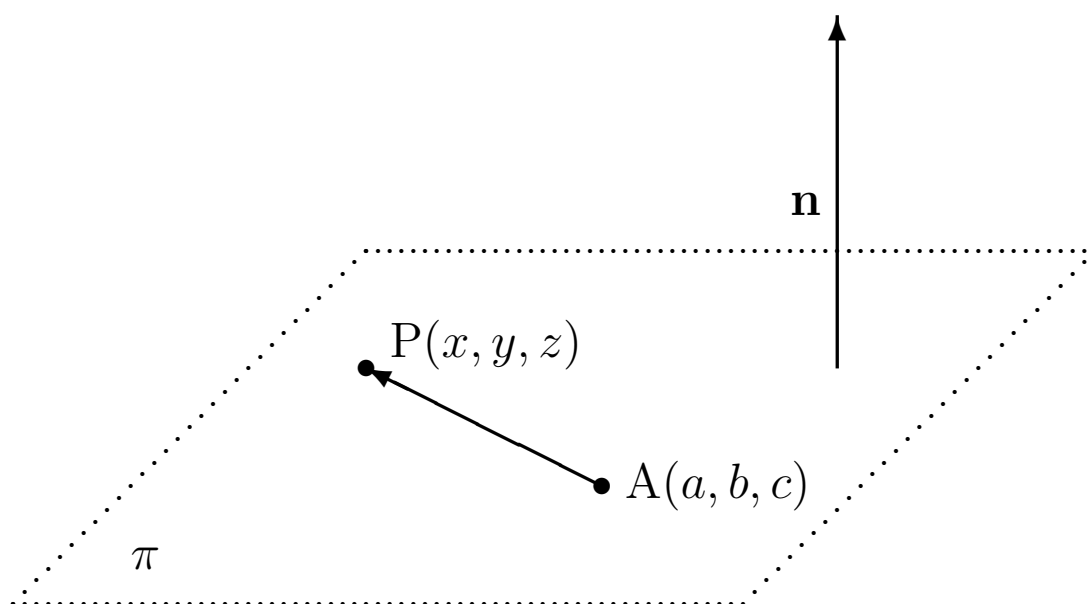


(NOTE Feb 2013: This is the old version of MathsTrack.
New books will be created during 2013 and 2014)

Topic 3

Vectors and Applications



$$\pi: n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$$



MATHS LEARNING CENTRE

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This Topic . . .

Vectors are quantities which have both magnitude (size) and direction. They are used in navigation, engineering, science, economics, etc.

The topic introduces vectors and vector operations. For convenience, examples and exercises use two and three dimensional vectors, however the ideas are applicable to vectors with any number of dimensions.

The topic has 3 chapters:

Chapter 1 introduces vectors and scalars. It gives examples of vectors and shows how vectors can be added and subtracted. Vector algebra is introduced and is used to solve problems in geometry.

Chapter 2 introduces vector components and unit vectors. These give rise to a powerful new computational approach to vector algebra. Three dimensional vectors are introduced.

Chapter 3 introduces the scalar product and uses it to find the angle between two vectors. The equation of a plane in 3-dimensions is introduced.

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Chapter 1

Vectors

1.1 Vectors and Scalars

In everyday language speed and velocity mean the same thing, but in mathematics *speed* is a scalar and *velocity* is a vector. A *vector* is a quantity which is completely characterised by two things: its magnitude (or size) and its direction.

Example

vectors

- The velocity of a car heading North at 60 km/h.¹
- A force exerted on an object.
- The magnetic field of the earth at a given place.

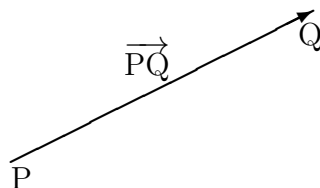
A *scalar* is a quantity which is completely characterised by its magnitude alone. Scalars are just numbers.

Example

scalars

- Length
- Area
- Volume
- Any number
- Mass
- Time
- Temperature

The simplest example of a vector is a *directed line segment* (or *arrow*). If P and Q are two points, then the directed line segment from P to Q is the straight line which begins at P and ends at Q. This vector is denoted by \overrightarrow{PQ} , and we say “the vector PQ”.



¹Velocity has both magnitude and direction, but speed has magnitude only. For example we would say the speed of the car is 60 km/h without referring to its direction.

The *magnitude or length* of \overrightarrow{PQ} is the length of segment PQ, and is represented by the symbol $|\overrightarrow{PQ}|$. For example, if \overrightarrow{PQ} is the velocity of a car, then $|\overrightarrow{PQ}|$ would be the speed of the car.

The *direction* of \overrightarrow{PQ} is given by the arrowhead. A direction can be described in a number of ways.

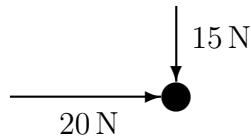
- In navigation, direction is given by a compass bearing, e.g. NE, $N45^\circ E$, $N45^\circ$ and 045° all refer to a bearing of 45° from north taken in a *clockwise* direction.
- In mathematics or engineering, direction is typically measured from a selected axis with angles taken in an *anticlockwise* direction.

All vectors can be represented by arrows or directed line segments.

Example

*vector
diagram*

This *vector diagram* shows two forces acting on an object.
(Scale: 1cm = 10 Newtons)



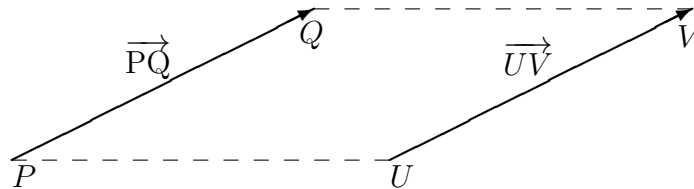
Definition 1.1.1

Two vectors are called **equal** if they have the same length and direction.

Example

*equal
vectors*

The vectors \overrightarrow{PQ} and \overrightarrow{UV} below are equal as they have the same length and the same direction. It doesn't matter that the vectors are in different places, so long as their magnitudes and directions are the same. We write $\overrightarrow{PQ} = \overrightarrow{UV}$.

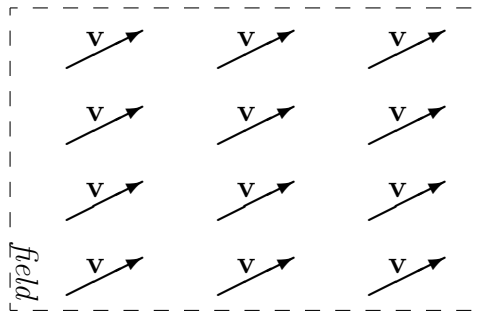


Vectors are usually represented in print by small boldface letters like **a**, **b**, **c**, ... , and in handwriting by \underline{a} , \underline{b} , \underline{c} ,

Example

*wind
velocity*

Each of the vectors below represents the velocity **v** of the wind moving across a field. The wind has the same strength and direction everywhere on the field.



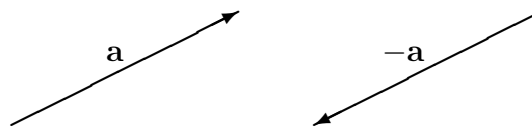
We think of negative numbers as being in the opposite direction to positive numbers on a numberline. The same is true for vectors.

Definition 1.1.2

The **negative** of a vector \mathbf{a} is a vector with the same length but the opposite direction. It is denoted by $-\mathbf{a}$.

Example

negative
of a
vector



When a car doubles its velocity, then we think of it as travelling in the same direction but at twice the speed. The same is true when we double a vector. This is an example of *scalar multiplication*: multiplying a vector by the scalar 2.

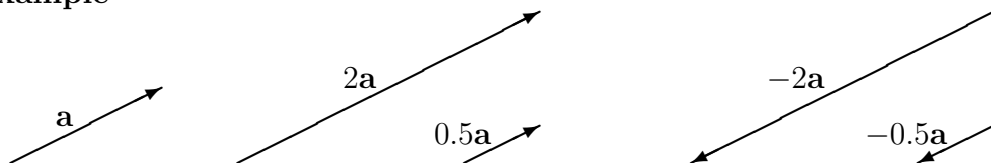
Definition 1.1.3

If \mathbf{a} is a vector and k is a scalar (a number), then $k\mathbf{a}$ is the vector with

- (if $k > 0$) length k times the length of \mathbf{a} and direction the same as \mathbf{a} .
- (if $k < 0$) length $|k|$ times the length of \mathbf{a} and direction opposite to \mathbf{a} .²

Example

scalar
multiplication



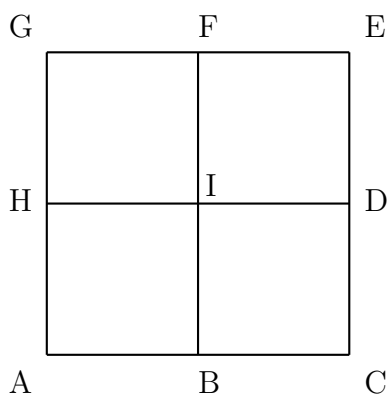
Exercise 1.1

1. After taking off, one plane flew East at 300 km/h and another flew 045° at 400 km/h.³ Sketch these vectors using the scale 1cm = 100 km/h.

² The same symbol “|” is used with vectors and scalars: $|k|$ is the absolute value of the number k , and $|\mathbf{a}|$ is the length of the vector \mathbf{a} .

³ 45° from North in a clockwise direction.

2. Sketch the following vectors:
- the velocity \mathbf{v} of a car travelling south at 60 km/h.
 - $1.5\mathbf{v}$
 - $0.5\mathbf{v}$
3. A car was travelling with velocity \mathbf{v} , then 5 minutes later its velocity was $1.1\mathbf{v}$ and then 10 minutes later its velocity was $-\mathbf{v}$. What happened?
4. The figure below consists of 4 squares. $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AH} = \mathbf{q}$.



Which of the following is true?

- | | | | |
|---------------------------------------------|-------------------------------------------------|---------------------------------------------|-----------------------------------------|
| (a) $\overrightarrow{FE} = \mathbf{p}$ | (b) $ \mathbf{p} = \mathbf{q} $ | (c) $\overrightarrow{BC} = \mathbf{p}$ | (d) $\overrightarrow{GF} = \mathbf{q}$ |
| (e) $\overrightarrow{GH} = \mathbf{q}$ | (f) $\overrightarrow{GE} = 2\mathbf{p}$ | (g) $ \overrightarrow{HD} = 2 \mathbf{q} $ | (h) $\overrightarrow{ED} = -\mathbf{q}$ |
| (i) $ \overrightarrow{FB} = 2\mathbf{p} $ | (j) $\overrightarrow{GI} = \overrightarrow{FD}$ | (k) $\overrightarrow{FG} = -\mathbf{p}$ | (l) $\mathbf{p} = \mathbf{q}$ |

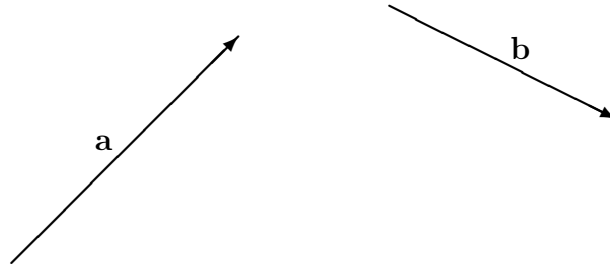
5. Is it true that $-(-\mathbf{a}) = \mathbf{a}$? Use a diagram to explain why.
-

1.2 Adding and Subtracting Vectors

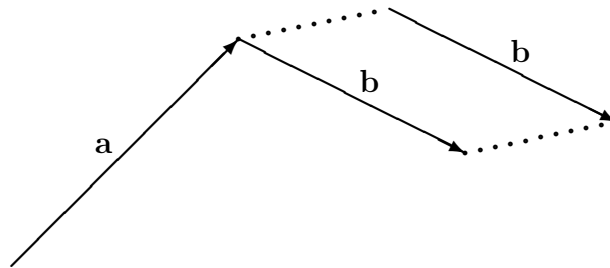
Although there are many different kinds of vectors, all behave in the same way. This is described in the definitions of vector addition and subtraction.

Definition 1.2.1

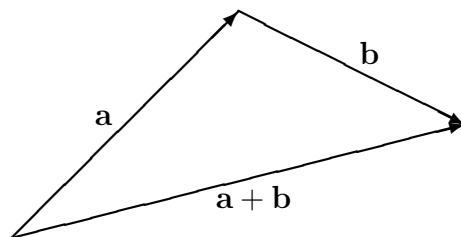
The **sum** $\mathbf{a} + \mathbf{b}$ of vectors \mathbf{a} and \mathbf{b} is the given by the triangle rule below.



Step 1: Shift \mathbf{b} without changing its length or direction, until its tail just touches the head of \mathbf{a} .



Step 2: The vector $\mathbf{a} + \mathbf{b}$ is equal to the vector that begins at the tail of \mathbf{a} and ends at the head of \mathbf{b} .

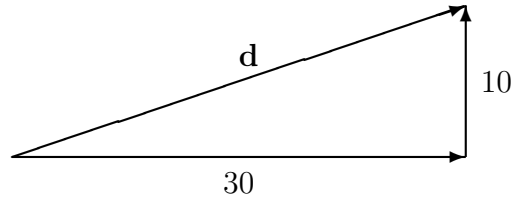


This method is called the *triangle method* or the *head-to-tail method* of addition.

Example

We can describe how an object is *displaced* by giving the distance and direction of the displacement. In the diagram below an object is first displaced towards the east by 30 km, and then north by 10 km.

*resultant
displacement*

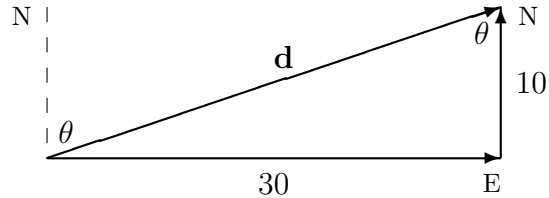


The vector \mathbf{d} shows the object's *resultant* displacement. This means the displacement resulting from combining the actions of the vectors.

The length or magnitude $|\mathbf{d}|$ of the vector \mathbf{d} can be estimated from accurately scaled vector diagrams or by using trigonometry.

Example (continued)

*finding
length
direction*



The length of \mathbf{d} is

$$|\mathbf{d}| = \sqrt{30^2 + 10^2} = \sqrt{1000} = 31.6 \text{ km.}$$

The direction of \mathbf{d} is $N\theta^\circ$, where

$$\tan \theta = \frac{30}{10} \implies \theta = 71.6^\circ.$$

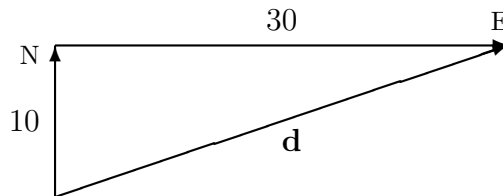
That is, the direction is the compass bearing of $N71.6^\circ$ (which is measured clockwise from North).

The resultant vector \mathbf{d} does not depend on the order in which vectors are added in the head-to-tail method.

Example

*order of
addition*

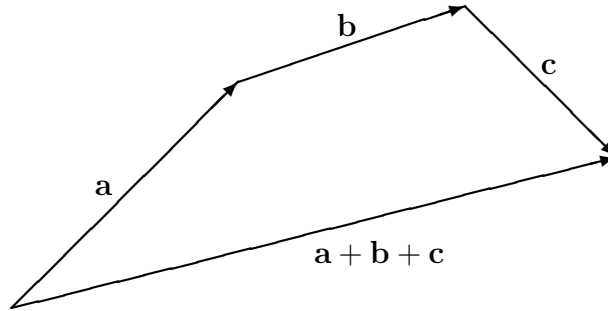
If the object is first displaced towards the north by 10 km, then east by 30 km, the resultant displacement \mathbf{d} is still the same as in the example above.



The head-to-tail method can be used for adding three or more vectors. It does not matter in which order these are combined.

Example

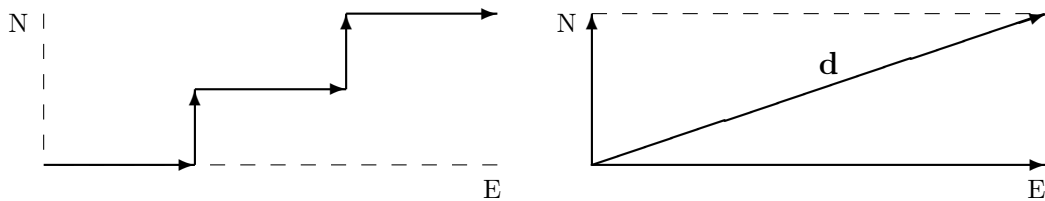
*adding
three
vectors*



Example

*rearranging
vectors*

A plane flew 30 km east and 10 km north in a zig-zag fashion. The diagrams below show that it doesn't matter how we rearrange and combine the zigs and zags, the total displacement to the east is 30 km, and the total displacement to the north is 10 km. The resultant displacement is always the same.

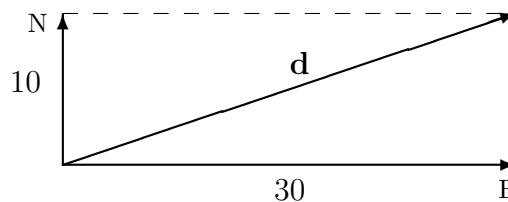


In the physical world many vectors interact simultaneously. For example, a plane may be displaced both towards the east and towards the north at the same time. In this case the resultant vector can still be found by adding the the individual vectors that produce the action. The example above suggests why if we imagine that the simultaneous displacements correspond to a large number of zig-zags.

Example (continued)

*simultaneous
displacements*

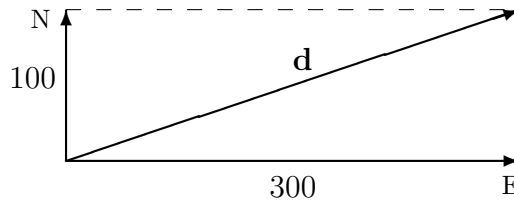
If a plane travels 30 km to the east, and is *simultaneously* blown off course by 10 km to the north, then we can think of the resultant displacement **d** as consisting of two parts, (a) the displacement 30 km east and (b) the displacement 20 km north.



resultant
velocity

Example

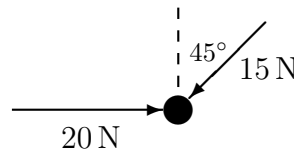
If a plane travels at 300 km/h towards the east, and it is *simultaneously* blown off course by a 100 km/h wind from the south, then we can think of its resultant velocity \mathbf{v} as consisting of two parts, the first a velocity of 300 km/h to the east and the second a 100 km/h to the north. The resultant velocity - both its length and direction - can be calculated from knowing these two velocities.



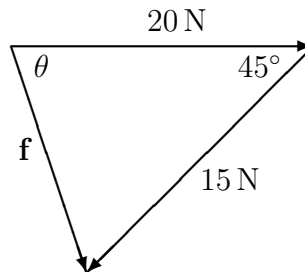
resultant
force

Example

The vector diagram shows two forces acting on a ball bearing. What is the resultant force?⁴



Answer



The length of \mathbf{f} can be found using the cosine rule.

$$|\mathbf{f}|^2 = 20^2 + 15^2 - 2 \times 20 \times 15 \times \cos 45^\circ = 200.74$$

$$|\mathbf{f}| = 14.17$$

The angle θ can be found from the sine rule.

$$\frac{\sin \theta}{15} = \frac{\sin 45^\circ}{14.17}$$

$$\sin \theta = 0.7485$$

$$\theta = 48.46$$

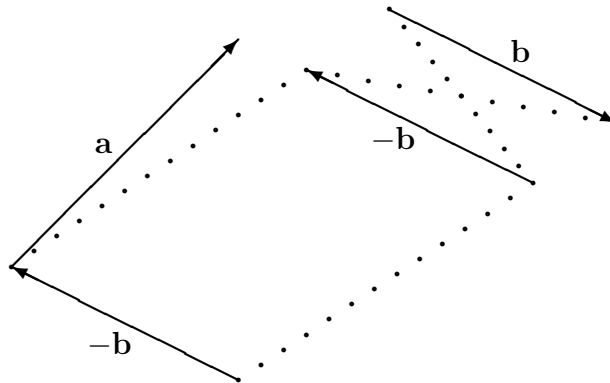
The resultant force \mathbf{f} has length 14.17 N and direction -48.46° relative to the horizontal.

⁴The resultant force can be thought of as a single force which is equivalent to the two forces shown in the diagram

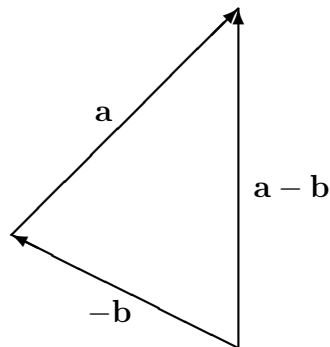
Definition 1.2.2

The difference $\mathbf{a} - \mathbf{b}$ of two vectors \mathbf{a} and \mathbf{b} is the sum of \mathbf{a} and $-\mathbf{b}$.

Step 1: Construct $-\mathbf{b}$, then shift it until its head just touches the tail of \mathbf{a} .



Step 2: The vector $\mathbf{a} - \mathbf{b}$ is equal to the vector that begins at the tail of $-\mathbf{b}$ and ends at the head of \mathbf{a} .



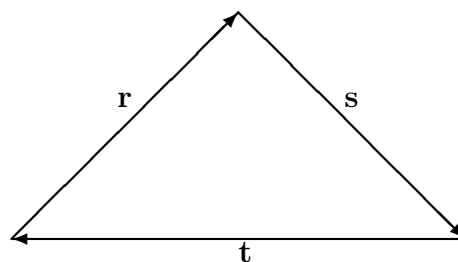
Definition 1.2.3

The zero vector $\mathbf{0}$ has length 0. Its direction is not defined.

Example

The vectors \mathbf{r} , \mathbf{s} and \mathbf{t} below satisfy the vector equation $\mathbf{r} + \mathbf{s} + \mathbf{t} = \mathbf{0}$.

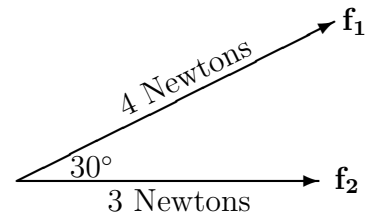
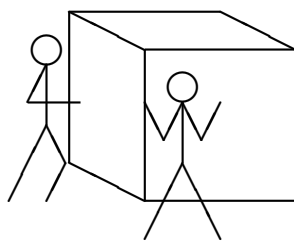
a vector equation



Exercise 1.2

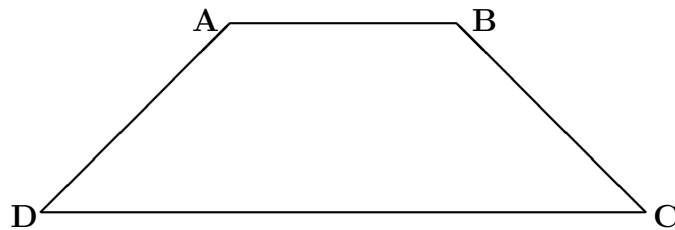
NOTE: These exercises use the rules of trigonometry, which are covered in MathsStart Topics 5 and 6.

- A plane is flying NE at its maximum velocity of 500 km/h according to its instruments. However there is a northerly⁵ wind of 100 km.
 - Sketch this situation on a vector diagram.
 - Calculate the resultant velocity of the plane.
- A yacht is cruising at 15 knots on a compass bearing of N124°. Find the actual speed and direction of the yacht if there is also an ocean current of 5 knots in the direction N58°.
- Two students push a heavy object. The vector diagram below shows the forces on that object.



Find the magnitude and direction of the resultant force.

- The four points A, B, C, D are vertices of the trapezium.

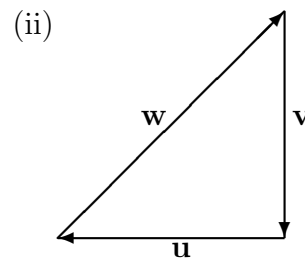
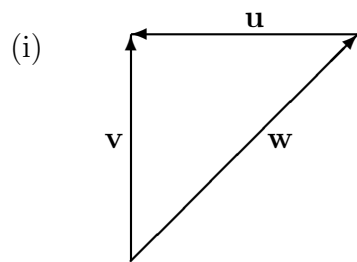


Simplify the following vector expressions as much as possible:

- $\overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC}$
- $\overrightarrow{DB} - \overrightarrow{AB}$
- $\overrightarrow{CA} - \overrightarrow{DA} + \overrightarrow{DB}$
- $\overrightarrow{DA} + \overrightarrow{AD}$

⁵Note that a northerly wind blows *from* the North!

5. Construct vector equations for \mathbf{u} , \mathbf{v} and \mathbf{w} if:



6. A pilot wants to fly due east at a speed of 400 km/h, however there is a 50km/h wind towards the SE. What direction must the pilot fly in, and at what speed?

7. Use a diagram to show that $\mathbf{a} - \mathbf{a} = \mathbf{0}$ for any vector \mathbf{a} .

1.3 Vector Algebra

We can perform a limited amount of algebra on vectors: adding, subtracting, scalar multiplication, simplifying by collecting and combining like terms, multiplying out brackets, and solving linear equations. These techniques contribute to the usefulness of vectors in applications.

Example

collecting
&
combining
like terms

In a vector expression like $2\mathbf{a} + \mathbf{b} + \mathbf{a} + 3\mathbf{b} - 4\mathbf{a}$, the terms $2\mathbf{a}$, \mathbf{a} and $-4\mathbf{a}$ are called *like terms* because each is a scalar multiple of the vector \mathbf{a} . We can combine like terms by adding their coefficients.

$$2\mathbf{a} + \mathbf{a} - 4\mathbf{a} = (2 + 1 - 4)\mathbf{a} = -\mathbf{a}$$

In a similar way we can combine the multiples of \mathbf{b} .

We simplify $2\mathbf{a} + \mathbf{b} + \mathbf{a} + 3\mathbf{b} - 4\mathbf{a}$ by collecting and combining like terms, in exactly the same way as we do in ordinary algebra:

$$2\mathbf{a} + \mathbf{b} + \mathbf{a} + 3\mathbf{b} - 4\mathbf{a} = -\mathbf{a} + 3\mathbf{b}.$$

Example

expanding
brackets

Vector expressions may also contain brackets such as $2(\mathbf{a} + 3\mathbf{b})$. These can be multiplied out as in ordinary algebra.

$$2\mathbf{a} + \mathbf{b} - 2(\mathbf{a} + 3\mathbf{b}) = 2\mathbf{a} + \mathbf{b} - 2\mathbf{a} - 6\mathbf{b} = -5\mathbf{b}$$

Example

solving
vector
equations

We can solve vector equations in *almost* the same way that we solve ordinary linear equations⁶, the main difference being that it is customary to *multiply but not divide* vectors by scalars.

Solve the equation $2\mathbf{x} + \mathbf{a} - \mathbf{b} + \mathbf{c} = 2\mathbf{a} - 3\mathbf{b} + \mathbf{c}$ for the unknown vector \mathbf{x} .

Answer

$$\begin{aligned} 2\mathbf{x} + \mathbf{a} - \mathbf{b} + \mathbf{c} &= 2\mathbf{a} - 3\mathbf{b} + \mathbf{c} \\ 2\mathbf{x} &= \mathbf{a} - 2\mathbf{b} && \dots \text{ now multiply both sides by } \frac{1}{2}. \\ \mathbf{x} &= \frac{1}{2}(\mathbf{a} - 2\mathbf{b}) \text{ or } \frac{1}{2}\mathbf{a} - \mathbf{b} \end{aligned}$$

As you can see, the methods we use for simplifying linear expressions in ordinary algebra carry intuitively over to vectors. *However*, vectors are different from numbers, and we should confirm that what we are doing is valid.

⁶A *linear* equation is an equation in which the unknown only occurs to the first power.

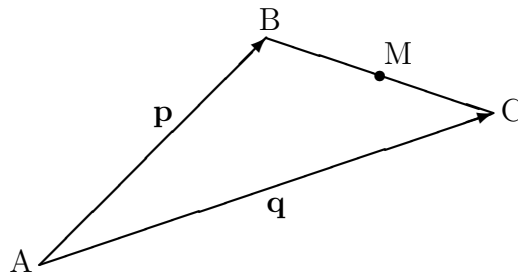
1.4 Geometric Problems

Vector algebra can be used to solve geometric problems.

Example

*midpoint
formula*

In the vector diagram M is the midpoint of BC. Vector algebra can be used to find an expression for \overrightarrow{AM} in terms of \mathbf{p} and \mathbf{q} .



The diagram shows that

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \mathbf{p} + \overrightarrow{BM}$$

As M is the midpoint of BC, you can see that

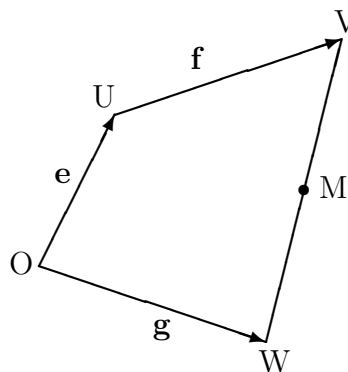
$$\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\mathbf{q} - \mathbf{p})$$

Substituting this into the previous equation:

$$\begin{aligned}\overrightarrow{AM} &= \mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p}) \\ &= \frac{1}{2}(\mathbf{p} + \mathbf{q})\end{aligned}$$

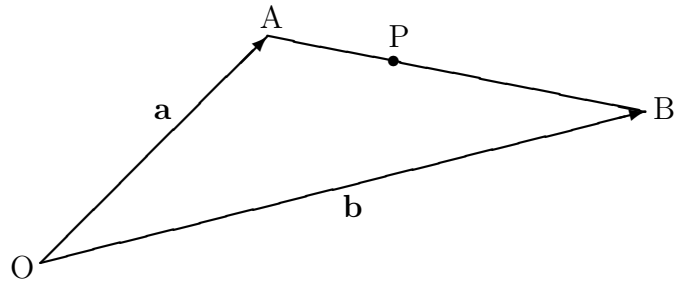
Exercise 1.4

1. In the diagram M is the midpoint of VW.



Find expressions for \overrightarrow{OV} , \overrightarrow{VW} , \overrightarrow{UM} and \overrightarrow{OM} in terms of \mathbf{e} , \mathbf{f} , \mathbf{g} .

2. In the diagram P divides AB in the ratio 1:2. Find an expression for \vec{OP} in terms of \mathbf{a} and \mathbf{b} .



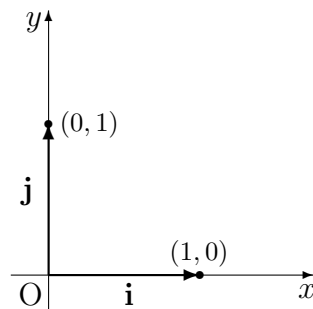
Chapter 2

Vector Components

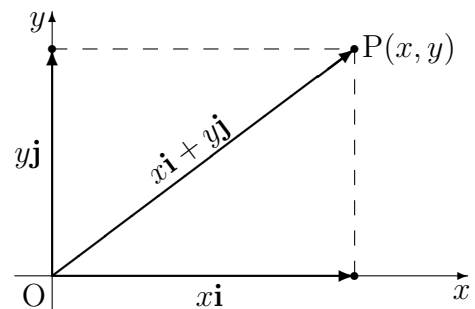
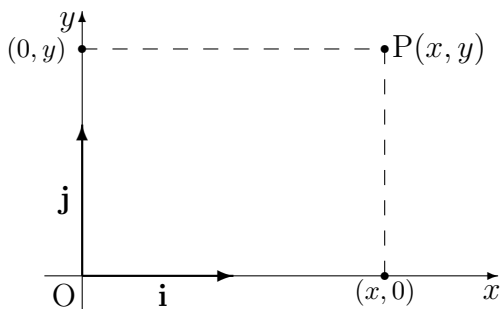
2.1 Vectors and Co-ordinates

If a Cartesian co-ordinate system was added to a vector diagram, then co-ordinates could be assigned to the top and tail of each vector. We could then describe vectors in terms of these co-ordinates, and use this information in calculations.

It is convenient to introduce vectors \mathbf{i} and \mathbf{j} on the x - and y -axes, each with tail at the origin and pointing in the positive x - and y -directions, and each with length 1. These are called *unit vectors* as they have length 1. When talking about them, we just call them “ i ” and “ j ”.



We can use co-ordinates to describe vectors ... and we can use vectors to locate points on the Cartesian plane.



Definition 2.1.1

If $P(x, y)$ is a point on the Cartesian plane, and if O is the origin then $\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$ is called the **position vector** of the point P .

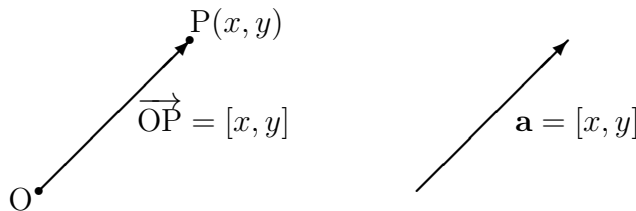
The notation $[x, y]$ is commonly used as an abbreviation for the vector $x\mathbf{i} + y\mathbf{j}$. The numbers x and y are called the *vector components* of $x\mathbf{i} + y\mathbf{j}$, x being the x -component and y being the y -component.

Example

*vector
components*

(a) $\mathbf{i} = [1, 0]$ (b) $\mathbf{j} = [0, 1]$ (c) $2\mathbf{i} - 3\mathbf{j} = [2, -3]$ (d) $\mathbf{0} = [0, 0]$

If \mathbf{a} is a vector which is equal to position vector $\overrightarrow{OP} = [x, y]$, then we can also write $\mathbf{a} = [x, y]$, and also say that the numbers x and y are the vector components of \mathbf{a} .

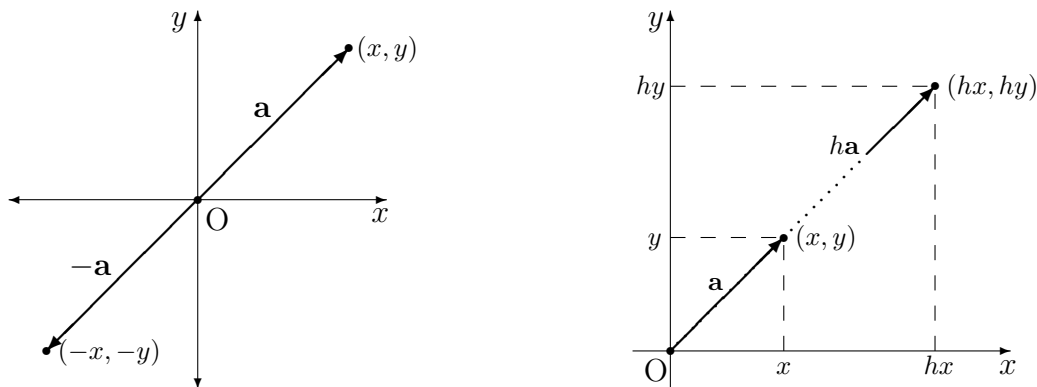


Working with components is very intuitive.

Properties of Components

- If $\mathbf{a} = [x, y]$, then $-\mathbf{a} = [-x, -y]$
- If $\mathbf{a} = [x, y]$ and h is a scalar, then $h\mathbf{a} = [hx, hy]$
- If $\mathbf{a} = [x_1, y_1]$ and $\mathbf{b} = [x_2, y_2]$, then $\mathbf{a} \pm \mathbf{b} = [x_1 \pm x_2, y_1 \pm y_2]$

The first two properties are illustrated below for position vectors, but are true for *all* vectors.



The three properties can be *proved* using vector algebra.

Example

adding
vector
components

Show that if $\mathbf{a} = [x_1, y_1]$ and $\mathbf{b} = [x_2, y_2]$, prove that $\mathbf{a} + \mathbf{b} = [x_1 + x_2, y_1 + y_2]$.

Answer

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= [x_1, y_1] + [x_2, y_2] \\ &= (x_1\mathbf{i} + y_1\mathbf{j}) + (x_2\mathbf{i} + y_2\mathbf{j}) \\ &= (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} \\ &= [x_1 + x_2, y_1 + y_2]\end{aligned}$$

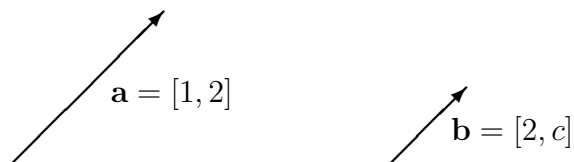
Example

parallel
vectors

Find c if $\mathbf{a} = [1, 2]$ and $\mathbf{b} = [2, c]$ are parallel.

Answer

Two vectors can only be parallel if one is a scalar multiple of the other.



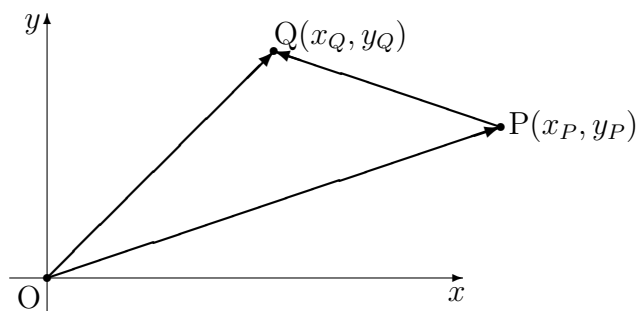
If \mathbf{a} and \mathbf{b} are parallel, then $\mathbf{a} = h\mathbf{b}$ for some scalar h .

$$\begin{aligned}\text{Put } [1, 2] &= h[2, c], \\ \text{then } 2h &= 1 \text{ and } ch = 2, \\ \text{so } h &= \frac{1}{2} \text{ and } c = 4.\end{aligned}$$

Theorem¹ If $P(x_P, y_P)$ and $Q(x_Q, y_Q)$ are any two points in a plane, then

$$\overrightarrow{PQ} = [x_Q - x_P, y_Q - y_P].$$

Proof.



¹A *theorem* is an important or useful result.

It can be seen from the diagram that the position vectors \overrightarrow{OP} and \overrightarrow{OQ} are given by

$$\begin{aligned}\overrightarrow{OP} &= [x_P, y_P] = [x_P\mathbf{i} + y_P\mathbf{j}] \\ \text{and } \overrightarrow{OQ} &= [x_Q, y_Q] = [x_Q\mathbf{i} + y_Q\mathbf{j}].\end{aligned}$$

$$\begin{aligned}\text{So } \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= x_Q\mathbf{i} + y_Q\mathbf{j} - ([x_P\mathbf{i} + y_P\mathbf{j}]) \\ &= (x_Q - x_P)\mathbf{i} + (y_Q - y_P)\mathbf{j} \\ &= [x_Q - x_P, y_Q - y_P].\end{aligned}$$

In words: the components of a directed line segment are the co-ordinates of the top minus the co-ordinates of the tail.

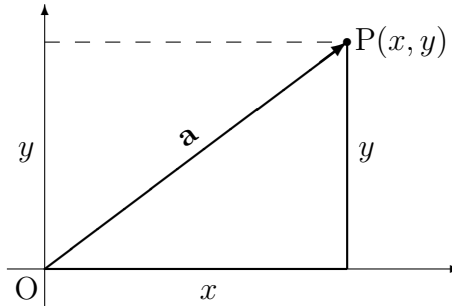
Exercise 2.1

1. If $\mathbf{a} = [2, -3]$ and $\mathbf{b} = [3, 4]$, find the vector components of
 - (i) $3\mathbf{a} + 4\mathbf{b}$
 - (ii) $2\mathbf{b} - \mathbf{a}$
 2. Find c if $\mathbf{a} = [5, -2]$ and $\mathbf{b} = [10, c]$ are parallel.
 3. If $\mathbf{a} = [2, -1]$, $\mathbf{b} = [3, 4]$ and $\mathbf{c} = [p, 15]$, find p so that \mathbf{c} is parallel to $\mathbf{b} - \mathbf{a}$.
 4. If $P(1, -2)$, $Q(-3, 1)$, $R(2, 4)$ and $S(-1, 6)$ are points the Cartesian plane, find the components of the vector $\overrightarrow{PQ} + 2\overrightarrow{RS}$.
-

2.2 Formula for Length

The magnitude or length of any vector can be found from its components using Pythagoras' theorem.

The diagram below shows that the length of $\mathbf{a} = [x, y]$ is $|\mathbf{a}| = \sqrt{x^2 + y^2}$.



Theorem If \mathbf{a} is a vector and h is a scalar, then $|h\mathbf{a}| = |h||\mathbf{a}|$.

Proof. Let $\mathbf{a} = [x, y]$, then $h\mathbf{a} = [hx, hy]$ and

$$|h\mathbf{a}| = \sqrt{(hx)^2 + (hy)^2} = h\sqrt{x^2 + y^2} = |h||\mathbf{a}|.$$

Finding a unit vector

Corollary If \mathbf{a} is a vector, then $\frac{1}{|\mathbf{a}|}\mathbf{a}$ has length 1 (ie. it is a unit vector having the same direction as \mathbf{a}).^{2,3}

Proof. The theorem shows that the length of $\frac{1}{|\mathbf{a}|}\mathbf{a}$ is $|\frac{1}{|\mathbf{a}|}\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$.

Exercise 2.2

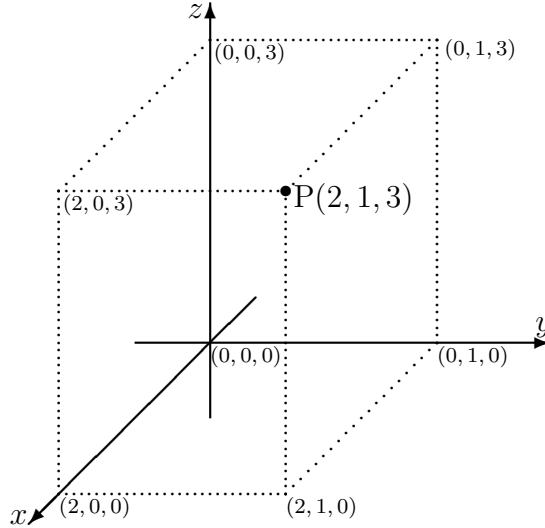
1. If $\mathbf{a} = [1, 3]$ and $\mathbf{b} = [2, -1]$, find $|\mathbf{a} - 2\mathbf{b}|$.
 2. If $\mathbf{a} = [1, 4]$, $\mathbf{c} = [2, 1]$ and $\mathbf{a} + \mathbf{b} = \mathbf{c}$, find $|\mathbf{b}|$.
 3. Find unit vectors having the (i) same direction and (ii) opposite direction to
(i) $\mathbf{a} = [3, -4]$ (ii) $\mathbf{b} = [-5, 12]$ (iii) $\mathbf{c} = [-6, 8]$ (iv) $\mathbf{O} = [0, 0]$.
 4. If $P(1, -1)$, $Q(-2, 1)$, $R(2, 4)$ and $S(-1, 5)$ are points, find the unit vector parallel to $\overrightarrow{PQ} + 2\overrightarrow{RS}$.
-

²A *corollary* is a deduction from a theorem.

³Unit vectors like this are used in astronomy, where the direction of a star is may be known but its distance from earth is not.

2.3 Vectors in Three Dimensions

All of the previous ideas can be extended to three dimensions. The diagram below shows a three dimensional co-ordinate system.



If you arrange your thumb and the first two fingers on your right hand in the same directions as the three axes in the diagram, with the thumb pointing in the z -direction, then first finger point will point in the x -direction and the second finger will point in the y -direction. When a screw is screwed in the z -direction, your right hand will rotate (anti-clockwise) from the x -axis towards the y -axis.

The point P has co-ordinates $(2, 1, 3)$. It can be reached by moving 2 units along the x -axis from the origin $(0, 0, 0)$, then moving 1 unit parallel to the y -axis, then moving 3 units parallel to the z -axis.

The *projection* of point $P(2, 1, 3)$ onto the (x, y) -plane is $(2, 1, 0)$, the projection onto the (y, z) -plane is $(0, 1, 3)$, and the projection onto the (z, x) -plane is $(2, 0, 3)$.

Example

In three dimensions, \mathbf{i} , \mathbf{j} and \mathbf{k} represent the unit vectors which have tails at the origin and which point along the x -, y - and z -axes in a positive direction.

The position vector of the point $P(2, 1, 3)$ above is $\vec{PQ} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

\vec{PQ} has components $[2, 1, 3]$ and length $|\vec{PQ}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$.

In general, the length of the three dimensional vector $[x, y, z]$ is $\sqrt{x^2 + y^2 + z^2}$.

Exercise 2.3

1. Draw a three dimensional co-ordinate system and show the points:
(i) $(3, 0, 0)$ (ii) $(3, 2, 0)$ (iii) $(3, 2, 1)$ (iv) $(-1, -1, -1)$
 2. How far is $P(3, 4, 5)$ from the origin?
 3. If $\mathbf{a} = [1, 3, 2]$ and $\mathbf{b} = [2, -1, 4]$, find $|\mathbf{a} - 2\mathbf{b}|$.
 4. If $\mathbf{a} = [1, 0, 4]$, $\mathbf{c} = [2, 1, 0]$ and $\mathbf{a} + 2\mathbf{b} = \mathbf{c}$, find $|\mathbf{b}|$.
 5. Find a unit vector having the same direction as $\mathbf{a} = [3, -4, 5]$
 6. A rectangular prism (box) has sides 2m, 3m and 4m long. What is the maximum distance between 2 corners?
-

Chapter 3

The Scalar Product

3.1 Scalar Products and Angles

Scalar products enable the angle between two vectors to be calculated.

Definition 3.1.1

1. If $\mathbf{a} = [a_1, a_2]$ and $\mathbf{b} = [b_1, b_2]$, then the scalar product of \mathbf{a} and \mathbf{b} is

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2.$$

2. If $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$, then the scalar product of \mathbf{a} and \mathbf{b} is

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Scalar products are also called *dot products*.

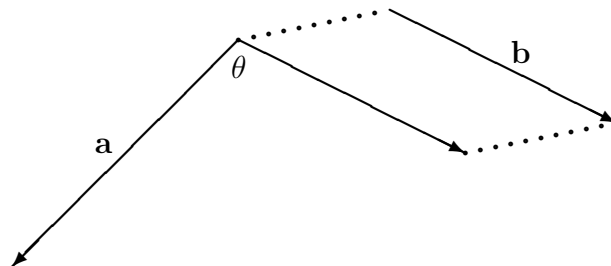
Example

(a) If $\mathbf{a} = [1, 4]$ and $\mathbf{b} = [2, -3]$, then $\mathbf{a} \bullet \mathbf{b} = 1 \times 2 + 4 \times -3 = -10$.

(b) If $\mathbf{u} = [1, 1, 2]$ and $\mathbf{v} = [1, 1, -1]$, then $\mathbf{u} \bullet \mathbf{v} = 1 \times 1 + 1 \times 1 + 2 \times -1 = 0$.

*scalar
or dot
products*

The angle between two position vectors is the *smallest* angle they make at the origin. If the vectors are not position vectors, then we can think of the angle between them as being the angle between their equivalent position vectors. Alternatively, it can be thought of as the smallest angle between them if they are shifted so that their tails touch.



The scalar product is important because of the following theorem.

Theorem If θ is the angle between two vectors \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$

Example

*finding
an angle*

Find the angle between the vectors $\mathbf{a} = [3, 2]$ and $\mathbf{b} = [2, 1]$

Answer

The dot product is $\mathbf{a} \bullet \mathbf{b} = 3 \times 2 + 2 \times 1 = 8$.

The lengths are $|\mathbf{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}$ and $|\mathbf{b}| = \sqrt{2^2 + 1^2} = \sqrt{5}$.

The angle is given by

$$\cos \theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{8}{\sqrt{13}\sqrt{5}} = \frac{8}{\sqrt{65}}.$$

So $\theta = 7.12^\circ$

Example

*perpendicular
vectors*

Show that the position vectors $[1, 1, 2]$ and $[1, 1, -1]$ are perpendicular.

Answer

The dot product is $\mathbf{a} \bullet \mathbf{b} = 1 \times 1 + 1 \times 1 + 2 \times -1 = 0$.

This shows that $\cos \theta = 0$, so the vectors are perpendicular.

Exercise 3.1

1. Find the angle between $[1, 2]$ and $[3, -3]$.
 2. Find the angle that points $P(1, 1)$ and $Q(1, 2)$ subtend at the origin.
 3. Find the angle at which the line between $P(1, 1)$ and $Q(2, 3)$ intersects the x -axis. (Hint: Use the unit vector \mathbf{i} in your calculations)
 4. A cube has sides 1 m long. If three of the sides are taken to be the x -, y - and z - axes, find the angle between z -axis and the diagonal running between the origin and the furthest edge.
 5. A rectangular box has dimensions 1m, 2m and k m. What should the value of k be if the angle between longest diagonal and the longest edge of the box needs to be 20° ?
-

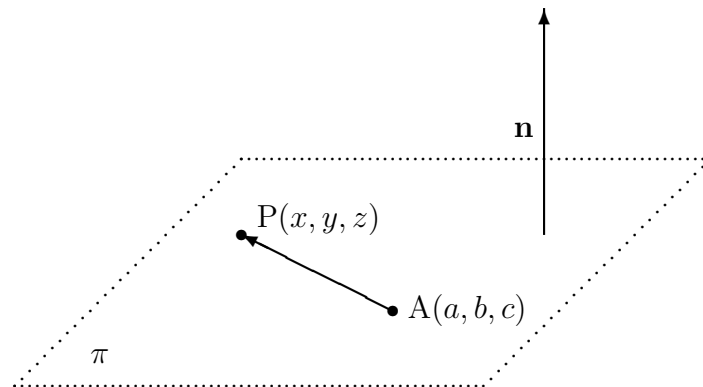
3.2 Planes in 3-dimensions

A vector is said to be normal to a plane if it is perpendicular to every vector that lies in the plane.

The equation of a plane can be found from:

- a point $A(a, b, c)$ on the plane.
- a normal vector $\mathbf{n} = [l, m, n]$ to the plane

In the diagram below, (x, y, z) is a general point on the plane π .



As $AP = [x - a, y - b, z - c]$ and $\mathbf{n} = [l, m, n]$ are perpendicular, their scalar product is zero and

$$l(x - a) + m(y - b) + n(z - c) = 0$$

The equation of π is $l(x - a) + m(y - b) + n(z - c) = 0$.

Example

*equation
of a
plane*

Find the equation of the plane with normal vector $[1, 0, -2]$ that passes through the point $(1, 1, 3)$, and find one point on the plane.

Answer

- The equation is $1(x - 1) + 0(y - 1) - 2(z - 3) = 0$, that is $x - 2z + 5 = 0$
- Let $z = 0$, then $x = -5$. the equation imposes no restrictions on the value of y , so take $y = 1$. One point is $(-5, 1, 0)$.

Example

*normal
to a
plane*

Find a normal vector to the plane $x + 2y - z = 4$

Answer

A normal vector is $[1, 2, -1]$.

Exercise 3.2

1. Find the equation of a plane with normal vector $[2, 3, -1]$ that passes through $(1, 1, 1)$.
 2. $A(1, 2, -1)$ and $B(2, 1, -1)$ are two points. Find the equation of the plane which passes through A and is normal to AB .
 3. Find a unit vector normal to the plane $x + y + z = 3$.
 4. Find the equation of the plane perpendicular to the z -axis and passing through the point $(1, -2, 5)$
-

Appendix A

Answers

Exercise 1.1

1. check with tutor
2. check with tutor
3. After 5 minutes, it increased its velocity by 10 percent, then 10 minutes later it turned in the opposite direction and moved at the same velocity it started with.
4. (d), (e) and (l) are false
5. check with tutor

Exercise 1.2

1. (a) check with tutor
(b) Magnitude = $\sqrt{500^2 + 100^2 - 2 \times 500 \times 100 \times \cos 45} = 435.1$,
Direction = $45^\circ + \theta$ from North, where

$$\frac{\sin \theta}{100} = \frac{\sin 45}{435.1} \Rightarrow \theta = 9.353^\circ$$

That is, the direction is a compass bearing of N54.353°.

2. (a) Magnitude = $\sqrt{5^2 + 15^2 - 2 \times 5 \times 15 \times \cos 114} = 17.64$
(b) Direction = N109.0°
3. (a) Magnitude = $\sqrt{3^2 + 4^2 - 2 \times 3 \times 4 \times \cos 150} = 6.766$
(b) Direction = 17.19° above horizontal.
4. (a) \vec{DC} (b) \vec{DA} (c) \vec{CB} (d) $\mathbf{0}$
5. (a) $\mathbf{u} - \mathbf{v} + \mathbf{w} = \mathbf{0}$ (b) $\mathbf{u} + \mathbf{w} + \mathbf{v} = \mathbf{0}$

6. (a) Magnitude = 436.9 km/h
 (b) Direction = 4.6° S of E (or a compass bearing of N 94.6°).
7. check with tutor

Exercise 1.4

1. (a) $\overrightarrow{OV} = \mathbf{e} + \mathbf{f}$
 (b) $\overrightarrow{VW} = -\mathbf{f} - \mathbf{e} + \mathbf{g}$
 (c) $\overrightarrow{UM} = \mathbf{f} + \frac{1}{2}(-\mathbf{f} - \mathbf{e} + \mathbf{g}) = \frac{1}{2}\mathbf{f} - \frac{1}{2}\mathbf{e} + \frac{1}{2}\mathbf{g}$
 (d) $\overrightarrow{OM} = \mathbf{e} + \mathbf{f} + \frac{1}{2}(-\mathbf{f} - \mathbf{e} + \mathbf{g}) = \frac{1}{2}\mathbf{e} + \frac{1}{2}\mathbf{f} + \frac{1}{2}\mathbf{g}$
2. $\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

Exercise 2.1

1. (i) [18, 7] (ii) [4, 11]
2. $c = -4$
3. $p = 3$
4. [-10, 7]

Exercise 2.2

1. $\sqrt{34}$
2. $\sqrt{10}$
3. (i) $\pm\frac{1}{5}[3, -4]$
 (ii) $\pm\frac{1}{13}[5, -12]$
 (iii) $\pm\frac{1}{5}[-3, 4]$
 (iv) the direction is not defined.
4. $\frac{1}{\sqrt{97}}[-9, 4]$

Exercise 2.3

1. check with tutor
2. $\sqrt{50}$
3. $\sqrt{70}$
4. $\sqrt{\frac{9}{2}}$
5. $\frac{1}{\sqrt{50}}[3, -4, 5]$

6. $\sqrt{29}$ m

Exercise 3.1

1. 108.4°

2. 18.43°

3. 63.44°

4. 54.74°

5. 6.143

Exercise 3.2

1. $2(x - 1) + 3(y - 1) - (z - 1) = 0$

2. $1(x - 1) - 1(y - 2) = 0$

3. $\frac{1}{3}[1, 1, 1]$

4. $z = 5$