

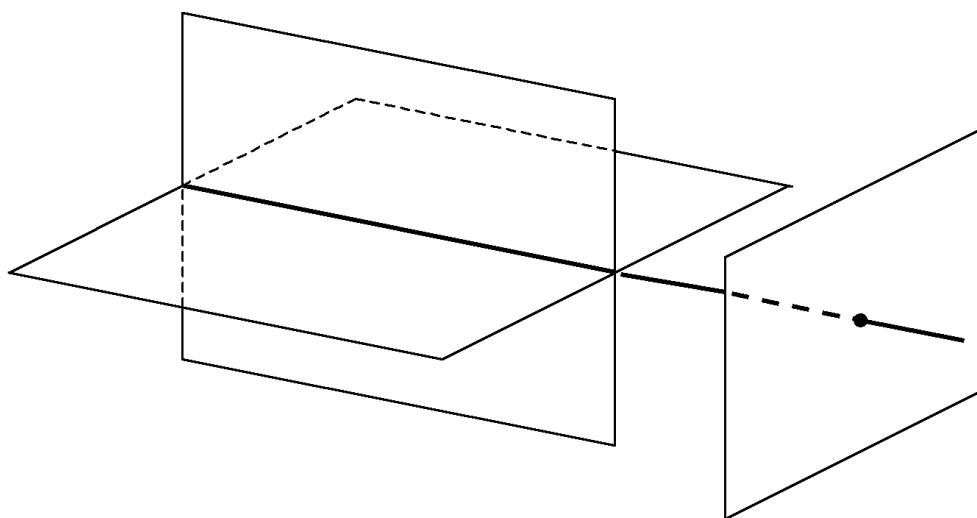
MathsTrack



(NOTE Feb 2013: This is the old version of MathsTrack.
New books will be created during 2013 and 2014)

Topic 4

Systems of Linear Equations



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This Topic . . .

Many practical problems in economics, engineering, biology, electronics, communication, etc can be reduced to solving a system of linear equations. These equations may contain thousands of variables, so it is important to solve them as efficiently as possible.

The Gauss-Jordan method is the most efficient way for solving large linear systems on a computer, and is used in specialist mathematical software packages such as MATLAB. The Gauss-Jordan method can also be used to find the complete solution of a system of equations when there are infinitely many solutions.

This topic introduces the Gauss and Gauss-Jordan methods. For convenience, the examples and exercises in this module use small systems of equations, however the methods are applicable to systems of any size.

The topic has 3 chapters:

Chapter 1 introduces systems of linear equations and elementary row operations.

It begins by showing how solving a pair of simultaneous equations in two variables using algebra is related to Gauss's method for solving a large system of linear equations, and then explains the difference between the Gauss and the Gauss-Jordan methods.

After reading this chapter, you will have a good understanding of how to solve a large system of linear equations using elementary row transformations.

Chapter 2 examines systems of linear equations that do not have a unique solution¹. The chapter shows how to recognise when systems have no solutions or have infinitely many solutions, and how to describe the solutions when there are infinitely many.

Chapter 3 explains how to use Gauss-Jordan elimination to find the inverse of a matrix.

¹A system of linear equations can have one solution, no solutions or infinitely many solutions. When it has exactly one solution, it is said to have a unique solution.

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Chapter 1

Systems of Linear Equations

1.1 Linear Equations

If a, b, c are numbers, the graph of an equation of the form

$$ax + by = c$$

is a straight line. Accordingly this equation is called a *linear equation* in the variables x and y .

When an equation has only 2 or 3 variables, we usually denote the variables by the letters x, y and z , but when there are more it is often convenient to denote the variables by x_1, x_2, \dots, x_n .

In general, a linear equation in variables x_1, x_2, \dots, x_n is one that can be put in the form

$$(1.1) \quad a_1x_1 + a_2x_2 + \dots + a_nx_n = b ,$$

where a_1, a_2, \dots, a_n are the coefficients of the variables. Notice that the variables occur only to the first power in the equation, that they do not appear in the argument of any function such as a logarithm or exponential or any other sort of function, and that the variables are not multiplied together.

Example

The equations

$$2x + y = 7 \quad z = 2x - 6y \quad 3x_1 - 2x_2 + 5x_3 + x_4 = 4$$

are all linear equations as they can be put in the form (1.1) above, whereas

$$2x^2 + \sqrt{y} = 7 \quad z = 2 \ln x - 6 \exp y \quad 3x_1 - 2x_2 + 5x_3x_4 = 4$$

are not linear equations.

*linear &
nonlinear
equations*

A *solution* of an equation is a set of numerical values for the variable which satisfies the equation.

Example

*solution
of linear
equation*

- Two solutions of $2x + y = 7$ are $x = 3, y = 1$ and $x = 1, y = 5$.
- One solution of $3x_1 - 2x_2 + 5x_3 + x_4 = 4$ is $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$.

We often need to solve a number of linear equations at the same time. A collection of linear equations is called a *system of linear equations* or a *linear system*. The linear system below has n variables (or unknowns) x_1, x_2, \dots, x_n in m equations.

$$(1.2) \quad \begin{array}{cccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & a_{1n}x_n & = & b_{1n} \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & a_{2n}x_n & = & b_{2n} \\ \vdots & & & & & \vdots & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & a_{mn}x_n & = & b_{mn} \end{array}$$

A *solution of a linear system* is a set of numbers which satisfies each of the equations simultaneously. A linear system has either one solution, no solutions, or infinitely many solutions.¹

Example

*solution
of linear
system*

- The only solution of the system of equations

$$\begin{array}{r} 2x + y = 7 \\ x - y = 2 \end{array}$$

is $(x, y) = (3, 1)$. [Check by substitution.]

- The system of equations

$$\begin{array}{r} 2x + y = 7 \\ 2x + y = 6 \end{array}$$

has no solution as the left sides are equal but the right sides are not.

- The system of equations

$$\begin{array}{r} 2x + y = 7 \\ 2x + y = 7 \end{array}$$

has infinitely many solutions - all number pairs (x, y) satisfying $2x + y = 7$.

A system of equations with at least one solution is called *consistent* and a system with no solutions is called *inconsistent*.

¹If a linear system has two solutions then it must have infinitely many solutions.

1.2 Elementary Row Operations

Appendix A revises how to solve pairs of simultaneous equations. These methods are not easy to use on larger systems of equations so another method is needed. This is called *Gaussian elimination*.²

Gauss's method is to transform the original system of equations into another system of equations which have the same solution but which is easier to solve. The method is efficient, simple and easy to program on a computer.

Two systems of linear equations are called *equivalent* if they have the same solutions.

Example

*equivalent
linear
system*

The system of linear equations

$$\begin{aligned}x + 4y &= 17 \\2x + 3y &= 9\end{aligned}$$

has the same solutions as the system

$$\begin{aligned}x + 4y &= 17 \\y &= 5 ,\end{aligned}$$

so they are *equivalent* systems of equations. However, the second system is much easier to solve. Using *back-substitution*:

$$\begin{aligned}y &= 5 \\x &= 17 - 4y = 17 - 4 \times 5 = -3 .\end{aligned}$$

The first system of equations in this example can be written as a matrix equation

$$AX = B$$

where A is the coefficient matrix, i.e.

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 9 \end{bmatrix} .$$

(Verify by matrix multiplication that this represents the same pair of equations.)

In Gauss's method this matrix equation is represented in the form of a table

$$[A|B] = \left[\begin{array}{cc|c} 1 & 4 & 17 \\ 2 & 3 & 9 \end{array} \right]$$

²Carl Freidrich Gauss (1777-1855) is considered to be one of the three greatest mathematicians of all time (the others being Archimedes and Newton). He is said to have been the last mathematician to know everything in mathematics.

which is called an *augmented matrix*.³

An augmented matrix is a shorthand way of writing a linear system without using variables, and corresponds to the way information about the system is entered into a computer. Augmented matrices are called equivalent when the corresponding linear systems are equivalent.

Example

*equivalent
augmented
matrix*

The second linear system in the previous example

$$\begin{aligned}x + 4y &= 17 \\ y &= 5\end{aligned}$$

is represented by the equivalent augmented matrix

$$\left[\begin{array}{cc|c} 1 & 4 & 17 \\ 0 & 1 & 5 \end{array} \right].$$

The augmented matrix for the general linear system in (1.2) is

$$(1.3) \quad [A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{2n} \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{mn} \end{array} \right]$$

In Gauss's method we apply *elementary row operations* (see below) to rows of an augmented matrix. Each row corresponds to an equation in a system of equations, and the new system of equations formed after application of the elementary row operations is equivalent to the original system.

Elementary Row Operations (EROs)

1. Interchange two rows (equations).
2. Add a multiple of one row (equation) to another.
3. Multiply a row (equation) by a nonzero number.

Example

The system of equations

$$\begin{aligned}2x + 3y &= 9 \\ x + 4y &= 17\end{aligned}$$

is solved below by using elementary row operations to transform it into a simpler equivalent system.

³Augmented means enlarged.

Step 1: Represent the system of equations as an augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 9 \\ 1 & 4 & 17 \end{array} \right] \quad \begin{cases} 2x + 3y = 9 \\ x + 4y = 17 \end{cases}$$

Step 2: Interchange row 1 and row 2.

$$\left[\begin{array}{cc|c} 1 & 4 & 17 \\ 2 & 3 & 9 \end{array} \right] \quad \begin{cases} x + 4y = 17 \\ 2x + 3y = 9 \end{cases}$$

Step 3: Add $-2 \times$ row 1 to row 2 to create a zero in the (1, 2) position.⁴

$$\left[\begin{array}{cc|c} 1 & 4 & 17 \\ 0 & -5 & -25 \end{array} \right] \quad \begin{cases} x + 4y = 17 \\ -5y = -25 \end{cases}$$

Step 4: Multiply row 2 by $-\frac{1}{5}$.

$$\left[\begin{array}{cc|c} 1 & 4 & 17 \\ 0 & 1 & 5 \end{array} \right] \quad \begin{cases} x + 4y = 17 \\ y = 5 \end{cases}$$

The solution can now be found from the last equivalent system of equations by using back substitution:

$$\begin{aligned} y &= 5 \\ x &= 17 - 4 \times y = -3 \end{aligned}$$

The most efficient way of solving a system of linear equations is to transform the associated augmented matrix into an equivalent matrix having only zeros below the *leading diagonal*. When an augmented matrix is in this form the corresponding system of equations can be solved by back substitution.

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{4} \end{array} \right]$$

The most systematic way of performing this transformation is to first obtain zeros in the first column below the leading diagonal, then obtain zeros in the second column, etc. If you are working by hand then it is best to avoid introducing fractions until the final stages.

Note. It is good practice to record each row operation so you can retrace your steps if there is a mistake.

⁴This is an example of the second ERO. It just means that we subtract $2 \times$ row 1 from row 2

Example

Solve

$$\begin{aligned}x_1 + 4x_2 + x_3 &= 3 \\2x_1 - 3x_2 - 2x_3 &= 5 \\2x_1 + 4x_2 + 2x_3 &= 6\end{aligned}$$

Answer

Step 1: Represent linear system as an augmented matrix.

$$(*) \quad \left[\begin{array}{ccc|c} 1 & 4 & 1 & 3 \\ 2 & -3 & -2 & 5 \\ 2 & 4 & 2 & 6 \end{array} \right]$$

Step 2: Add $-2 \times$ row 1 to row 2, then add $-2 \times$ row 1 to row 3.

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 3 \\ 0 & -11 & -4 & -1 \\ 0 & -4 & 0 & 0 \end{array} \right] \quad \begin{array}{l} R2 = R2 - 2 \times R1^5 \\ R3 = R3 - 2 \times R1 \end{array}$$

If we next divide row 2 by -11 , then we will introduce the fractions $\frac{4}{11}$ and $\frac{1}{11}$. It is better to interchange rows 2 and 3.

Step 3: Interchange rows 2 and 3, then divide the new row 2 by -4 .⁶

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -11 & -4 & -1 \end{array} \right] \quad \begin{array}{l} R2, R3 = R3, R2 \\ R2 = R2 \div (-4) \end{array}$$

Step 4: Multiply row 2 by 11 and add it to row 3.

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & -1 \end{array} \right] \quad R3 = R3 + 11 \times R2$$

Step 5: Divide row 3 by (-4) .

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{4} \end{array} \right] \quad R3 = R3 \div (-4)$$

Step 6: Solve by back substitution

$$\begin{aligned}x_3 &= 1/4 \\x_2 &= 0 \\x_1 &= 3 - 4x_2 - x_3 = -5\end{aligned}$$

⁵This records that the new row 2 is obtained from (*) by subtracting $2 \times$ row 1 from row 2.

⁶This is an ERO as it is the same as multiplying by $-\frac{1}{4}$.

Exercise 1.2

1. Represent the following systems of equations as augmented matrices.

$$\begin{array}{ll} \text{(a)} & 2x - 3y + 2z = 4 \\ & x + 5y + 4z = 1 \\ & 7x + 2y - 3z = 3 \end{array} \qquad \begin{array}{l} \text{(b)} \\ \\ \\ \end{array} \begin{array}{l} 3x_2 + 2x_3 = 7 \\ x_1 + 4x_2 = 3 \\ 3x_1 + 3x_2 + 8x_3 = 1 \end{array}$$

2. Write down the system of equations in unknowns p, q which has augmented matrix

$$\left[\begin{array}{cc|c} 3 & -2 & 8 \\ 2 & 5 & 1 \end{array} \right].$$

3. (a) Which of the following operations are elementary row operations?

- i. Multiply row 2 by 2.
- ii. Multiply row 1 by 3 and add it to row 2.
- iii. Multiply row 2 by -3 and add it to row 1.
- iv. Replace row 2 by the sum of row 1 and row 2.
- v. Subtract row 2 from row 1.
- vi. Add 5 to each element in row 1.
- vii. Interchange column 1 and column 2.

(b) Perform operations i-v on the augmented matrix

$$\left[\begin{array}{cc|c} 3 & -2 & 8 \\ 2 & 5 & 1 \end{array} \right].$$

4. Use back-substitution to solve

$$\begin{array}{r} x_1 + 4x_2 - x_3 = 3 \\ x_2 + 2x_3 = -1 \\ x_3 = 4. \end{array}$$

5. Solve each of the linear systems below by using EROs.

$$\begin{array}{ll} \text{(a)} & x - 2y = 1 \\ & 2x + y = 7 \end{array} \qquad \begin{array}{l} \text{(b)} \\ \\ \end{array} \begin{array}{l} 2r - 3s = 6 \\ r - 7s = 25 \end{array}$$

$$\begin{array}{ll} \text{(c)} & x_1 - 2x_2 - 3x_3 = 7 \\ & 2x_1 + 3x_2 - 5x_3 = -13 \\ & 3x_1 - 4x_2 - 7x_3 = 15 \end{array} \qquad \begin{array}{l} \text{(d)} \\ \\ \\ \end{array} \begin{array}{l} 2x + y + 3z = 11 \\ 4x + 3y - 2z = -1 \\ 6x + 5y - 4z = -4 \end{array}$$

1.3 Echelon Form

It is useful to have a standard method for solving to all linear systems. This will allow us to systematically describe and solve systems of equations, and interpret the solutions. A standard procedure is also better for computer computation.

There are two standard methods for solving linear systems: Gaussian elimination and Gauss-Jordan⁷ elimination. In each method it is the final augmented matrix that is important rather than the sequence of EROs leading to it.

Gaussian elimination consists of transforming an augmented matrix into *row-echelon form*, and Gauss-Jordan elimination consists of transforming an augmented matrix into *reduced row-echelon form*.

Definition 1.3.1

A matrix is said to have **row-echelon form**⁸ if

1. All rows that contain only zeros (called **zero rows**) are at the bottom of the matrix.
2. The first nonzero element in each nonzero row is a 1 (called the **leading 1** or **pivot**).
3. The leading 1 in any nonzero row is to the right of the leading 1 in any row above it.

Row-echelon matrices have a 'staircase' appearance, e.g.

$$\begin{bmatrix} 0 & \boxed{1} & * & * & * & * & * \\ 0 & 0 & 0 & \boxed{1} & * & * & * \\ 0 & 0 & 0 & 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the astrisks stand for numbers, the leading 1's proceed downward and to the right, and all elements below and to the left of the leading ones are zero.

Example

*row-echelon
form*

(a) The following matrices are in row-echelon form:

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) The matrices below are not in row-echelon form:

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

⁷Camille Jordan (1838-1922) was an eminent French mathematician

⁸*Echelon* comes from the French word *échelle* meaning ladder. Solving a set of equations by back-substitution is like climbing up a ladder.

Definition 1.3.2

An augmented matrix has **reduced row-echelon form** if it has row-echelon form and if

4. Each leading 1 is the only nonzero entry in its column.

Example

reduced
row-echelon
form

- (a) The following matrices are in reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- (b) The matrices below are not in reduced row-echelon form:

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Gaussian and Gauss-Jordan elimination are both important and you will need to know understand each method. One difference between the two methods is that when an augmented matrix is transformed into row-echelon form, it is possible to arrive at different row-echelon forms by choosing different sequences of EROs, in other words *the row-echelon form is not unique*. In contrast, when an augmented matrix is transformed into reduced row-echelon form, the final matrix is always the same no matter what sequence of EROs was used, in other words *the reduced row-echelon form of a matrix is unique*.

Every row-echelon matrix can be transformed into a reduced row-echelon matrix by applying more EROs. In pen and paper calculations it is best to transform matrices into row-echelon form first and then, if required, into reduced row-echelon form by eliminating the zeros above the leading ones in a column-by-column fashion. Appendix C describes further refinements.

Exercise 1.3

1. Which of the following matrices are in (i) row-echelon form (but not reduced row-echelon form), (ii) reduced row-echelon form, or (iii) neither?

$$\begin{array}{lll} \text{(a)} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{(b)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} & \text{(c)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \text{(d)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \text{(e)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \text{(f)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \\ \text{(g)} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} & \text{(h)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{(i)} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 4 \end{bmatrix} \end{array}$$

2. Use elementary row operations to reduce the following matrices to row-echelon form and reduced row-echelon form.

$$(a) \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad (c) \begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

3. Use Gaussian elimination to solve

$$(a) \begin{aligned} 2x - 3y &= 1 \\ x - 2y &= 2 \end{aligned} \quad (b) \begin{aligned} 2x_1 - x_2 - x_3 &= 2 \\ x_1 + x_2 - x_3 &= 3 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

4. Use Gauss-Jordan elimination to solve

$$(a) \begin{aligned} 2p + 3q &= 3 \\ p + 2q &= 3 \end{aligned} \quad (b) \begin{aligned} x_1 - x_2 - x_3 &= 1 \\ x_1 + x_2 - x_3 &= 3 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

Chapter 2

Consistent and Inconsistent systems

2.1 The Geometry of Linear Systems

Systems of linear equations can have one solution, no solutions or infinitely many solutions. It is important to understand the reason for this.

The system of two linear equations in two unknowns in (2.1) can be thought of geometrically as a pair of straight lines. This helps us interpret the solutions.

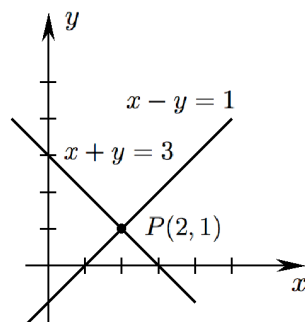
$$(2.1) \quad \begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

There are three ways in which two lines can meet (or not meet).

- (i) The lines could intersect in *a single point*.

Example

*system
with a
unique
solution*

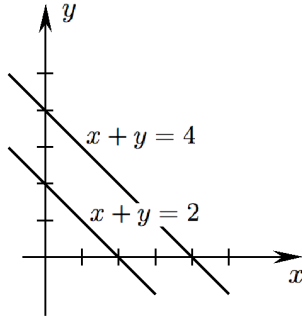


... the simultaneous equations
 $x - y = 1$
 $x + y = 3$
have the unique solution $(x, y) = (2, 1)$.

(ii) The lines could be *parallel*.

Example

*system
with no
solutions*

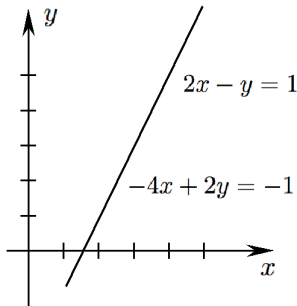


... the system of equations
 $x + y = 4$
 $x + y = 2$
 has no solutions as the lines never meet.

(iii) The lines could *coincide*.

Example

*infinitely
many
solutions*



... the pair of equations
 $2x - y = 1$
 $-4x + 2y = -1$
 has infinitely many solutions - every point on the line gives a solution.

The geometric interpretations above allow us to understand why some systems can have no solutions and others have infinitely many solutions.

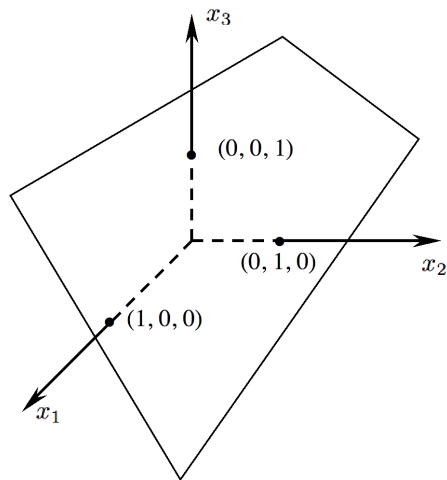
Systems of linear equations in three variables are more complex:

$$(2.2) \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 . \end{aligned}$$

We can think of a linear equation in three variables as the equation of a plane in three dimensional geometry (each variable representing a co-ordinate).

Example

The plane $x_1 + x_2 + x_3 = 3$ is shown below. Each variable represents a co-ordinate. The plane extends in all directions (this is difficult to represent in a diagram).

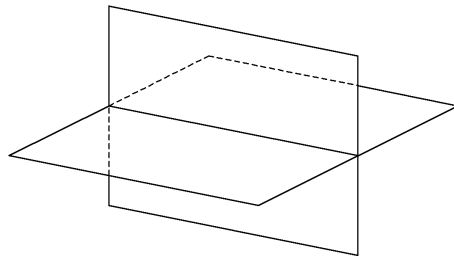


If we just consider the first two linear equations in (2.2)

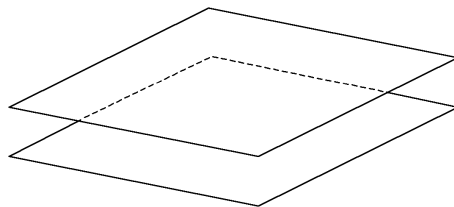
$$(2.3) \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2, \end{aligned}$$

then the corresponding planes may:

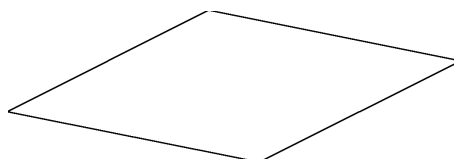
(i) intersect in a line,



(ii) be parallel,



or (iii) coincide.

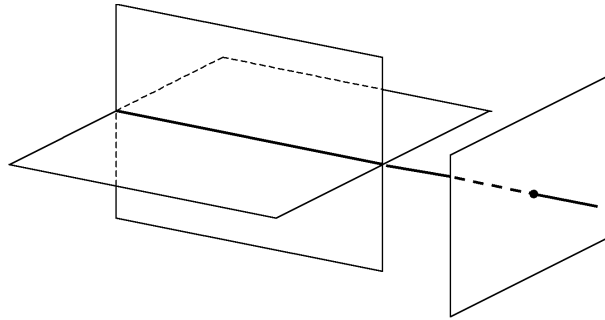


Each of the equations in (2.3) is a *constraint* on the values that (x_1, x_2, x_3) can take. When the planes meet in a straight line in (i), the values of (x_1, x_2, x_3) that satisfy both constraints correspond to the points on the line of intersection, and when the planes are parallel in (ii), there are no values of (x_1, x_2, x_3) that satisfy both constraints.

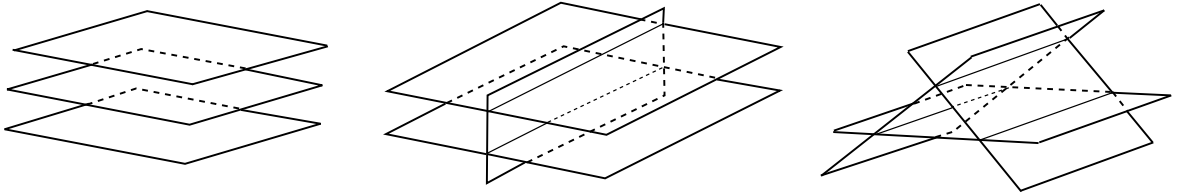
If a third constraint is added, as in (2.2), then the linear system may have (a) a unique solution, (b) no solutions or (c) infinitely many solutions, according to whether the three planes (a) intersect in a point, (b) don't intersect at all or (c) either intersect in a line or coincide.

The diagrams below show some of these possibilities.

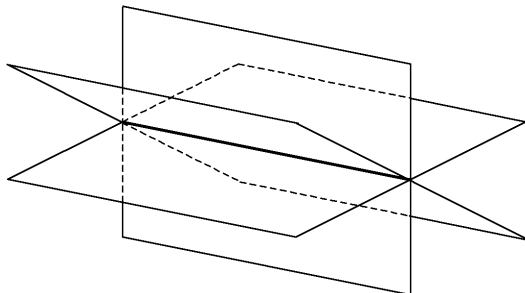
(a) Three planes meeting in a single point.



(b) Three planes without a common point. (This can happen in multiple ways.)



(c) Three planes meeting in a line.



A system of linear equations has either a unique solution, no solutions or infinitely many solutions

Exercise 2.1

1. The linear system below has a unique solution. Interpret this geometrically.

$$2x - 3y = 1$$

$$x + y = 3$$

$$x + 6y = 8$$

2. For any numbers $a, b \neq 0$, the line

$$a(2x - 3y - 1) + b(x + y - 3) = 0$$

passes through the intersection of the lines

$$2x - 3y - 1 = 0 \quad \text{and} \quad x + y - 3 = 0.$$

Find the values of a, b that give the third equation in exercise 1.

3. Sketch all the ways three planes can be arranged so that:
- i. the planes have no point that is common to all.
 - ii. the planes have at least two points common to all.
-

2.2 Systems Without Unique Solutions

A system of linear equations is said to be *consistent* if it has at least one solution and *inconsistent* if it has no solutions. We can tell if a system of linear equations is consistent or inconsistent by transforming its augmented matrix into either row echelon or reduced-row form.

A system is inconsistent if any row has a leading one in the last column:

$$\left[\begin{array}{ccc|c} \dots & 0 & & 1 \end{array} \right]$$

This is because this row is equivalent to the equation

$$0x_1 + \dots + 0x_n = 1$$

which has no solution. Note that when doing row operations, it is possible to tell if the system is inconsistent before getting to row echelon form. In particular, if the final row contains all zeros except for the final entry, then this final entry does not have to be a 1 in order for the system to be inconsistent – it could be any nonzero number.

Example

*inconsistent
system*

The linear system

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 2 \\ -x_1 - x_2 + x_3 &= 4 \end{aligned}$$

has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 2 \\ -1 & -1 & 1 & 4 \end{array} \right],$$

and this can be transformed into row echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

As there is a leading 1 in the last column, the system is inconsistent.

Note. A system is consistent when it is not inconsistent.

Example

*consistent
or
inconsistent?*

For what value of k is the following linear system consistent?

$$\begin{aligned}x_1 + x_2 - x_3 &= 2 \\x_1 - x_2 + x_3 &= 2 \\-x_1 - x_2 + kx_3 &= 4\end{aligned}$$

Answer. The system has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 2 \\ -1 & -1 & k & 4 \end{array} \right],$$

and this can be transformed into echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k-1 & 6 \end{array} \right].$$

The system is inconsistent when $k = 1$, and is consistent when $k \neq 1$.

If a linear system is consistent, then it may have a unique solution or may have infinitely many solutions.

Definition 2.2.1

*In a linear system, the variables that correspond to columns with leading 1's are called **basic variables**. Any remaining variables are called **free variables**.*

If a linear system is consistent, and:

- if every variable is a basic variable, for example

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{array} \right],$$

then the system has a one unique solution and this can be found by back-substitution.

- if there is at least one free variable, for example

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

then the system has infinitely many solutions.

*inconsistent
system*

Example

The system of linear equations

$$\begin{aligned}x_1 - x_2 + x_3 &= 3 \\2x_1 - x_2 + 4x_3 &= 7 \\3x_1 - 5x_2 - x_3 &= 7\end{aligned}$$

has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{array} \right],$$

and this can be transformed into reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

This shows that:

- i. the system is consistent as there is no leading 1 in the last column.
- ii. there are infinitely many solutions as x_3 is a free variable.

The original linear system has the same solutions as

$$\begin{aligned}x_1 + 3x_3 &= 4 \\x_2 + 2x_3 &= 1,\end{aligned}$$

...and solutions can be found by giving values to the free variable x_3 then evaluating the basic variables x_1 and x_2 .

To describe the general solution, assign the arbitrary value t to x_3 . Back-substitution now shows that

$$x_2 = 1 - 2x_3 = -2t + 1 \quad \text{and} \quad x_1 = 4 - 3x_3 = -3t + 4.$$

So the general solution to this system of equations is

$$(x_1, x_2, x_3) = (-2t + 1, -3t + 4, t)$$

where $t \in \mathbf{R}^1$ can have any value, and the set of all solutions is

$$\{(-2t + 1, -3t + 4, t) \mid t \in \mathbf{R}\}$$

In this example the arbitrary value $t \in \mathbf{R}$ is called a parameter, and the general solution $(-2t + 1, -3t + 4, t)$ is called a parametric solution.

¹ \mathbf{R} is the set of all real numbers

Exercise 2.2

1. Find all solutions (x, y, z) of the linear systems that have been transformed into the row echelon forms below.

$$(a) \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$(b) \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$(c) \quad \left[\begin{array}{ccc|c} 1 & -1 & 3 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$(d) \quad \left[\begin{array}{ccc|c} 1 & 5 & 2 & 8 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

2. The two planes

$$x + y + z = 1$$

$$x - y + 3z = 7$$

intersect in a straight line.

(a) Find the points on this line.

(b) Where does the line meet the the xy -plane ($z = 0$).

3. For what value of k does the linear system

$$x_1 + 2x_2 + x_3 = 4$$

$$5x_1 + 2x_2 - 3x_3 = k$$

$$-x_1 + 2x_2 + 3x_3 = 4$$

have a solution. Find all solutions for this value of k .

4. Show that the system of linear equations

$$2x + 3y + 4z = 3$$

$$x + y - 8z = 1$$

$$5x + 6y - 20z = u$$

can be solved if and only if $u = 6$. Interpret this fact geometrically.

5. Find a condition on b_1 , b_2 and b_3 for system of equations

$$u + 2v + 3w = b_1$$

$$2u + v - w = b_2$$

$$4u + 5v + 5w = b_3$$

to have a solution in u , v and w .

6. Consider the system of linear equations

$$\begin{aligned}x_1 - 2x_2 + 3x_3 &= 1 \\x_1 + kx_2 + 2x_3 &= 2 \\-2x_1 + k^2x_2 - 4x_3 &= 3k - 4\end{aligned}$$

where k is a real constant.

- (a) Write the system of equations in augmented matrix form.
(b) Use row operations to transform the augmented matrix into the form

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & k+2 & -1 & 1 \\ 0 & 0 & k & 2k \end{array} \right]$$

- (c) State the value of k for which the system has an infinite number of solutions. Represent these solutions in parametric form and give a geometric interpretation.
(d) State the values of k for which the system has no solution.
(e) For what values of k does the system have a unique solution?
-

Chapter 3

Matrix Inverses

Gauss-Jordan elimination can be used to find the inverse of a square matrix.

If A is a square matrix of order n and I_n is the identity matrix of order n , then:

1. form the augmented matrix $[A \mid I_n]$,
2. use EROs to transform $[A \mid I_n]$ to reduced row echelon form.

If the transformed matrix has the form $[I_n \mid B]$, then $B = A^{-1}$. If the first $n \times n$ submatrix is not I_n , then A does not have an inverse.

Example

*invertible
3 × 3
matrix*

Find the inverse of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -1 & -3 \\ 3 & 1 & 2 \end{bmatrix}.$$

Answer. The augmented matrix is

$$[A \mid I_n] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ -4 & -1 & -3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array} \right].$$

Transformation to reduced row echelon form gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -3 \\ 0 & 1 & 0 & -1 & 4 & 6 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right] = [I_n \mid B],$$

where B is the inverse of A .

*uninvertible
matrix*

Example

Is the matrix

$$H = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

invertible?

Answer. The augmented matrix is

$$[H \mid I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array} \right].$$

This is equivalent to

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right],$$

so H does not have an inverse.

Note. The 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has inverse matrix

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if and only if its determinant

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is not zero.

There are similar formulas for the inverse of a general $n \times n$ matrix, but its usually more efficient to use Gauss-Jordan elimination.

Exercise 3

1. Use Gauss-Jordan elimination to calculate the inverses of the following matrices (if possible).

(a) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

2. Find the values of u for which

$$A = \begin{bmatrix} u & 2 \\ 1 & u - 1 \end{bmatrix}$$

is invertible.

Appendix A

Simultaneous Equations in Two Unknowns

A pair of linear equations in two unknowns is usually solved by (i) eliminating one of the unknowns from an equation, then (ii) solving for the other unknown.

Example

*substitution
method*

Solve the simultaneous equations

$$2x + y = 7 \quad \dots (1)$$

$$x - y = 2 \quad \dots (2)$$

Answer. Equation (2) shows that

$$x = 2 + y \quad \dots (3)$$

... we can use this to eliminate x from equation (1).

Substitute $x = 2 + y$ into (1):

$$2x + y = 7$$

$$2(2 + y) + y = 7$$

$$4 + 2y + y = 7$$

$$3y = 3$$

$$y = 1$$

To find x , substitute $y = 1$ into equation (3):

$$x = 2 + y$$

$$= 2 + 1$$

$$= 3$$

A second method for solving pairs of equations is to combine the left and right sides of the equations.

Example

*elimination
method*

Solve the simultaneous equations

$$2x + y = 7 \quad \dots (1)$$

$$x - y = 2 \quad \dots (2)$$

Answer. The unknown y can be eliminated by adding the left and right sides of the equations together.

$$2x + y = 7 \quad \dots (1)$$

$$x - y = 2 \quad \dots (2)$$

$$(1) + (2) \Rightarrow (2x + y) + (x - y) = 7 + 2$$

$$3x = 9$$

$$x = 3$$

To find y , we can substitute $x = 3$ into equation (1).

$$2x + y = 7$$

$$6 + y = 7$$

$$y = 1$$

Alternatively, we could have eliminated the unknown x first by subtracting twice equation (2) from equation (1).

$$2x + y = 7 \quad \dots (1)$$

$$x - y = 2 \quad \dots (2)$$

$$(1) - 2 \times (2) \Rightarrow (2x + y) - 2 \times (x - y) = 7 - 2 \times 2$$

$$3y = 3$$

$$y = 1$$

... and then find x by substituting $y = 1$ into equation (2).

$$x - y = 2$$

$$x - 1 = 2$$

$$x = 3$$

These methods are not very efficient for solving larger systems of equations.

Appendix B

Answers

Exercise 1.2

$$1(a) \quad \left[\begin{array}{ccc|c} 2 & -3 & 2 & 4 \\ 1 & 5 & 4 & 1 \\ 7 & 2 & -3 & 3 \end{array} \right] \quad 1(b) \quad \left[\begin{array}{ccc|c} 0 & 3 & 2 & 7 \\ 1 & 4 & 0 & 3 \\ 3 & 3 & 8 & 1 \end{array} \right]$$

$$2. \quad \begin{array}{l} 3p - 2q = 8 \\ 2p + 5q = 1 \end{array}$$

3(a) The elementary row operations are i, ii, iii, iv and v.

$$3(b) \text{ i. } \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 2 & 5 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 4 & 10 & 2 \end{array} \right] \quad \text{ii. } \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 2 & 5 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 11 & -1 & 25 \end{array} \right]$$

$$\text{iii. } \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 2 & 5 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} -3 & -17 & 5 \\ 2 & 5 & 1 \end{array} \right] \quad \text{iv. } \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 2 & 5 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 5 & 3 & 9 \end{array} \right]$$

$$\text{v. } \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 2 & 5 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} -1 & -12 & 6 \\ 2 & 5 & 1 \end{array} \right]$$

$$4. \quad (43, -9, 4)$$

$$5(a) (3, 1) \quad 5(b) (-3, -4) \quad 5(c) (2, -4, 1) \quad 5(d) \left(\frac{1}{2}, 1, 3\right)$$

Exercise 1.3

1(i) row-echelon: a, g 1(ii) row-reduced: c, d, f 1(iii) neither: b, e, h, i

$$2(a)^1 \quad \left[\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right] \quad 2(b) \quad \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array} \right] \quad 2(c) \quad \left[\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

¹Only the reduced row echelon form answer is given as it is unique.

$$3(a) (-4, -3) \quad 3(b) (1, 1, -1)$$

$$4(a) (-3, 3) \quad 4(b) (1, 1, -1)$$

Exercise 2.1

1. The three lines all pass through the point $(2, 1)$.

2. $a = -1, b = 3$

Exercise 2.2

$$1(a) \{(-t - 4, t, 3) \mid t \in \mathbf{R}\} \quad 1(b) \text{ no solution} \quad 1(c) (9, 4, 0)$$

$$1(d) \{(18s - 7, -4s + 3, s) \mid s \in \mathbf{R}\}$$

$$2(a) \{(-2r + 4, r - 3, r) \mid r \in \mathbf{R}\} \quad 2(b) (4, -3, 0)$$

$$3 \quad k = 4, \{(t, -t + 2, t) \mid t \in \mathbf{R}\}$$

4. The plane $5x + 6y - 20z = u$ is parallel to the line of intersection of the other two planes when $u \neq 6$, and meets the other two planes in this line when $u = 6$

5. $2b_1 + b_2 - b_3 = 0$

6(c) The system has infinitely many solutions when $k = 0$. The parametric form of the solutions is $(2 - 2t, \frac{1}{2} + \frac{1}{2}t, t), t \in \mathbf{R}$. The three planes intersect in a straight line and the solutions correspond to the points on this line.

6(d) The system has no solutions if $k = -2$.

6(e) The system has a unique solution when $k \neq 0$ and $k \neq -2$.

Exercise 3

$$1(a) \begin{bmatrix} 3/2 & -2 \\ -1/2 & 1 \end{bmatrix}$$

$$1(b) \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 0 & | & 2 & 1 \end{bmatrix} \Rightarrow \text{not invertible}$$

$$1(c) \begin{bmatrix} 1 & 2 & -3 & | & 1 & 0 & 0 \\ 0 & -4 & 4 & | & -1 & 1 & 0 \\ 0 & 0 & 0 & | & -2 & -3 & 1 \end{bmatrix} \Rightarrow \text{not invertible}$$

$$1(d) \quad \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$$

2. A is invertible if and only if $\det A = u(u - 1) - 2 = u^2 - u - 2 \neq 0$, i.e. if and only if $u \neq -1, 2$.