$h^4 + 3h^3 - 7h^2 + h - 10 \quad \text{and} \quad -3 + 4L + 2L^2 + L^5$$

are examples of a **quartic** polynomial (degree 4) in h and a **quintic** polynomial (degree 5) in L. Polynomials are written with powers either descending or ascending. The leading coefficient is 1 for each polynomial.

Note: if some of the exercises below seem a bit tricky, then see MathsStart Topic 3: Quadratic Functions.

Exercise 1.1

1. A simple **cost function** for a business consists of two parts

- *fixed costs* which need to be paid no matter how many items of a product are produced (eg. rent, insurance, business loans, etc).
- *variable costs* which depend upon the number of items produced.

A computer software company produces a new spreadsheet program which costs \$25 per copy to make, and the company has fixed costs of \$10,000 per month. Find the total monthly cost C as a function of the number x of copies of the product made.

2. The diagram represents a rectangular garden which is enclosed by 100 m of fencing.

Show that

a. the area A of the garden is is given by the quadratic function

$$A = 50x - x^2$$

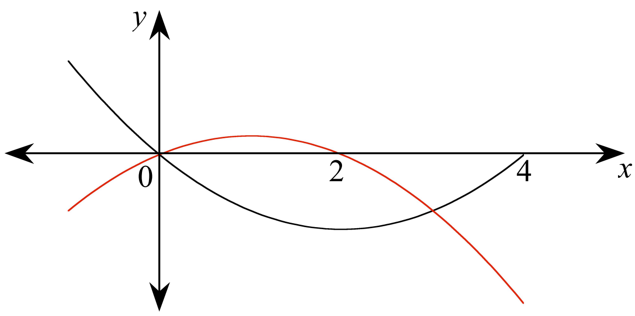
where x m is the length of one of the sides of the garden.

b. the area is a **maximum** when the garden is square.

3. The graphs of

$$y = x^2 - 4x \quad \text{and} \quad y = 2x - x^2$$

are sketched below.



Prove that

a. the graphs meet when $x = 0$ and $x = 3$

b. The vertical separation between the graphs is

$$6x - 2x^2$$

for $0 \leq x \leq 3$.

c. The vertical separation is a maximum when $x = 1\frac{1}{2}$.

1.2 Combining polynomials

Many functions in mathematics are constructed out of simpler 'building-block' functions. In this section we consider some of the ways polynomials can be combined to obtain new functions.

If f and g are two functions and c is a fixed number, then we can construct new functions using the **sum** $f+g$, the **difference** $f-g$, the (**scalar**) **multiple** cf, the **product** $f \cdot g$ and the **quotient** f/g.

1. Sums and Differences

Polynomials are added (or subtracted) by adding (or subtracting) **like terms**.[3]

adding & subtracting

Example

a. If $f(x) = 2x^2 + x - 3$ and $g(x) = -x^2 + 4x + 5$, then

$$f(x) + g(x) = (2x^2 + x - 3) + (-x^2 + 4x + 5) = x^2 + 5x + 2$$

b. If $p(t) = 4 - 5t$ and $q(t) = 2 + t - 2t^2$, then

$$p(t) - q(t) = (4 - 5t) - (2 + t - 2t^2) = 2 - 6t + 2t^2$$

2. (Scalar) Multiples

When a polynomial is multiplied by a number or a constant, each term is multiplied by that number.[4]

multiplying by a number or a constant

Example

a. If $f(x) = 2x^2 + x - 3$, then $10f(x) = 20x^2 + 10x - 30$

b. If $p(t) = 4 - 5t$ and $q(t) = 2 + t - 2t^2$, then

$$2p(t) - 3q(t) = 2(4 - 5t) - 3(2 + t - 2t^2) = 2 - 13t + 6t^2$$

3. Products

When one polynomial is multipled by another polynomial, each term in one polynomial is multiplied by each term in the other.

multiplying polynomials

Example

If $f(x) = 2x^2 + x - 3$ and $g(x) = -x^2 + 4x + 5$, then

$$\begin{aligned} f(x)g(x) &= (2x^2 + x - 3)(-x^2 + 4x + 5) \\ &= 2x^2(-x^2 + 4x + 5) + x(-x^2 + 4x + 5) - 3(-x^2 + 4x + 5) \\ &= (-2x^4 + 8x^3 + 10x^2) + (-x^3 + 4x^2 + 5x) + (3x^2 - 12x - 15) \\ &= -2x^4 + 7x^3 + 17x^2 - 7x - 15 \end{aligned}$$

[3] These are terms that have the same power of the variable.

[4] The number sometimes called a **scalar**.

4. Quotients

A function expressed as the quotient of two polynomials is called a **rational function**. The **domain** of a rational function excludes all values of the variable for which the denominator is zero.

rational functions

Example

If $p(x) = x - 3$ and $q(x) = x + 5$, then

$$\frac{p(x)}{q(x)} = \frac{x-3}{x+5}$$

provided $x \neq -5$.

This function can also be represented as

$$1 - \frac{8}{x+5}$$

In this example, $\dfrac{x-3}{x+5}$ and $\dfrac{8}{x+5}$ are rational functions but $1 - \dfrac{8}{x+5}$ is not.

Exercise 1.2

1. If $p(x) = 2x - 3$ and $q(x) = x^2 - 4x + 5$, simplify the expressions below by expanding brackets and collecting like terms:

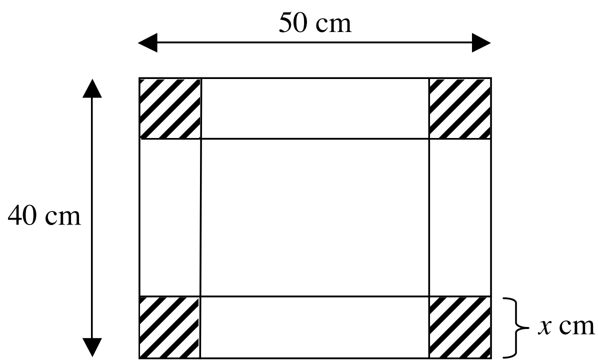
a. $5p(x) + 3q(x)$

b. $xp(x) - 2q(x)$

c. $p(x)q(x)$

d. $p(x)^2$

2. A manufacturer plans to make cake tins from 40 cm by 50 cm rectangular metal sheets. Squares will be cut from each corner and the metal will be folded as in the diagram below. The volume of the cake tin will depend upon its height.



a. Show that

i. If the height of the cake tin is x cm, then the volume is

$$V = x(40 - 2x)(50 - 2x) \text{ cm}^3.$$

ii. ...and that this is the cubic polynomial

$$V = 4x^3 - 180x^2 + 2000x \text{ cm}^3.$$

b. What are the restrictions on the x values?

1.3 Division of polynomials

When we divide 47 by 3 we get "15 with 2 left over", and write

$$\frac{47}{3} = 15 + \frac{2}{3}$$

This shows that 47/3 can be written as the sum of a whole number and a simple fraction between 0 and 1. The number 3 is called the **divisor**, the number 15 is called the **quotient**[5] and 2 is called the **remainder**.

As we shall see soon, when we divide $2x^2 + x + 1$ by $x + 2$ we obtain:

$$\frac{2x^2 + x + 1}{x + 2} = 2x - 3 + \frac{7}{x + 2}$$

Here $x + 2$ is called the **divisor**, $2x - 3$ is the **quotient** and 7 is the **remainder**.

This shows that the rational function

$$\frac{2x^2 + x + 1}{x + 2}$$

can be represented as the sum of the polynomial $2x-3$ and a simple rational function.

The rational function

$$\frac{7}{x + 2}$$

is considered to be a simple because the degree of the polynomial in the numerator is less than the degree of polynomial in the denominator. It can not be reduced to a simpler function.

This representation is important as it shows that

$$\frac{2x^2 + x + 1}{x + 2} \approx 2x - 3$$

when x has large values, since when x is large, then $\frac{7}{x+2}$ is very small.

The **algorithm**[6] for dividing one polynomial by another is similar to the algorithm for dividing whole numbers.

[5]The word quotient has two meanings (i) the number that results from the division of one number by another, and (ii) the whole number part of the result of dividing one number by another. The second meaning is used in this section

[6]An algorithm is a step-by-step procedure for solving a mathematical problem in a finite number of steps, often involving repetition of the same basic operation. Algorithms are frequently used to solve mathematical problems on computers.

Example

polynomial division algorithm

To divide $2x^2 + x + 1$ by $x + 2$, we perform the following steps ...

Step 1 The term with the highest power in $2x^2 + x + 1$ is $\boldsymbol{2x^2}$, and the term with the highest power in the divisor is $\boldsymbol{x}$. We ask "what do we multiply the x in the divisor by to get $2x^2$?"

The answer is $\boldsymbol{2x}$, so we can write:

$$\boldsymbol{2x^2} + x + 1 = \boldsymbol{2x}(\boldsymbol{x} + 2) + \text{ remainder.}$$

You can see that the remainder must be $-3x + 1$, and that

$$\begin{aligned}\frac{2x^2 + x + 1}{x + 2} &= \frac{2x(x + 2) + -3x + 1}{x + 2} \\ &= \frac{2x(x + 2)}{x + 2} + \frac{-3x + 1}{x + 2} \\ &= 2x + \frac{-3x + 1}{x + 2}\end{aligned}$$

The rational function

$$\frac{-3x + 1}{x + 2}$$

is simpler than the original function

$$\frac{2x^2 + x + 1}{x + 2},$$

as the polynomial in the numerator has degree 1. However, we can obtain a even simpler rational function that has only a constant in the numerator.

Step 2 We now look at the new function

$$\frac{-3x + 1}{x + 2}$$

and repeat what we did in step 1 ... we ask "what do we multiply $\boldsymbol{x}$ in the divisor by to get the $\boldsymbol{-3x}$ in the numerator?"

The answer is $\boldsymbol{-3}$, so we can write:

$$\boldsymbol{-3x} + 1 = \boldsymbol{-3}(\boldsymbol{x} + 2) + \text{ new remainder.}$$

You can see that the new remainder is 7, and that

$$\frac{-3x + 1}{x + 2} = \frac{-3(x + 2) + 7}{x + 2} = \frac{-3(x + 2)}{x + 2} + \frac{7}{x + 2} = -3 + \frac{7}{x + 2}$$

The original function can now be written as

$$\frac{2x^2+x+1}{x+2} = 2x + \frac{-3x+1}{x+2} = 2x - 3 + \frac{7}{x+2}$$

We stop using the algorithm now because

$$\frac{7}{x+2}$$

is a simple rational function with only a constant in its numerator.

The next example shows how to perform this division algorithm quickly and efficiently.

Example

finding the quotient & remainder

To divide $2x^2+x+1$ by $x+2$, begin with the following scheme:

$$x+2 \,\overline{\big)\; 2x^2 \;+\; x \;+\; 1}$$

In **Step 1**, we asked what do we multiply x by to obtain $2x^2$? Enter your answer on the top line as follows:

$$\begin{array}{r|l} & \boldsymbol{2x} \\ \cline{2-2} \boldsymbol{x}+2 & \boldsymbol{2x^2} \;+\; x \;+\; 1 \end{array}$$

We then found the remainder left over:

$$\boldsymbol{2x^2} + x + 1 = \boldsymbol{2x}(\boldsymbol{x}+2) + \text{ remainder.}$$

In the scheme, enter the product of $2x$ and $x+2$ under $2x^2+x+1$ and subtract to find the remainder $-3x+1$:

$$\begin{array}{rlllll} & 2x & & & & \\ \cline{2-6} x+2 \,\Big) & 2x^2 & + & x & + & 1 \\ - & \boldsymbol{2x^2} & + & \boldsymbol{4x} & & \\ \cline{2-4} & & & \boldsymbol{-3x} & \boldsymbol{+} & \boldsymbol{1} \end{array}$$

In **Step 2**, we repeated the process by asking what do we multiply x by to obtain $-3x$? Enter your answer on the top line of the scheme:

$$
\begin{array}{r|rcrcr}
 & 2x & - & \mathbf{3} & & \\
\cline{2-6}
\boldsymbol{x}+2 & 2x^2 & + & x & + & 1 \\
- & 2x^2 & + & 4x & & \\
\cline{2-4}
 & & & -\mathbf{3}\boldsymbol{x} & + & 1
\end{array}
$$

We then found the remainder left over:

$$-\mathbf{3}\boldsymbol{x}+1=-\mathbf{3}(\boldsymbol{x}+2)+\text{ new remainder.}$$

In the scheme, enter the product of -3 and $x+2$ under $-3x+1$ and subtract to find the new remainder 7:

$$
\begin{array}{r|rcrcr}
 & 2x & - & 3 & & \\
\cline{2-6}
x+2 & 2x^2 & + & x & + & 1 \\
- & 2x^2 & + & 4x & & \\
\cline{2-4}
 & & & -\mathbf{3}\boldsymbol{x} & + & \mathbf{1} \\
 & & - & -\mathbf{3}\boldsymbol{x} & - & \mathbf{6} \\
\cline{4-6}
 & & & & & \mathbf{7}
\end{array}
$$

The algorithm now ends as $x+2$ doesn't divide into the new remainder.

If a polynomial has a missing term, then it is useful to represent it as a term with coefficient 0 in the division algorithm.

missing terms

Example

Divide x^3+1 by $x-2$, expressing your answer in the form

$$q(x)+\frac{R}{x-2}$$

where $q(x)$ is a polynomial and R is a constant.

Answer.

$$
\begin{array}{r|rcrcrcr}
 & x^2 & + & 2x & + & 4 & & \\
\cline{2-8}
x-2 & x^3 & + & 0x^2 & + & 0x & + & 1 \\
- & x^3 & - & 2x^2 & & & & \\
\cline{2-4}
 & & & 2x^2 & + & 0x & + & 1 \\
 & & - & 2x^2 & - & 4x & & \\
\cline{4-6}
 & & & & & 4x & + & 1 \\
 & & & & - & 4x & - & 8 \\
\cline{6-8}
 & & & & & & & 9
\end{array}
$$

$$\implies \quad \frac{x^3+1}{x-2} = x^2 + 2x + 4 + \frac{9}{x-2}$$

The Remainder Theorem

A. *If $p(x)$ is a polynomial of degree n and $x-\alpha$ is a divisor, then*

$$\frac{p(x)}{x-\alpha} = q(x) + \frac{R}{x-\alpha} \ ,$$

where

- *the quotient $q(x)$ is a polynomial of degree $n-1$*
- *the remainder R is a constant.*

B. *This can also be rewritten as*

$$p(x) = (x-\alpha)q(x) + R.$$

where $R = p(\alpha)$.

(Substituting $x = \alpha$ into $p(x) = (x-\alpha)q(x) + R$ shows that $R = p(\alpha)$.)

If you consider the example above, the Remainder Theorem tells us that we could have found the remainder without doing the division by simply substituting $x = 2$ into $x^3 + 1$ to give $2^3 + 1 = 9$.

Exercise 1.3

1. Find the quotient and remainder for:

a. $\dfrac{x^3 - x + 1}{x-2}$

b. $\dfrac{x^2 - 4x + 5}{x+3}$ (Hint: $x + 3 = x - (-3)$.)

Use part B of the Remainder Theorem to check the remainders above.

2. Use the division algorithm to find polynomials $q(x)$ and $v(x)$ below

a. $x^3 + 1 = (x+1)q(x)$

b. $x^3 + x - 2 = (x-1)v(x)$

3. Explain why

$$\frac{x^2}{x-1} \approx x + 1$$

when x is large.

1.4 Zeros and Factors

If $p(x)$ is a polynomial and α is a number, then

1. α is a **zero** of the polynomial $p(x)$ if $p(\alpha) = 0$.
2. α is a **root of the equation** $p(x) = 0$ if it is a solution of $p(x) = 0$.
3. $x - \alpha$ is a **factor** of $p(x)$ if there is a polynomial $q(x)$ with $p(x) = (x - \alpha)q(x)$.

Example

- $x = 2$ is a zero of the polynomial $x^2 - 5x + 6$ as $2^2 - 5 \times 2 + 6 = 0$.
- $x = 2$ is a root of the equation $x^2 - 5x + 6 = 0$ as $2^2 - 5 \times 2 + 6 = 0$.
- $x - 2$ is a factor of $x^2 - 5x + 6$ as $x^2 - 5x + 6 = (x - 2)(x - 3)$.

The Factor Theorem

If $p(\alpha) = 0$, then $x - \alpha$ is a factor of $p(x)$.

Proof

The remainder theorem shows that $p(x)$ can be written as

$$p(x) = (x - \alpha)q(x) + R\ ,$$

and that R is equal to $p(\alpha)$. If $p(\alpha) = 0$, then $R = 0$ and $x - \alpha$ is a factor of $p(x)$.

Example

zeros & factors

a. Check that 2 is a zero of the cubic polynomial $p(x) = x^3 - 2x^2 - x + 2$.

b. Use this to find one factor of $p(x)$.

c. Use the division algorithm to find the other factors of $p(x)$.

Answer.

a. $p(2) = 2^3 - 2 \times 2^2 - 2 + 2 = 8 - 8 + 2 - 2 = 0$. So 2 is a zero.

b. The factor theorem shows that $x - 2$ is a factor of $p(x)$.

c. Using the division algorithm:

$$\begin{array}{rrrrrr}
 & x^2 & & & - & 1 \\ \cline{2-6}
x-2 & \big)\, x^3 & - & 2x^2 & - & x + 2 \\
- & x^3 & - & 2x^2 & & \\ \cline{2-6}
 & & & & & -x + 2 \\
 & & & & - & -x + 2 \\ \cline{6-6}
 & & & & & 0
\end{array}$$

This shows that

$$x^3 - 2x^2 - x + 2 = (x - 2)(x^2 - 1).$$

The quadratic factor $x^2 - 1$ can be factored as well. The *complete factorisation* is

$$x^3 - 2x^2 - x + 2 = (x - 2)(x - 1)(x + 1).$$

Theorem

If a polynomial $p(x)$ has integer coefficients and an integer zero, then the zero is a factor of the constant term.

integer zeros

Example

Find an integer zero for the polynomial $p(x) = x^3 + 2x^2 - 3x - 6$

Answer.

The constant term 6 has factors ± 1, ± 2, ± 3, ± 6.
Check to see if any are zeros:

$$\begin{array}{llll} p(1) = -6, & p(2) = 4, & p(3) = 30, & p(6) = 264 \\ p(-1) = -1, & p(-2) = 0, & p(-3) = -6, & p(-6) = -132 \end{array}$$

There is only one integer zero: $x = -2$. The polynomial has factors

$$x^3 + 2x^2 - 3x - 6 = (x + 2)(x^2 - 3).$$

Exercise 1.4

1. Each polynomial below has an integer zero. Find this zero, then factorise the polynomial as much as possible.

a. $x^3 - 2x^2 - 5x + 6$

b. $x^3 - 3x + 2$

c. $x^3 - x^2 - 5x - 3$

d. $x^3 - 3x^2 + 3x - 1$

e. $x^3 + 2x^2 + 3x + 2$

Chapter 2

Graphs of Polynomials

2.1 Continuity and Smoothness

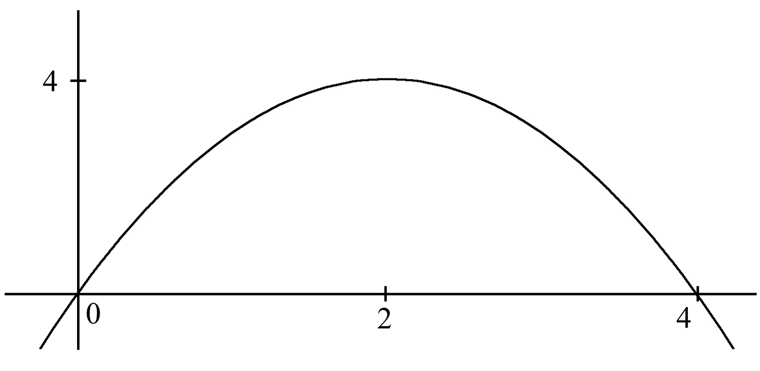
We use the word **continuous** in everyday language to describe changes that are gradual rather than sudden. It is used similarly in mathematics when we talk about **continuous functions**. Roughly speaking, a function is continuous when small changes in the independent variable (x) produce small changes in the value of the function (y). Geometrically, *a function is continuous if its graph has no sudden jumps as the independent variable changes.*

It is often said that a function is continuous if its graph can be drawn without lifting the pencil off the paper.

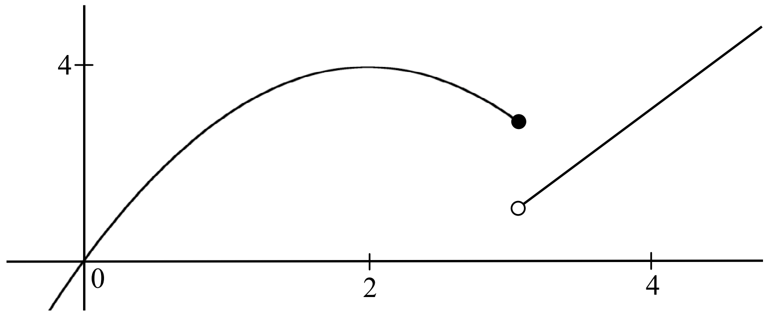
continuous & discontinuous functions

Example

a. This is the graph of a continuous function. The graph has no sudden jumps. When the independent variable x changes by a small amount, the corresponding value of y also changes by a small amount.



b. This is the graph of a discontinuous function. It is discontinuous at $x = 3$ as the curve has a jump there. You can see that when the independent variable x changes from 3 to a value slightly greater than 3, then the value of the function changes abruptly, and that this is true no matter how small the change in x is. While the graph is discontinuous at $x = 3$, it is continuous when $x < 3$ and when $x > 3$.

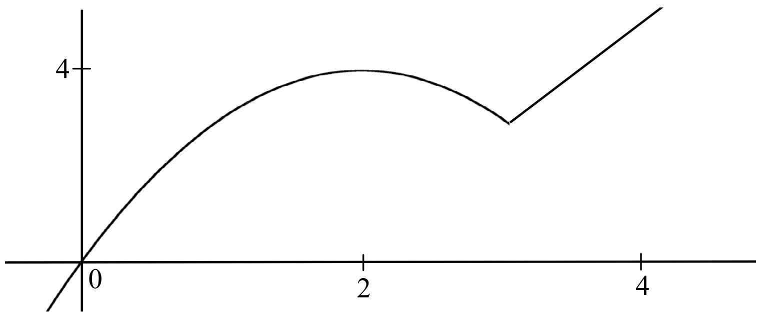


A **smooth curve** is a curve without corners. The continuous curve in example(a) is a smooth curve without corners. The diagram below shows a continuous curve which is not smooth.

curves can have corners

Example

This is a continuous curve, but not a smooth curve as it has a corner.



2.2 Shapes of graphs

Graphs give us a better understanding of the differences between functions. In this section we look at graphs 'locally' (ie. up close) and 'globally' (ie. their shape from a distance).

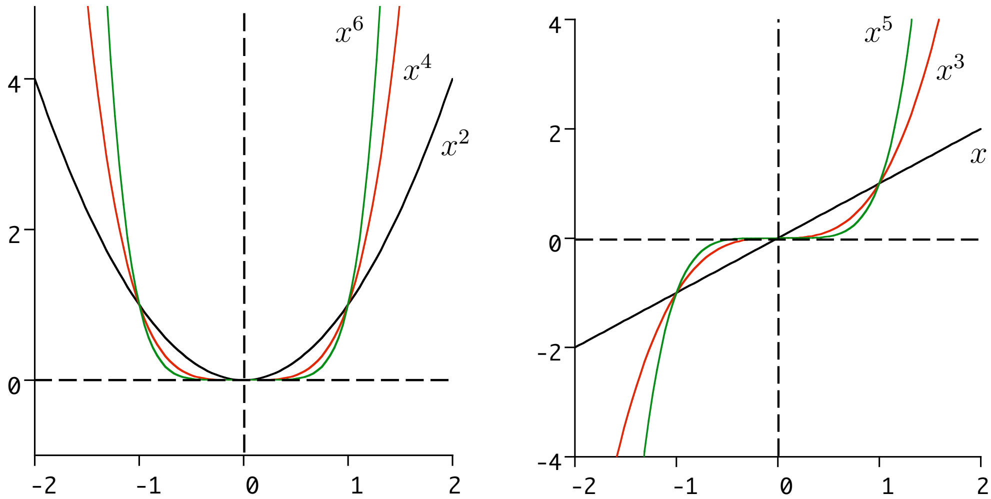
The simplest polynomials are the **power functions** $f(x) = x^n$ for $n = 1, 2, 3, \ldots$.

Example

graphs of power functions

The shape of power functions changes with their degree. You can see that:

- the graphs of *even-degree* power functions $x^2, x^4, x^6, \ldots$ are 'cup shaped'
- the graphs of *odd-degree* power functions $x, x^3, x^5, \ldots$ go 'from south west to northeast'



This example also shows that the graph of $y = x^n$ looks like a parabola when n is even and that, as n increases, the graphs becomes flatter near the origin and steeper when $|x| > 1$. Similarly, if n is odd, the graph of $y = x^n$ looks like that of $y = x^3$ when $n > 3$ and, as n increases, it becomes flatter near the origin and steeper for $|x| > 1$.

Why are the graphs of even- and odd-degree power functions so different?

The shape of a polynomial graph depends on its leading coefficient:

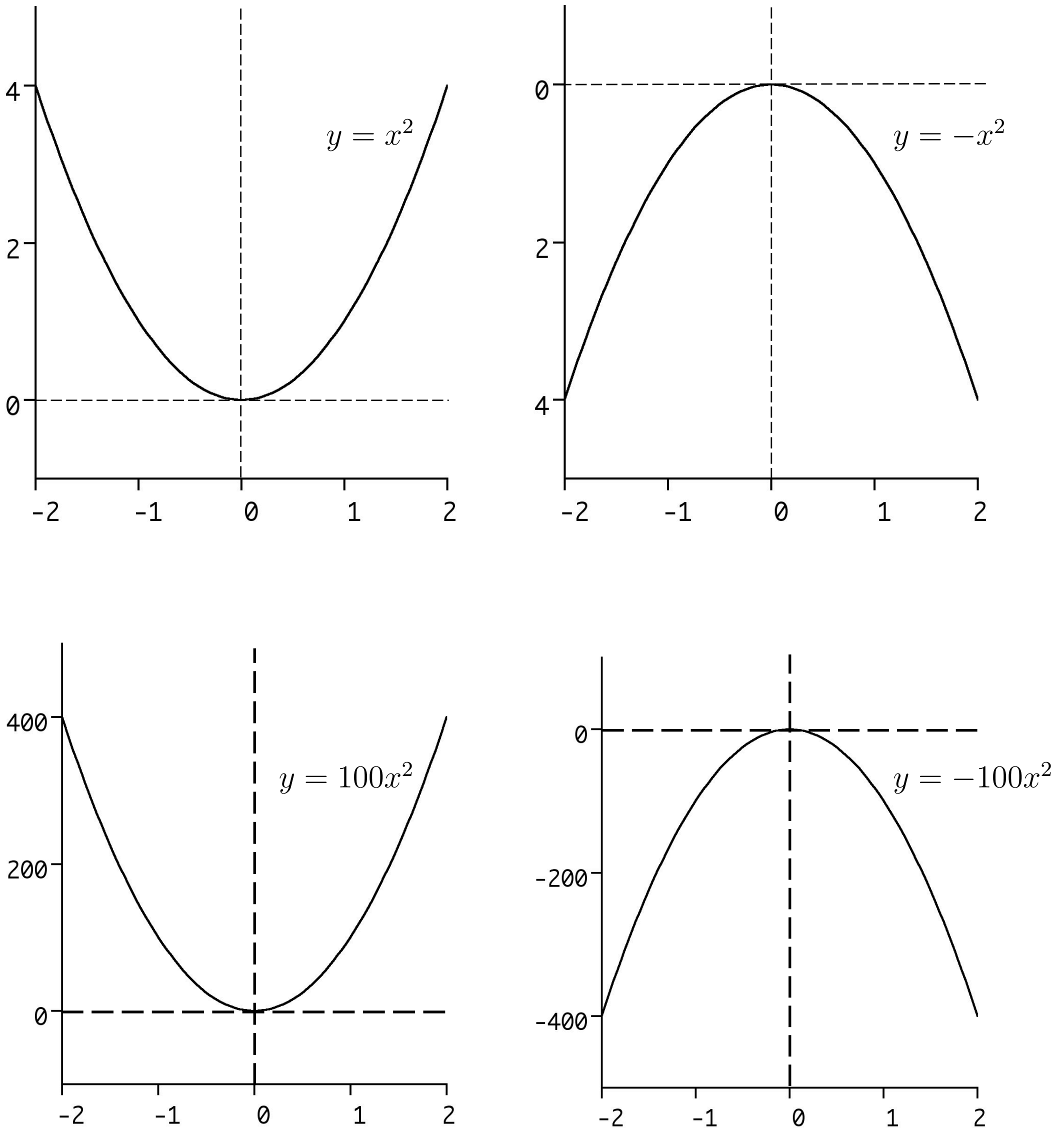
1. The *sign* of the leading coefficent determines its *orientation*.
2. The *magnitude* of the leading coefficient determines the *vertical scale*.

leading coefficients

Example

You can see below that:

- the graph of $-x^2$ is obtained by 'flipping over' the graph of x^2.
- the graph of $100x^2$ is obtained by stretching the graph of x^2 vertically.
- the graphs of $-100x^2$ obtained by stretching the graph of $-x^2$ vertically.
- the graphs of $-100x^2$ obtained by 'flipping over' the graph of $100x^2$.

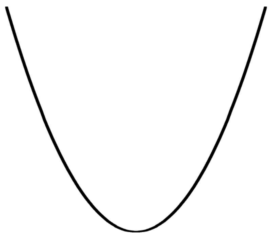


Polynomial graphs often have bends. The place where a graph 'turns around' is called a **turning point**. For example, the vertex of a parabola is its turning point.

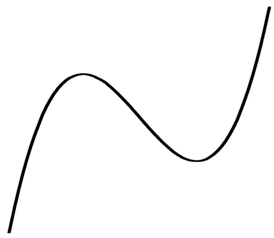
Example

bends & turning points

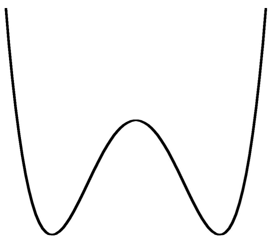
These graphs are typical polynomial graphs. Notice that the quadratic has one turning point, the cubic has two, the quartic has three and the quintic has four.



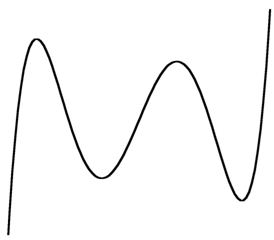
quadratic ($n = 2$)



cubic ($n = 3$)



quartic ($n = 4$)



quintic ($n = 5$)

The graph of a polynomial of degree n can have up to $n - 1$ turning points.

Two polynomial graphs may look very different when you look at them close up (locally) but look similar when you look at them from from a distance (globally). Their global shape depends upon the leading term of the polynomial.

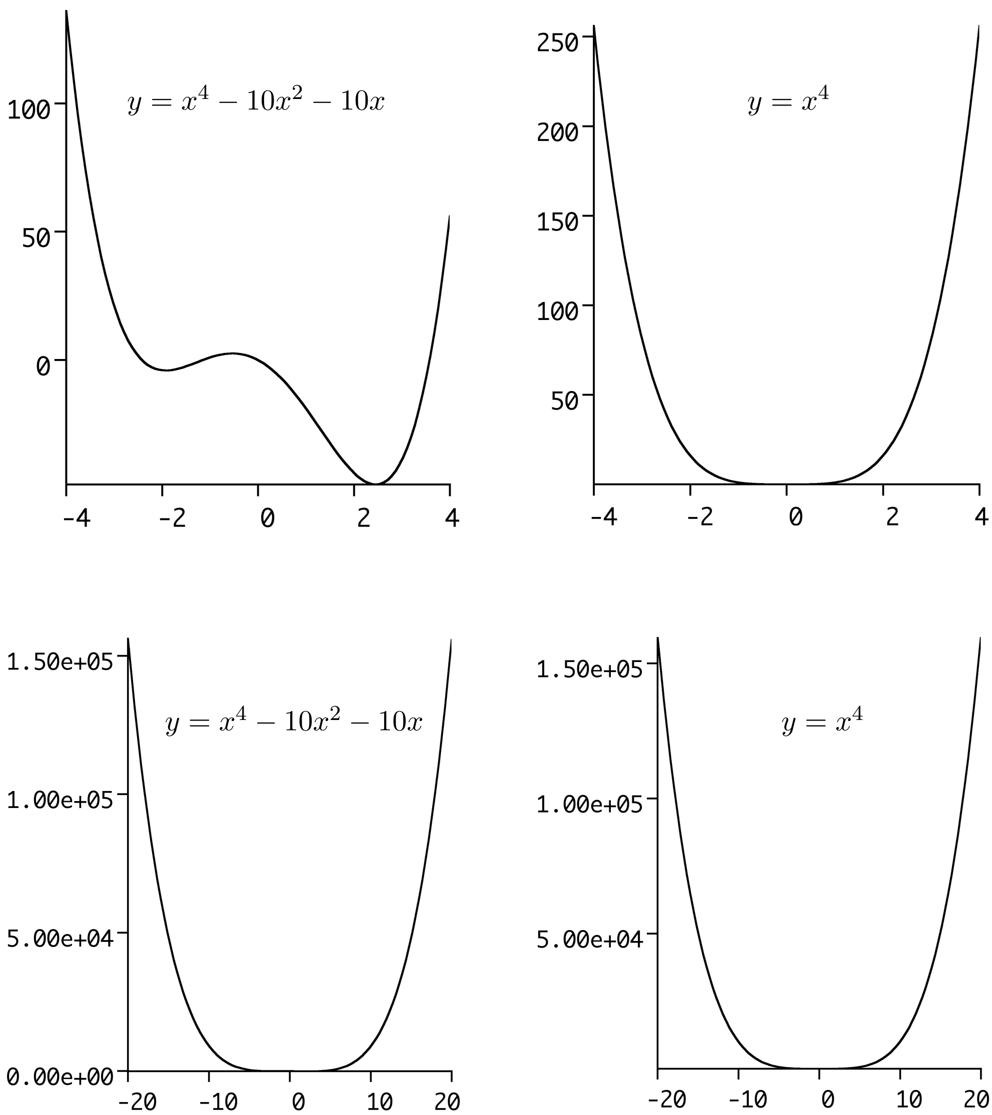
Example

local and global shapes

This example compares the local and global shapes of the two quartic curves

$$y = x^4 - 10x^2 - 10x \quad \text{and} \quad y = x^4$$

You can see that the two curves look very different locally for $-4 \leq x \leq 4$, but similar globally for $-20 \leq x \leq 20$. This is because the leading term dominates the other terms when x has large values. *Use your calculator to check this dominance.*



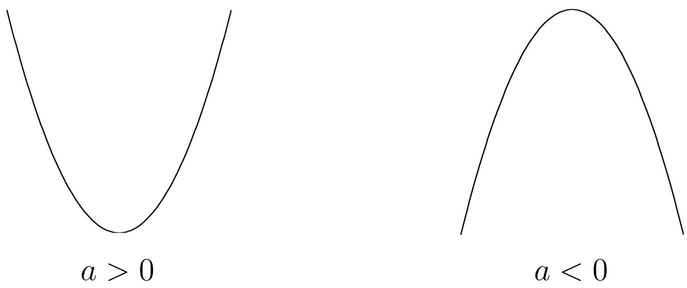
We can obtain a good idea of the global shapes of polynomial functions from knowledge of the shape of the power functions $f(x) = x^n$ for $n = 1, 2, 3, \ldots$.

2.3 Selected graphs

In this section, we explore the local shapes of graphs of polynomials that can be fully factorised.

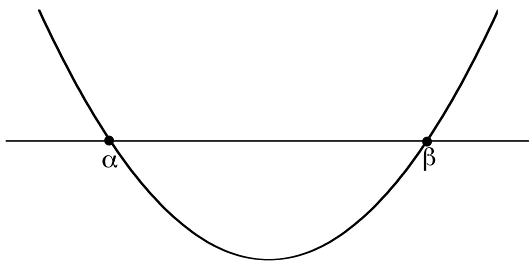
A. Graphs of the form $\boldsymbol{y = a(x - \alpha)(x - \beta)}$

- The orientation of $y = a(x - \alpha)(x - \beta)$ is **concave upward** when $a > 0$ and **concave down** when $a < 0$.

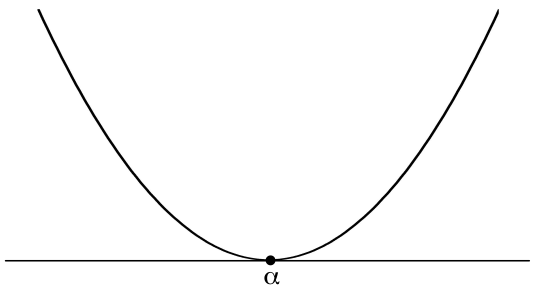


(The graph of $y = a(x - \alpha)(x - \beta)$ looks like the graph of $y = ax^2$ when x is large.)

- The x-intercepts of $y = (x - \alpha)(x - \beta)$ are $x = \alpha$ and $x = \beta$.



- In the special case $y = (x - \alpha)^2$, the parabola touches the x-axis at a single point (the turning point).



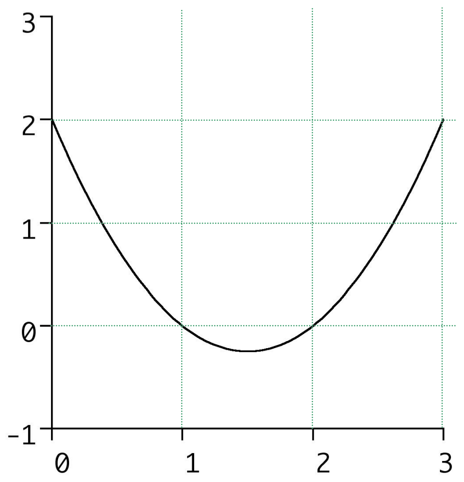
One way of thinking about this situation is to imagine that the two x-intercepts in $y = (x - \alpha)(x - \beta)$ are moved closer and closer together until they coincide.

A squared factor always corresponds to a turning point.

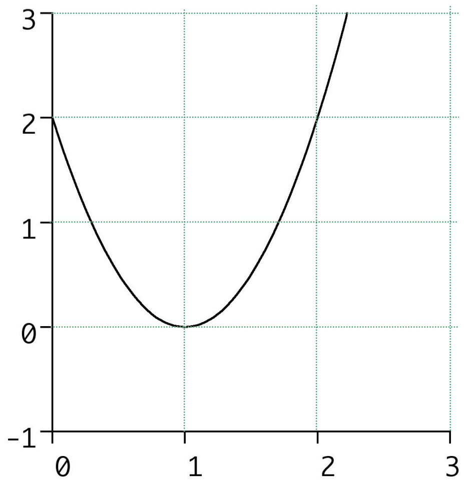
Exercise 2.3

1. What are the equations of the parabolas below.

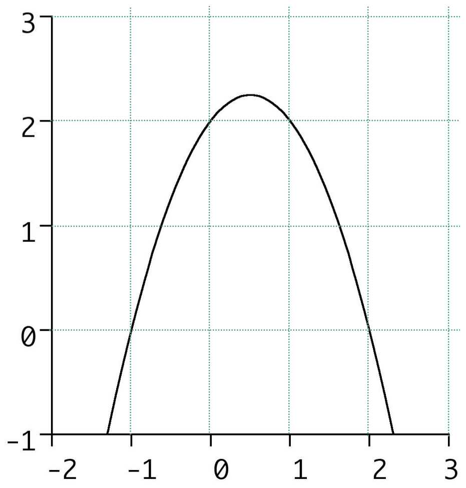
a.



b.



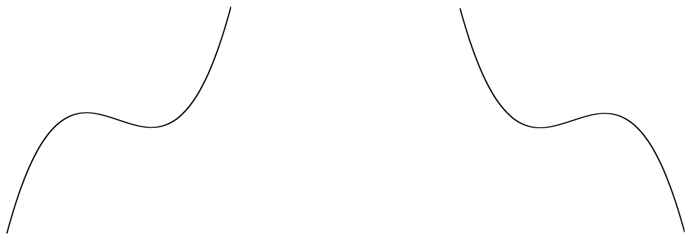
c.



2. Sketch the graph of $y = -2(x-1)(x+1)$, showing the x- and y-intercepts

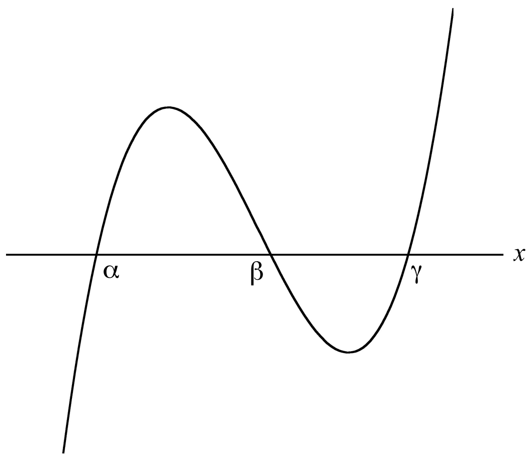
B. Graphs of the form $y = a(x-\alpha)(x-\beta)(x-\gamma)$

- The orientation of $y = a(x-\alpha)(x-\beta)(x-\gamma)$ is from south-west to north-east when $a > 0$ and from north-west to south-east when $a < 0$.

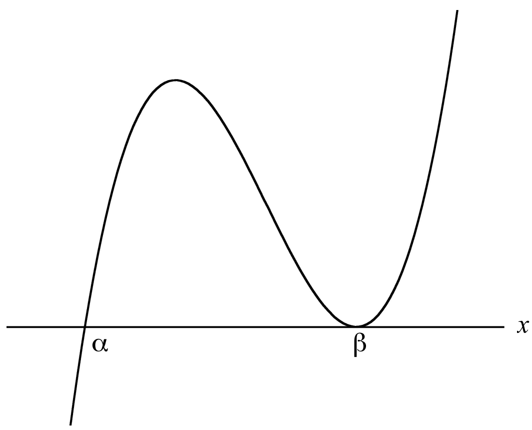


$a > 0$ $a < 0$

- The x-intercepts of $y = (x-\alpha)(x-\beta)(x-\gamma)$ are $x = \alpha$, $x = \beta$ and $x = \gamma$.

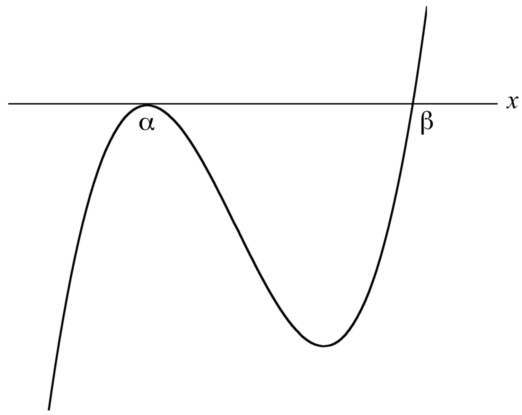


- The x-intercepts of $y = (x-\alpha)(x-\beta)^2$ are $x = \alpha$ and $x = \beta$, and the cubic has a turning point at $x = \beta$.



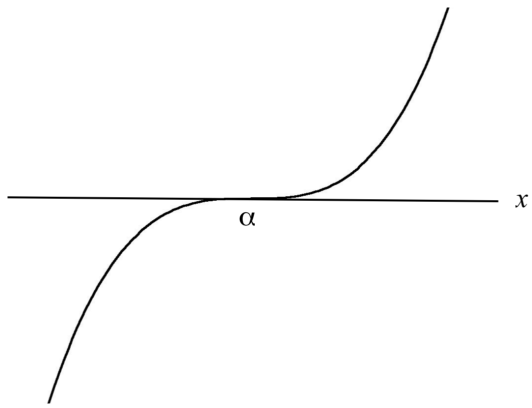
If the cubic is $y = (x-1)(x-2)^2$, then $y(0.9) = -0.121 < 0$, $y(1.1) = 0.081 > 0$, $y(1.9) = 0.009 > 0$ and $y(2.1) = 0.011 > 0$.

- The x-intercepts of $y = (x-\alpha)^2(x-\beta)$ are $x = \alpha$ and $x = \beta$, and the cubic has a turning point at $x = \alpha$. (Note that you always get a turning point on the x-axis when you have a squared factor in your polynomial.)



If the cubic is $y = (x-1)^2(x-2)$, then $y(0.9) = -0.121 < 0$, $y(1.1) = -0.009 < 0$, $y(1.9) = -0.001 < 0$ and $y(2.1) = 0.121 > 0$.

- The cubic $y = (x-\alpha)^3$ has a single x-intercept at $x = \alpha$. It is not a turning point.



One way of thinking about this situation is to imagine that the three x-intercepts in $y = (x-\alpha)(x-\beta)(x-\gamma)$ are moved closer and closer together until they all coincide.

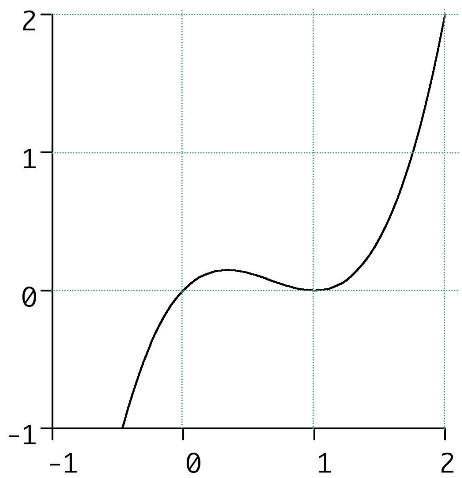
If the cubic is $y = (x-1)^3$, then $y(0.9) = -1.331 < 0$, $y(1.1) = 1.331 > 0$.

When sketching curves, simple calculations like those above are useful for exploring and confirming their shapes.

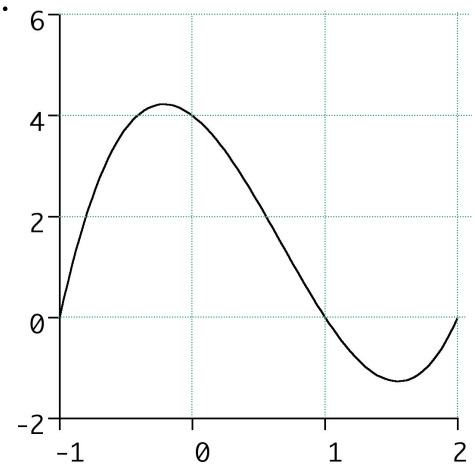
Exercise 2.3

3. What are the equations of the cubic curves below.

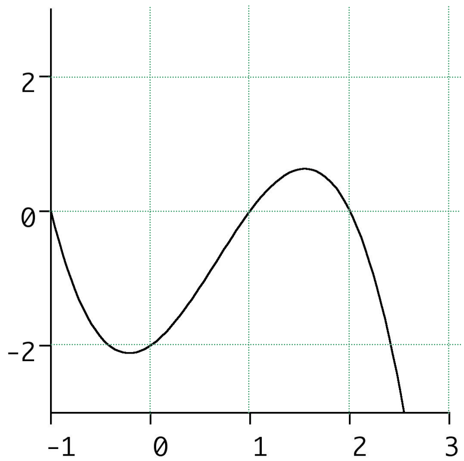
a.



b.



c.



4. Sketch the graph of $y = 2(x^3 + 2x^2 - x - 2)$, showing the x- and y-intercepts

Appendix A

Factorisation (revision)

A quadratic polynomial is a polynomial of the form $ax^2 + bx + c$. This appendix gives two methods for factorising quadratic polynomials: simple factorisation and the quadratic formula.

The first method is emphasised to give you more practice at algebra, but you are more likely to use the quadratic method in real applications.

A.1 Simple factorisation

To understand how to factorise a polynomial like $x^2 + 5x + 4$ into a product of two factors $(x + \alpha)(x + \beta)$, we expand the product $(x + \alpha)(x + \beta)$ giving

$$(x + \alpha)(x + \beta) = x^2 + \alpha x + \beta x + \alpha\beta = x^2 + (\alpha + \beta)x + \alpha\beta \ .$$

We can write $x^2 + 5x + 4$ in the form $(x + \alpha)(x + \beta)$ if we can find two numbers α and β with $\alpha + \beta = 5$ and $\alpha\beta = 4$. *We now try to guess the numbers α and β.*

- The product of α and β is 4, so the numbers could be 1 and 4 or 2 and 2.
- The sum of α and β is 5, so α and β must be 1 and 4.

This shows that $x^2 + 5x + 4 = (x + 1)(x + 4)$.

If we couldn't guess these numbers, then we would need to use another method!

Example

products and sums

Write $x^2 + x - 6$ as a product of two factors.

Answer.
The numbers α and β have product -6, and sum 1.

Pairs of numbers with product -6 are

$$1 \text{ and } -6,\ \ 2 \text{ and } -3,\ \ 3 \text{ and } -2,\ \ 6 \text{ and } -1.$$

The two numbers have sum 1, so α and β must be 3 and -2, and

$$x^2 + x - 6 = (x+3)(x-2) \ .$$

A **common factor** is a factor which is a factor of every term. It is a good idea to factorise out the any common factors first of all.

Example

common factor

Factorise $10x^2 + 10x - 60$.

Answer.
Taking out the common factor first, then factorising:

$$\begin{aligned} 10x^2 + 10x - 60 &= 10(x^2 + x - 6) \\ &= 10(x+3)(x-2) \end{aligned}$$

Exercise A.1

1. Factorise the following quadratic polynomials

a. $x^2 + 4x + 3$	**b.** $x^2 + 6x + 5$	**c.** $a^2 + 8a + 7$
d. $x^2 + x - 2$	**e.** $b^2 - b - 2$	**f.** $n^2 - 4n - 5$
g. $s^2 - s - 12$	**h.** $x^2 + x - 12$	**i.** $x^2 + 8x + 12$
j. $t^2 - 11t - 12$	**k.** $x^2 - 9x + 14$	**l.** $y^2 - 13y - 14$

A.2 The quadratic formula

The quadratic formula is useful for solving quadratic equations when the equation has complicated coefficients. If you only want an approximate answer, then the quadratic formula is best.

The Quadratic Formula

1. If $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has no solutions.
2. If $b^2 - 4ac > 0$, then equation $ax^2 + bx + c = 0$ has two solutions given by
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ .$$
3. If $b^2 - 4ac = 0$, then the two solutions are the same.

When the quadratic equation equation $ax^2 + bx + c = 0$ has solutions, the corresponding factorisation is

$$ax^2 + bx + c = a(x + \frac{b + \sqrt{b^2 - 4ac}}{2a})(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}) \ .$$

Appendix B

Answers

Exercise 1.1

1. $C = 10000 + 25x$ 2. self checking   3. self checking

Exercise 1.2

1(a) $3x^2 - 2x$ 1(b) $5x - 10$ 1(c) $2x^3 - 11x^2 + 22x - 15$ 1(d) $4x^2 - 12x + 9$

2(a) self checking   2(b) $0 < x < 20$

Exercise 1.3

1(a) $x^2 + 2x + 3$, 7   1(b) $x - 7$, 26

2(a) $q(x) = x^2 - x + 1$ 2(b) $v(x) = x^2 + x + 2$

3. $\dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$

Exercise 1.4

1(a) $(x-1)(x-3)(x+2)$ 1(b) $(x-1)^2(x+2)$ 1(c) $(x-3)(x+1)^2$

1(d) $(x-1)^3$ 1(e) $(x+1)(x^2+x+2)$

Exercise 2.3

1(a) $y = (x-1)(x-2)$ 1(b) $y = 2(x-1)^2$ 1(c) $y = -(x+1)(x-2)$

2. Intercepts are $(1,0)$, $(-1,0)$ and $(0,2)$. Check sketch with lecturer.

3(a) $y = x(x-1)^2$ 1(b) $y = 2(x-1)(x-2)(x+1)$ 1(c) $y = -(x-1)(x-2)(x+1)$

4. Intercepts are $(1,0)$, $(-1,0)$, $(-2,0)$ and $(0,-4)$. Check sketch with lecturer.