Tests for Convergence

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Order in which to try the tests:

1. Check if it's a known series
   (including rewriting it to look like a geometric series)
2. Divergence test
   (particularly useful for terms that are rational functions).
3. Alternating series test
   (specifically for alternating series).
4. Ratio test
   (particularly useful for terms involving n in the power or as a factorial).

Known series:

harmonic series	$\sum_{n=1}^{\infty} \frac{1}{n}$	DIVERGES	
alternating harmonic series	$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$	CONVERGES	
p-series "power of harmonic"	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p \leq 1$, DIVERGES	
		If $p > 1$, CONVERGES	
geometric series	$\sum_{n=0}^{\infty} x^n$	If $\lvert x \rvert \geq 1$, DIVERGES	
		If $\lvert x \rvert < 1$, CONVERGES	and converges to $\frac{1}{1-x}$
MacLaurin series for e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	For any x, CONVERGES	and converges to e^x

Test for divergence (check limit of individual terms):

Given a series $\sum_{n=0}^{\infty} a_n$	If	$\lim_{n \to \infty} a_n \neq 0$	then the series DIVERGES.
	If	$\lim_{n \to \infty} a_n = 0$	then you don't know if the series converges or diverges.

Alternating series test:

Given a series $\sum_{n=0}^{\infty} a_n$	If	0. It's an alternating series: $a_n = (-1)^n b_n$	then the series CONVERGES.
		1. $\lim_{n\to\infty} a_n = 0$	
		2. For some N, $\lvert a_N \rvert > \lvert a_{N+1} \rvert > \lvert a_{N+2} \rvert > \cdots$	

Ratio test:

Given a series $\sum_{n=1}^{\infty} a_n$	1. Calculate $\left\lvert \frac{a_{n+1}}{a_n} \right\rvert$	If $L < 1$, then the series CONVERGES
	2. Find $L = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$	If $L > 1$, then the series DIVERGES
		If $L = 1$, then you don't know if the series converges or diverges

Intervals of convergence for power series:

Given a series $\sum_{n=1}^{\infty} a_n$. It is a **power series** when $a_n = b_n(x-a)^n$, where b_n is an expression in n.

(Note there may be a coefficient next to the x inside the bracket and so you may need to rewrite it so that this coefficient becomes part of the b_n.)

The **interval of convergence** is the set of x-values which produce a series that converges.

To find the interval of convergence of a power series, do the ratio test. The working will look like this:

Calculate

$$\left\lvert \frac{a_{n+1}}{a_n} \right\rvert = \left\lvert \frac{b_{n+1}(x-a)^{n+1}}{b_n(x-a)^n} \right\rvert = \cdots$$

Find

$$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = \cdots = \cdots = L\lvert x-a \rvert$$

In order to converge, we need

$$L\lvert x-a \rvert < 1$$
$$\lvert x-a \rvert < R \leftarrow \text{Radius of convergence}$$
$$-R < x-a < R$$
$$b < x < c$$

Test endpoints:

If $x = b$, … … which converges/diverges
If $x = c$, … … which converges/diverges

So the interval of convergence is $[b, c]$ or $(b, c]$ or $[b, c)$ or (b, c).