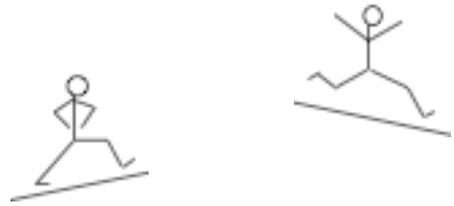
MathsTrack

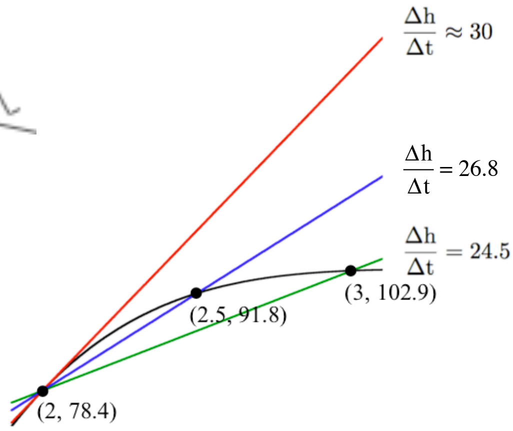
(NOTE Feb 2013: This is the old version of MathsTrack.
New books will be created during 2013 and 2014)

Topic 6

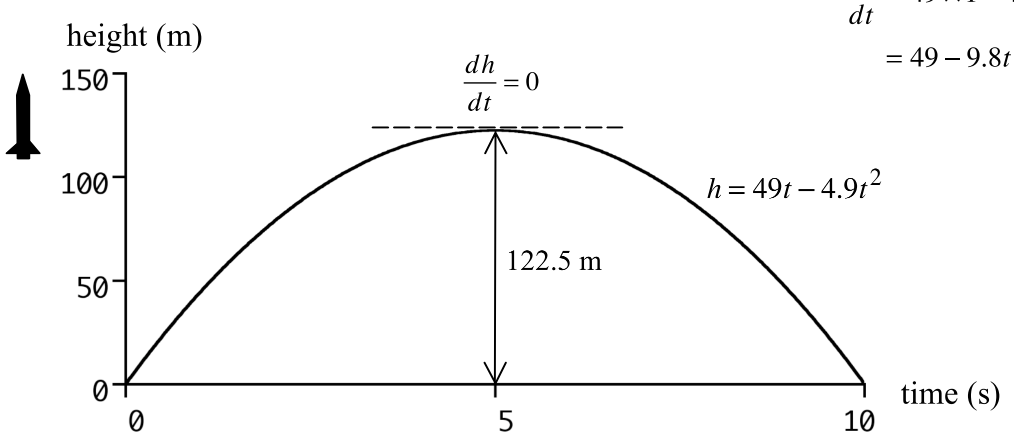
Differentiation



$$\frac{\Delta\,\text{height}}{\Delta\,\text{time}} = \frac{\text{change in height}}{\text{change in time}}$$



$$\frac{dh}{dt} = 49 \times 1 - 4.9 \times 2t^{2-1}$$
$$= 49 - 9.8t$$



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This Topic . . .

This Topic begins by introducing the gradient of a curve. This concept was invented by Pierre de Fermat in the 1630s and made rigorous by Sir Issac Newton and Gottfried Wilhelm von Leibniz in the 1670s.

The process of finding the gradient by algebra is called *differentiation*. It is a powerful mathematical technique and many scientific discoveries of the past three centuries would have been impossible without it. Newton used these ideas to discover the Law of Gravity and to find equations describing the orbits of the planets around the sun.

Differentiation remains a powerful technique today and has many theoretical and practical applications.

The Topic has 2 chapters:

Chapter 1 explores the rate at which quantities change. It introduces the gradient of a curve and the rate of change of a function. Examples include motion and population growth.

Chapter 2 introduces derivatives and differentiation. Derivatives are initially found from *first principles* using limits. They are then constructed from known results using the *rules of differentiation* for addition, subtraction, multiples, products, quotients and composite functions. Implicit differentiation is also introduced. Applications include finding tangents and normals to curves.

Auhor: Dr Paul Andrew Printed: February 27, 2013

Contents

1 Gradients of Curves **1**

1.1 Describing change . . . 1

1.2 Gradients of curves . . . 4

1.3 The rate of change of a function . . . 9

2 Differentiation **11**

2.1 From first principles 11

2.1.1 Terminology and Notation . . . 14

2.2 Constructing derivatives . . . 17

2.2.1 Powers . . . 17

2.2.2 Polynomials . . . 19

2.2.3 Products and quotients . . . 22

2.2.4 Composite functions and the chain rule . . . 24

2.2.5 Implicit Differentiation . . . 28

A First Principles **32**

B Answers **36**

Chapter 1

Gradients of Curves

1.1 Describing change

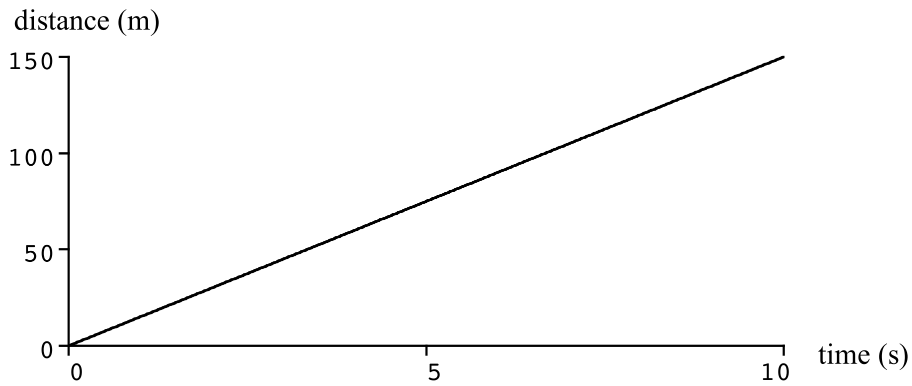
How can we describe the rate at which quantities change?

Example

constant velocity

gradient of a line

The *distance-time graph* below shows the distance travelled by a car that has a constant velocity of 15 m/s.[1]



Velocity measures how distance changes with time. You can see that the car travelled 15m in 1 second, 30m in 2 seconds, etc. When the velocity is constant, it is calculated using the ratio:[2]

$$\frac{\Delta\,\text{distance}}{\Delta\,\text{time}} = \frac{\text{change in distance}}{\text{change in time}} = 15\ m/s$$

This ratio is the gradient of the line on the graph, so the constant velocity of the car can also be thought of as the gradient of a line.

[1] meters per second

[2] Δ is an uppercase Greek letter called "Delta". It is used in mathematics to mean 'change in'.

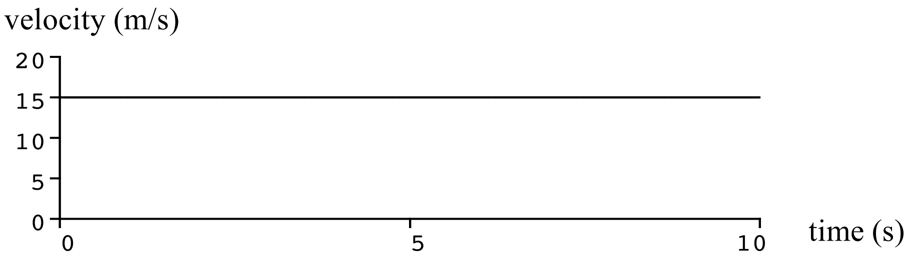
Observe:

- the distance-time graph provides information about velocity
- the distance-time graph is a straight line with gradient 15
- The velocity of the car is 15 m/s

Example

velocity
acceleration

Here is the *velocity-time graph* for the same car ...



Acceleration measures how velocity changes with time. When the acceleration is constant, it can be calculated using the ratio:

$$\frac{\Delta\,\text{velocity}}{\Delta\,\text{time}} = \frac{\text{change in velocity}}{\text{change in time}} = 0\ m/s^2$$

Observe:

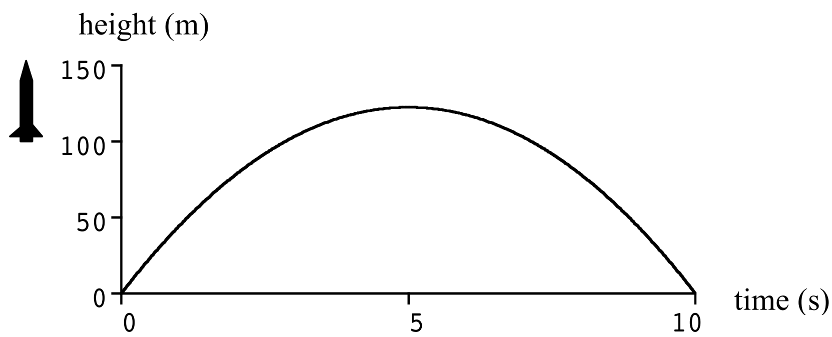
- the velocity-time graph provides information about acceleration
- the velocity-time graph is a horizontal line with gradient 0
- the acceleration of the car is 0 m/s^2.

Example

shape of graph

vertical velocity

This is the *height-time graph* of an experimental rocket.



The rocket climbs to about 120 m and then returns to the ground again. The graph is curved, so the change in height with time is not constant.

What does the shape of the graph tell us about the vertical velocity of the rocket? Can the velocity be found from the graph?

Questions like these motivated Fermat, Newton and Leibniz in their exploration of the gradient of a curve and led to the invention of differentiation.[3]

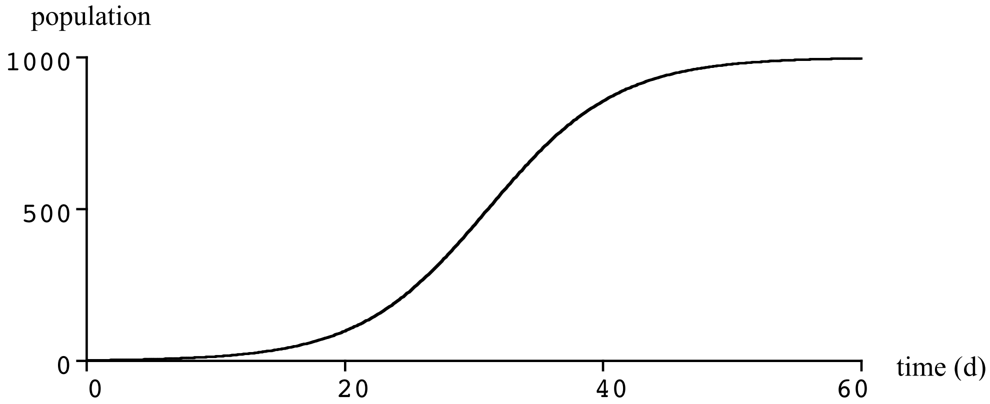
The graph shows that the rocket climbs 44 m between $t = 0$ and $t = 1$, but only 5m between $t = 4$ and $t = 5$.

The *shape* of the curve shows that the rocket *slows down* as it climbs, until it reaches its maximum height at $t = 5$, ... and that it then *speeds up* as it falls to the ground.

Example

population growth

This *population-time graph* models the growth of aphids on broad bean plants over 60 days.



The graph shows that the aphid population grows slowly for the first 20 days, grows faster from day 20 to day 40, then grows more slows again and finally levels off.

The graph shows that the population is growing faster at day 30 than it is at day 50. But exactly how fast is the population growing on each of these days?

This is another example of the type of question that led to the invention of differentiation.

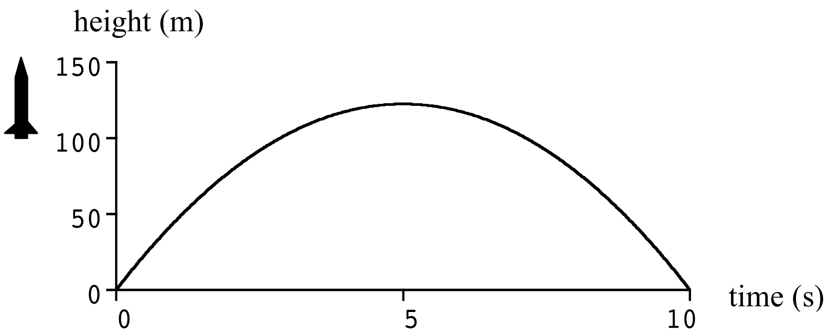
Exercise 1.1

1. A car starting from rest travelled the first $300\,m$ in 10 seconds at a constant velocity.
   (a) Represent this on a distance-time graph.
   (b) What is the constant velocity of the car?
   (c) Draw the corresponding velocity-time graph
   (d) What is the acceleration of the car?

[3] Pierre de Fermat (1601 - 1665), Sir Isaac Newton (1643 - 1727), Gottfried Wilhelm von Leibniz (1646 - 1716)

1.2 Gradients of curves

How can we determine the vertical velocity of an experimental rocket from its height-time graph? What is its velocity at *exactly* $t = 2$ seconds?



The second question can be approached by estimating the velocity *near* $t = 2$:

- at $t = 2$, the rocket's height is 78.4 m, and at $t = 3$ the height is h = 102.9 m, so the vetrical velocity between $t = 2$ and $t = 3$ is approximately:

$$\frac{\Delta \text{ height}}{\Delta \text{ time}} = \frac{\text{change in height}}{\text{change in time}} = \frac{102.9 - 78.4}{3 - 2} = 24.5 \; m/s$$

  ...the velocity at t = 2 is about 24.5 m/s.

- at $t = 2.5$ the height is 91.8 m, so the vertical velocity between $t = 2$ and $t = 2.5$ is approximately:

$$\frac{\Delta\text{height}}{\Delta\text{height}} = \frac{\text{change in height}}{\text{change in height}} = \frac{91.8 - 78.4}{2.5 - 2} = 26.8 \; m/s$$

  ...so 26.8 m/s is a better estimate of the velocity at $t = 2$.

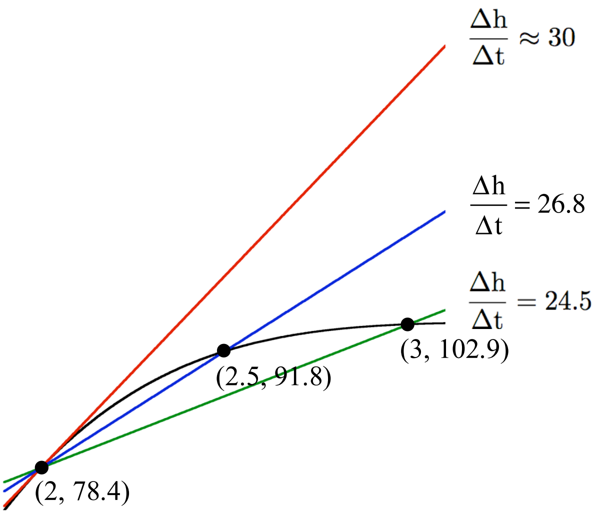
Smaller time intervals will give better estimates of the velocity at $t = 2$.

The table below shows how the estimate improves when smaller time intervals are used.

Interval	**Δtime**	**Δheight**	$\frac{\Delta\textbf{height}}{\Delta\textbf{time}}$
$t = 2$ $(h = 78.4) \to t = 3.0$ $(h = 102.9)$	1	24.5	24.5
$\to t = 2.5$ $(h = 91.9)$	0.5	13.4	26.8
$\to t = 2.1$ $(h = 81.3)$	0.1	2.9	29
$\to t = 2.01$ $(h = 78.7)$	0.01	0.3	30

You can see that the velocity of the rocket at $t = 2$ will be very close to 30 m/s.

These estimates can be interpreted as gradients:



The first estimate of the velocity was 24.5 m/s.
  . . . it is the gradient of the line from $(2, 78.4)$ *to* $(3, 102.9)$.

The second estimate was 26.8 m/s.
  . . . it is the gradient of the line from $(2, 78.4)$ *to* $(2.5, 91.8)$.

The third estimate was 30 m/s.
  . . . it is the gradient of the line from $(2, 78.4)$ *to* $(2.01, 78.7)$, *and is very close to the gradient of the line that just touches the curve at* $(2, 78.4)$.

The velocity of the rocket at $t = 2$ is equal to the gradient of the straight line that just touches the curve at $t = 2$.

Terminology

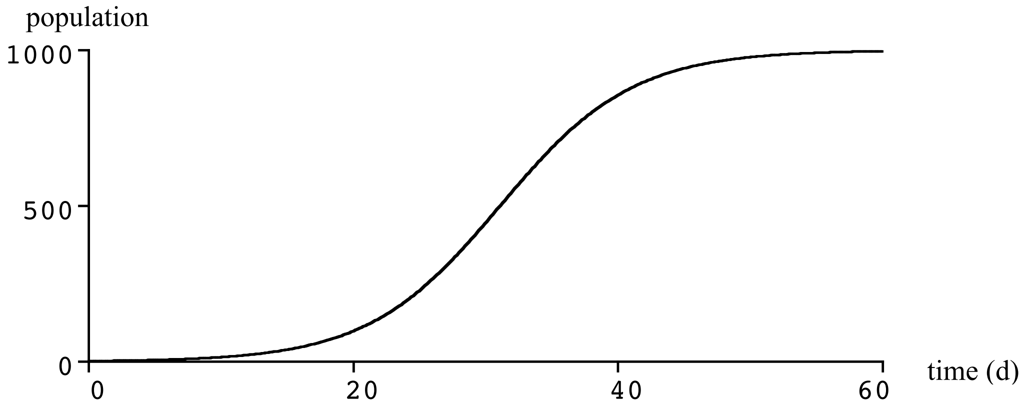
- A straight line that just touches a curve is called a *tangent line*.
- The *gradient of a curve* at a point is the gradient of the tangent line to the curve at that point.

> *The gradient of the curve at a point (or the gradient of the tangent line) measures the rate of change of the quantity at the point.*

Example

changing growth rate

The growth rate of an aphid population can be found from the gradient of the *population-time* graph. This observation is used to describe how the population changes over 60 days.



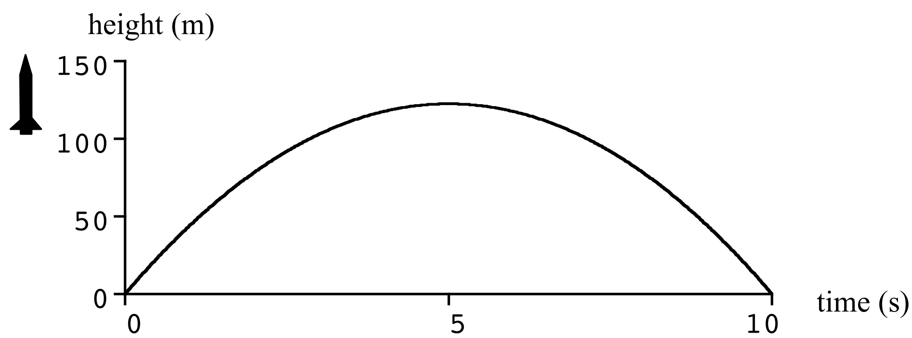
In the graph ...

- the gradient of the curve increases from t = 0 to t = 30, so
  ... *the growth rate increases from t = 0 to t =30*
- the gradient of the curve stops increasing at t = 30, begins to decrease and becomes close to zero after t = 50, so
  ... *the growth rate stops increasing at t = 30, begins to decrease and becomes close to zero after t = 50.*[4]

Example

changing velocity

The vertical velocity of an experimental rocket can be found from the gradient of the *height-time* graph. When the rocket is climbing the vertical velocity is positive, and when it is returning the vertical velocity is negative.



In this graph ...

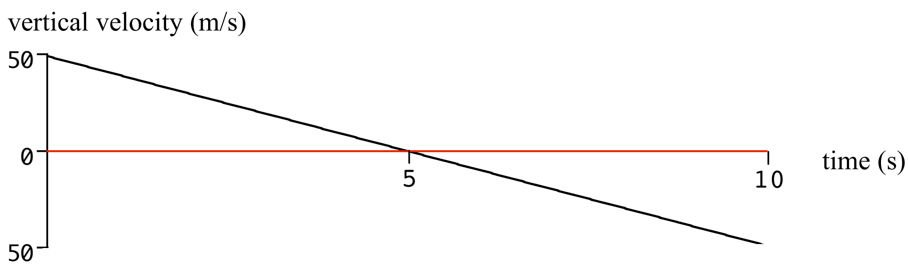
[4] The size of the population is increasing even though the growth rate is decreasing. The population levels off as the growth rate falls to 0.

- the gradient of the curve is positive but reducing from $t = 0$ to $t = 5$, so ... *the velocity is positive (climbing) and decreasing for* $0 \leq t < 5$.
- the gradient of the curve is zero at $t = 5$, so ... *the (vertical) velocity is zero at* $t = 5$.
- the gradient of the curve is negative and growing for $5 < t \leq 10$, so ... *the velocity is negative (returning) and increasing for* $0 \leq t < 5$.

Example

vertical velocity & acceleration

Here is the *velocity-time* graph for the same rocket:



You can see that ...

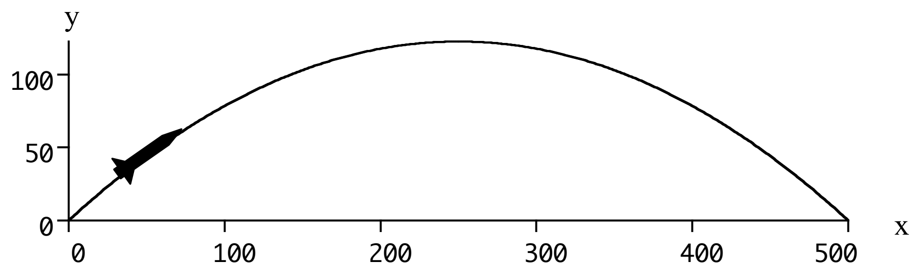
- the initial velocity is about 50 m/s
- the velocity is positive for $0 \leq t < 5$ (climbing)
- the velocity is zero at $t = 5$ (maximum height reached)
- the velocity is negative for $5 < t \leq 10$ (returning)
- the velocity is changing at a constant rate which is equal to the gradient of the line (-9.8)

Acceleration measures how velocity changes with time. In this example, the rocket has a vertical acceleration of $-9.8\,m/s^2$, due to the downward pull of gravity.

Example

change in height with distance

The experimental rocket followed a parabolic path. The height (metres) is given on the y-axis, and the distance travelled (metres) on the x-axis.

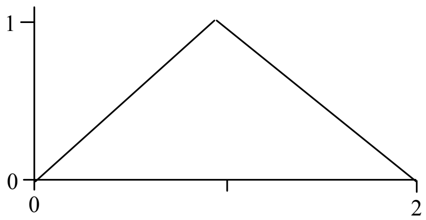


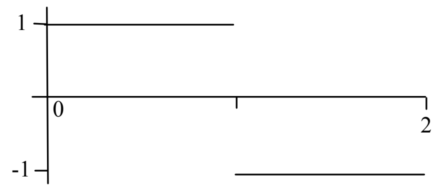
In this *height-distance* graph, the gradient of the curve is interpreted as the rate at which height changes with distance travelled.

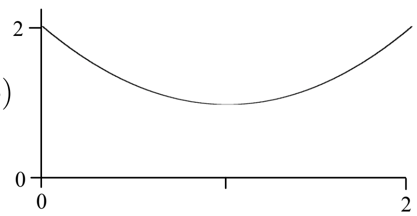
Exercise 1.2

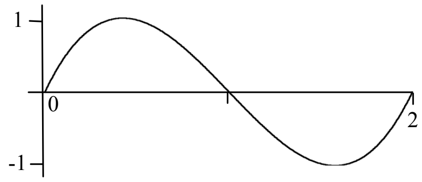
1. Sketch the graph of $y = x^2$ for $0 \leq x \leq 2$, then draw chords[5] from $P(1, 1)$ to the points $Q(0.5, 0.25)$ and $R(1.5, 2.25)$.
   (a) What are the gradients of the chords
      i. PQ
      ii. PR
   (b) Show the tangent to $y = x^2$ at $x = 1$ has gradient between 1.5 and 2.5 .
   (c) By selecting other chords on $y = x^2$, estimate the gradient of the tangent to $y = x^2$ at $x = 1$ to within ± 0.1.

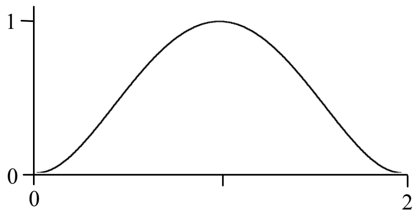
2. Match up the graphs below. Each graph on the right shows the gradient of a curve on the left. *Hint: Observe where the gradients to the curves are constant, positive, negative, and zero.*

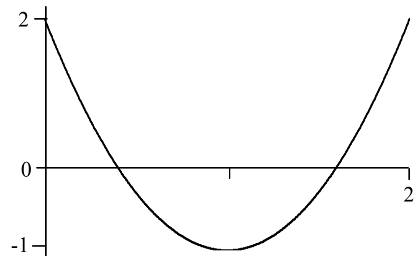
(a)


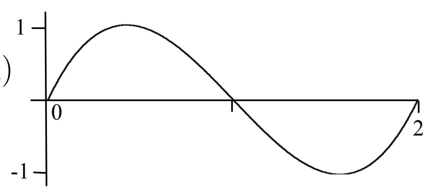
(i)


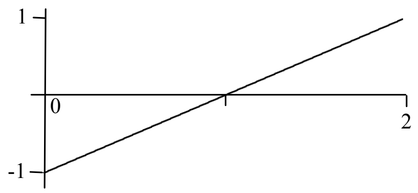
(b)


(ii)


(c)


(iii)


(d)


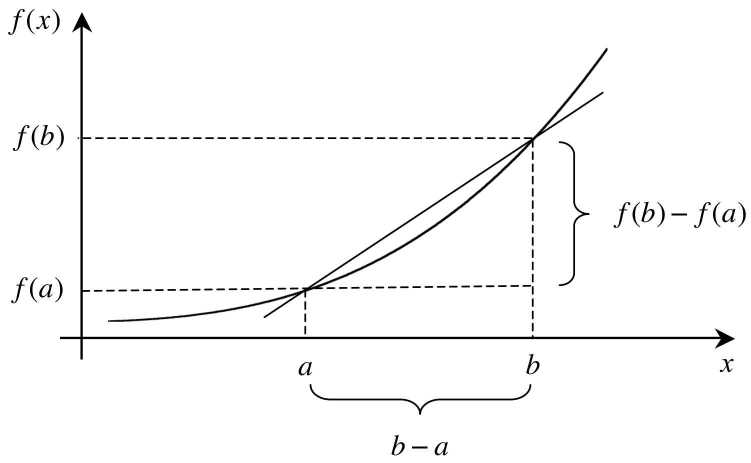
(iv)


[5] A chord is a line segment joining two points on a curve.

1.3 The rate of change of a function

Velocity measures how distance changes with time. Growth rate measures how a population changes with time. How can we measure the way a function $f(x)$ changes with x?

The graph below shows how a function $f(x)$ might change between $x = a$ and $x = b$.



The *average rate of change* of the function $f(x)$ from $x = a$ to $x = b$ is

$$\frac{\Delta f}{\Delta x} = \frac{\text{change in } f(x)}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}.$$

This is equal to the gradient of the chord[6] from $(a, f(a))$ to $(b, f(b)$.

As the width of the interval $[a, b]$ decreases, the approximation

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

becomes closer to the *rate of change of* $f(x)$ *at* $x = a$.[7]

You can see that the rate of change of $f(x)$ at $x = a$ is equal to the gradient of the *tangent line* to the graph of $f(x)$ at $x = a$.

> The gradient of the curve $y = f(x)$ at a point $x = a$ (the gradient of the tangent line) measures the rate of change of the function $f(x)$ with respect to x at the point $x = a$.

The phrase *instantaneous rate of change* is often used to emphasise that the rate of change of the function $f(x)$ is for a specific value of x. This contrasts with the previous use of *average rate of change.*

[6] A chord is a line segment joining two points on a curve.

[7] The phrase *rate of change* is used describe to the change in $f(x)$ as x changes, even though x may not be a time variable.

Exercise 1.3

1. Sketch the graph of $y = x^2$ for $0 \leq x \leq 4$ and draw the chords from $P(2, 4)$ to the points $Q(1, 1)$, $R(3, 9)$ and $S(4, 16)$.
2. What is the average rate of change of x^2 between
   (a) $x = 1$ and $x = 2$
   (b) $x = 2$ and $x = 3$
   (c) $x = 2$ and $x = 4$
3. Use your answer to question 2 to deduce that the rate of change of x^2 with respect to x at $x = 2$ is between 3 and 5.
4. Estimate the rate of change of x^2 with respect to x at $x = 2$ to within ± 0.1.
5. The graph of $y = x^2 + 1$ can be obtained by shifting the graph of $y = x^2$ upwards by one unit. Use this together with your answer to question 4 to estimate the instantaneous rate of change of $x^2 + 1$ with respect to x at $x = 2$.

Chapter 2

Differentiation

The *gradient of a curve* shows the rate at which a quantity changes on a graph.

If the quantity is described by a function[1] then the rate of change of the function can be found directly by algebra without drawing a graph. This process is called *differentiation.* We call the rate of change of a function the *derivative of the function.*

There are two ways of finding derivatives of functions:

- from first principles, following the footsteps of the early mathematicians. This is mainly of historical interest, however it introduces the important idea of a *limit* and explains the notation used in differentiation.
- constructing derivatives from known results using *the rules of differentiation.*

2.1 From first principles ...

The gradient at a point on a curve can be found exactly by algebra when the equation of the curve is known.

This section shows how the early mathematical explorers calculated gradients and derivatives.

Example

from first principles

To find the gradient of the parabola $y = x^2$ at the point $P(1, 1)$:

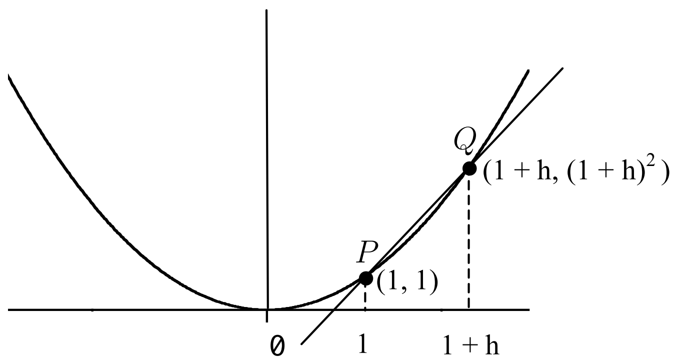
1. Find the gradient of the line through $P(1, 1)$ and a second point Q on the parabola. (See diagram on page 12.)

   We take the second point to be $Q(1 + h, (1 + h)^2)$ *where* h *stands for a number.*[2]

---

[1] A function is a formula that has only one value for each input value, for example $y = x^2$. The *function keys* on a calculator give one value for each input value.

[2] If $x = 1 + h$, then $y = x^2 = (1 + h)^2$.



You can see that when h is a very small number, the line through the points P and Q will be very close to the tangent line at $(1, 1)$.

2. The gradient of the line from $P(1, 1)$ to $Q(1 + h, (1 + h)^2)$ is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{(1+h)^2 - 1}{1+h-1} \\ &= \frac{(1+2h+h^2)-1}{1+h-1} \\ &= \frac{2h+h^2}{h} \\ &= \frac{h(2+h)}{h} \\ &= 2+h \qquad \ldots \text{as long as } h \neq 0.\end{aligned}$$

3. When h is very small, the gradient

$$\frac{\Delta y}{\Delta x} = 2 + h$$

will be very close to the gradient of the tangent line at (1, 1).

We can deduce that ....

$$\textit{as h becomes smaller and smaller, } \frac{\Delta y}{\Delta x} \text{ becomes closer to 2.}$$

This shows that the gradient of the parabola $y = x^2$ at $(1, 1)$ is *exactly* 2 .

Alternatively, we can say the derivative of the function x^2 at $x = 1$ is 2.

When the early mathematicians explored differentiation they discovered some simple relationships between functions and their derivatives that we can use to calculate derivatives quickly.

For example, the derivative of the function x^2 is always equal to $2x$ for every value of x. This is shown in the next example, using the method of first principles.

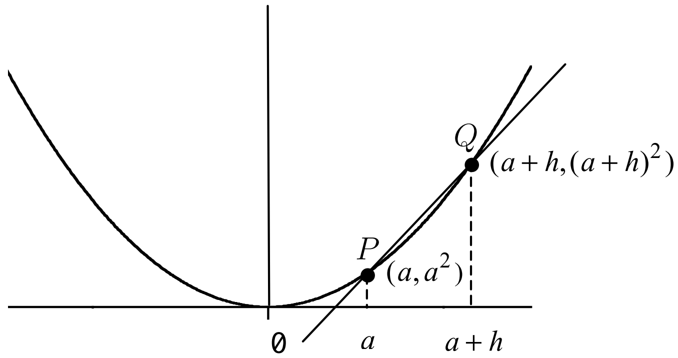
Example

gradient of $y = x^2$ at $x = a$

To find the gradient of the parabola $y = x^2$ at $x = a$ means to find the gradient at the point (a, a^2) on the parabola:

1. Find the gradient of the line through $P(a, a^2)$ and a second point Q on the parabola.

   We take the second point to be $Q(a + h, (a + h)^2)$ where h stands for a number.[3]

   When h is a very small number, the line going through the points P and Q will be very close to the tangent line at (a, a^2).

2. The gradient of the line from $P(a, a^2)$ to $Q(a + h, (a + h)^2)$ is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{(a+h)^2 - a^2}{a + h - a} \\ &= \frac{(a^2 + 2ah + h^2) - a^2}{a + h - a} \\ &= \frac{2ah + h^2}{h} \\ &= \frac{h(2a + h)}{h} \\ &= 2a + h \qquad \ldots \text{ as long as } h \neq 0\end{aligned}$$

3. When h is very small, the gradient

$$\frac{\Delta y}{\Delta x} = 2a + h$$

   will be very close to the gradient of the tangent line at (a, a^2).

   We can deduce that ....

   as h becomes smaller and smaller, $\dfrac{\Delta y}{\Delta x}$ becomes closer to 2a.

[3]If $x = a + h$, then $y = x^2 = (a + h)^2$.

This shows that the gradient of the parabola $y = x^2$ at (a, a^2) is *exactly* $2a$. *Alternatively*, we can say the derivative of the function x^2 at $x = a$ is $2a$.

> Note: Instead of saying
>
> *the derivative of* x^2 *at* $x = a$ *is* $2a$,
>
> we normally just say
>
> *the derivative of* x^2 *is* $2x$,
>
> So when $x = 1$, the derivative of x^2 is 2, ...and so on

See **Appendix A** *for more examples of differentiation using first principles.*

2.1.1 Terminology and Notation

(a) The argument that ...

as h becomes smaller and smaller, $\dfrac{\Delta y}{\Delta x}$ becomes closer to $2a$.

is called 'taking the limit', and is expressed more precisely by ...

as h approaches 0, $\dfrac{\Delta y}{\Delta x}$ approaches $2a$.

(b) An arrow is commonly used to represent the word 'approaches', e.g.

$$\textit{as } h \to 0,\ \frac{\Delta y}{\Delta x} \to 2a.$$

...which is read as

as h approaches 0, $\dfrac{\Delta y}{\Delta x}$ approaches $2a$.

(c) This limit can be written more compactly as ...

$$\lim_{h \to 0} \frac{\Delta y}{\Delta x} = 2a$$

...which is read as

the limit, as h approaches 0, of $\dfrac{\Delta y}{\Delta x}$ is $2a$.

(d) The symbol $\lim\limits_{h \to 0} \dfrac{\Delta y}{\Delta x}$ in (c) is the origin of the traditional symbol $\dfrac{dy}{dx}$ that is used to represent a derivative. Instead of writing

the derivative of $y = x^2$ *is* $2x$

we just write

$$\frac{dy}{dx} = 2x \qquad \text{or} \qquad \frac{d}{dx}(x^2) = 2x$$

These are read aloud as

dee y dee x equals $2x$ or *dee dee x of x squared equals $2x$*

This notation is adjusted when different variables are used. For example, if the relationship between population (P) and time (t) is given by $P = t^2$, then the population growth rate is $2t$ and we can write either

$$\frac{dP}{dt} = 2t \qquad \text{or} \qquad \frac{d}{dt}(t^2) = 2t$$

(d) These traditional symbols are clumsy to type without special software and are often replaced by dashes. For example, we can write

$$y' \text{ or } y'(x) \ \text{ instead of } \frac{dy}{dx} \qquad \text{and} \qquad P' \text{ or } P'(t) \ \text{ instead of } \frac{dP}{dt}$$

This is particularly convenient when evaluating derivatives for specific values of the variables. For example,

the derivative of $y = x^2$ at $x = 1$ is 2

can be written briefly as

$$y'(1) = 2,$$

. . . which is read aloud as either

the derivative of y at 1 is equal to 2 or *y dash at 1 is equal to 2*

and

the population growth rate when $t = 10$ is 100

can be written as

$$P'(10) = 100.$$

. . . and read aloud as either

the derivative of P at 10 is equal to 100 or *P dash at 10 is equal to 100*

Note: All of these notations will be used in this Topic.

Exercise 2.1

1. Find the derivative of $y = x^2$ at $x = 3$ from first principles by:
   (a) sketching the parabola $y = x^2$
   (b) marking the points $R(3, 9)$ and $S(3 + h, (3 + h)^2)$ on it, where h is some number.
   (c) finding the gradient $\frac{\Delta y}{\Delta x}$ of the line RS
   (d) evaluating the limit $\lim_{h\to 0} \frac{\Delta y}{\Delta x}$

2. Find the derivative of $y = x^2 - 2x$ at $x = 2$ from first principles by:
   (a) sketching the parabola $y = x^2 - 2x$
   (b) marking the points $U(2, 0)$ and $V(2+h, (2+h)^2 - 2(2+h))$ on it, where h is some number.
   (c) finding the gradient $\frac{\Delta y}{\Delta x}$ of the line UV
   (d) evaluating the limit $\lim_{h\to 0} \frac{\Delta y}{\Delta x}$

3. If $y = x^2 - 2x$, then it is known that $\frac{dy}{dx} = 2x - 2$. Use this to:
   (a) find the gradient of the parabola at the y-intercept
   (b) find the equation of the tangent line at the y-intercept

4. A fish population is increasing according to the quadratic model[4]

$$P(t) = 600t - t^2 \text{ fish/day.}$$

   (a) Sketch this model for $0 \le t \le 600$.

   If $P'(t) = 600 - 2t$:

   (b) find the population growth rate when $t = 100$
   (c) find when the population growth rate is zero
   (d) find the maximum size of the population

[4]formulae representing real life situations are frequently called *models*.

2.2 Constructing derivatives . . .

The early mathematicians discovered that

- the derivatives of basic functions have patterns that can be easily remembered
- the derivative of any function can be constructed from the derivatives of basic functions

2.2.1 Powers

The most common functions are powers, and their derivatives have this pattern ...[5]

$$\boxed{\text{If } y = x^a \text{ for } \textit{any} \text{ power } a\text{, then } \frac{dy}{dx} = ax^{a-1}.}$$

derivatives of powers

Example

a. If $y = x^2$, then $\dfrac{dy}{dx} = 2x^{2-1} = 2x^1 = 2x$

b. If $y = x^{100}$, then $\dfrac{dy}{dx} = 100x^{100-1} = 100x^{99}$

c. If $y = \sqrt{x}$, then $y = x^{1/2}$ and $\dfrac{dy}{dx} = \dfrac{1}{2}x^{1/2-1} = \dfrac{1}{2}x^{-1/2}$ or $\dfrac{1}{2\sqrt{x}}$

d. If $y = \dfrac{1}{x}$, then $y = x^{-1}$ and $\dfrac{dy}{dx} = -1x^{-1-1} = -x^{-2}$ or $-\dfrac{1}{x^2}$

velocity v
distance s
time t

Example

The displacement s (meters) travelled by a car in time t seconds is given by the model

$$s = t^2.$$

What is the velocity of the car after 5 seconds?

Answer:
The velocity is the rate of change in distance by time:

$$v = \frac{ds}{dt} = 2t^{2-1} = 2t$$

When $t = 5$, $v = 2t = 10\, m/s$.

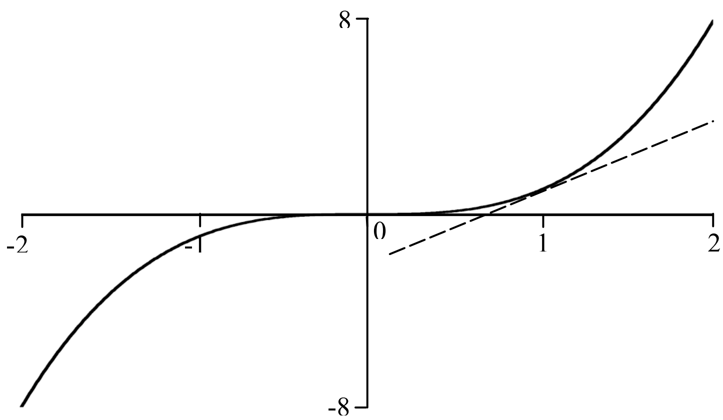
[5] Any variables can be used, not just x's and y's.

There are two powers that occur frequently and whose derivatives are worth memorising: $1\,(= x^0)$ and $x\,(= x^1)$.

If $y = 1\,(= x^0)$, then $\dfrac{dy}{dx} = 0.$	If $y = x\,(= x^1)$, then $\dfrac{dy}{dx} = 1.$

Exercise 2.2.1

1. Differentiate the following functions
   (a) $y = x^{20}$
   (b) $y = \dfrac{1}{x^2}$
   (c) $v = u^3$
   (d) $v = \dfrac{1}{\sqrt{u}}$
2. What is $h'(2)$ if $h(t) = t^4$?
3. Use differentiation to find the gradient of the curve $y = x^3$ at (1,1).



2.2.2 Polynomials

Many mathematical functions are built from simpler functions such as powers. We use this to construct their derivatives.

The general form of a *polynomial* of degree n in x is

$$ax^n + bx^{n-1} + \ldots + dx + e \ ,$$

where a, b, $\ldots$, d, e are numbers. It is constructed by adding or subtracting multiples of powers of a single variable (x in this case), and a constant term.

It can be differentiated by using the following rules:

Rule 1 (constants)
The derivative of a constant is zero.

$$f(x) = c \Longrightarrow f'(x) = 0$$

Rule 2 (multiples)
The derivative of a constant multiple is the multiple of the derivative.

$$y = cf(x) \Longrightarrow y' = cf'(x)$$

Rule 3 (sums)
The derivative of a sum of terms is the sum of their derivatives.

$$y = f(x) + g(x) + \ldots \Longrightarrow y' = f'(x) + g'(x) + \ldots$$

Example

applying the rules for differentiation

1. If $y = 100x^2$, then by rule 2

$$y' = 100 \times 2x = 200x$$

2. If $y = x^2 + 2x + 3$, then by rules 3 and 2

$$y' = 2x + 2 \times 1 + 0 = 2x + 2.$$

3. If $y = (x + 3)(x + 7)$, expand the brackets first, then apply rules 3 and 2

$$\begin{aligned} y &= (x+3)(x+7) \\ &= x^2 + 10x + 21 \\ y' &= 2x + 10 \end{aligned}$$

Rule 3 can also be applied to to differences.[6] This is because a difference such as $f(x) - g(x)$ can be rewritten as the sum of $f(x)$ and $(-1)g(x)$. We don't bother to write down every detail when differentiating, but take it for granted that Rule 2 and 3 imply that:

the derivative of a sum (or difference) of terms is the sum (or difference) of their derivatives.

Example

derivatives of differences

1. If $y = -x^2 + 2x + 3$, then $y' = -2x + 2$.

2. If $y = x^2 - 2x + 3$, then $y' = 2x - 2$.

3. If $y = x^2 + 2x - 3$, then $y' = 2x + 2$.

4. If $y = (x+3)(x-7)$, then

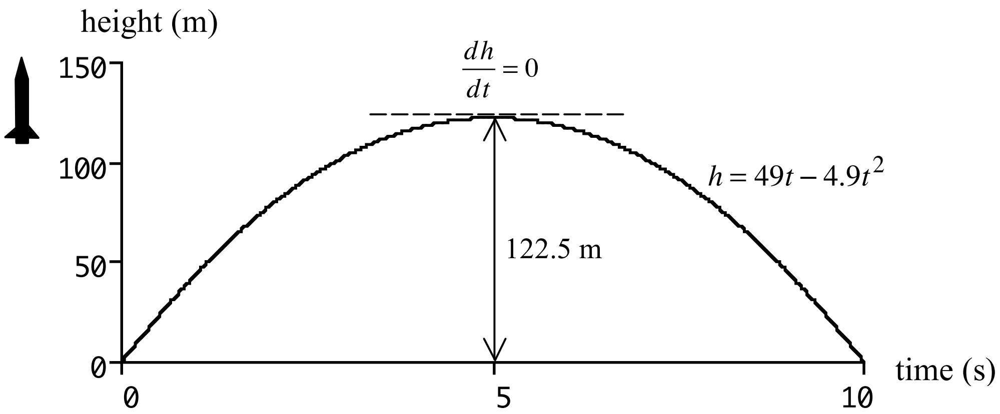
$$\begin{aligned} y &= (x+3)(x-7) \\ &= x^2 - 4x - 21 \\ y' &= 2x - 4 \end{aligned}$$

Example

vertical velocity & maximum height

The height h (metres) of an experimental rocket after t seconds is given by

$$h = 49t - 4.9t^2 \, m/s.$$



The vertical velocity (rate of change of height with time) is

$$\frac{dh}{dt} = 49 \times 1 - 4.9 \times 2t^1 = 49 - 9.8t \, m/s$$

At $t = 0$, the initial velocity is $49 - 9.8 \times 0 = 49 \, m/s$.
At $t = 2$, the velocity is $49 - 9.8 \times 2 = 29.4 \, m/s$.

[6] In mathematics, we use the smallest number of rules that are necessary.
This makes it easier to see how they can be altered or extended when exploring new situations.

The rocket reaches its maximum height when its vertical velocity is 0 m/s. This occurs when

$$\frac{dh}{dt} = 49 - 9.8t = 0 \Longrightarrow t = \frac{49}{9.8} = 5\,s$$

The maximum height reached by the rocket is

$$h(5) = 49 \times 5 - 4.9 \times 5^2 = 122.5\,m$$

Example

square roots

If $y = 2 - 3\sqrt{x}$, then

$$\begin{aligned} y &= 2 - 3\sqrt{x} \\ &= 2 - 3x^{1/2} \\ \\ y' &= 0 - 3 \times \tfrac{1}{2}x^{1/2-1} \\ &= -\frac{3}{2}x^{-1/2} \\ &= -\frac{3}{2\sqrt{x}} \end{aligned}$$

Note. The final line uses the same notation as in original question, a square root symbol rather than a half power.

Exercise 2.2.2

1. Differentiate the following functions
   (a) $y = 3x^3 - 2x + 120$
   (b) $y = 20(x + 2)(x - 5)$
   (c) $y = 2(x + 1)^2 + 10$
   (d) $y = 1 + \dfrac{1}{x}$
2. What is the gradient of the parabola $y = (x - 1)(x - 3)$ at $(1, 0)$ and $(3, 0)$?
3. Repeat the example of the experimental rocket above, assuming that the height is given by
$$h(t) = 98t - 4.9t^2.$$

2.2.3 Products and quotients

When functions are built from the products and quotients of simpler functions, we can use the following rules for constructing their derivatives.

Rule 4 (products)
The derivative of a product is the derivative of the first function multiplied by the second function, plus the first function multiplied by the derivative of the second function.

$$y = f(x)g(x) \Longrightarrow y' = f'(x)g(x) + f(x)g'(x)$$

Example

products $f'g + fg'$

(a) If $y = (x+1)(x^2+2)$, then

$$\begin{aligned} y' &= 1 \times (x^2+2) + (x+1) \times 2x \\ &= 3x^2 + 2x + 3 \end{aligned}$$

(b) If $f(x) = 15 - 3(x+1)(x^2+2)$, then

$$\begin{aligned} f'(x) &= 0 - 3[1 \times (x^2+2) + (x+1) \times 2x] \\ &= -3(x^2 + 2x + 3) \end{aligned}$$

Exercise 2.2.3

1. Use the product rule to differentiate the following
   (a) $y = x^2(2x-1)$
   (b) $y = (x+1)(x^3+3)$
   (c) $y = (x^3+6x^2)(x^2-1)+20$
   (d) $u = (7x+3)(2-3x)+(x+3)^2$
   (e) $u = 80(x^2+7x)(x^2+3x+1)$
   (f) $f(x) = 2-(x^2-5x+1)(2x+3)$
   (g) $g(t) = (t+\frac{1}{t})(5t^2-\frac{1}{t^2})$
   (h) $h(x) = (x^2+1)(3x-1)(2x-3)$

Rule 5 (quotients)
The derivative of a quotient is the derivative of the numerator multiplied by the denominator, less the numerator multiplied by the derivative of the denominator, all divided by the square of the denominator.

$$y = \frac{f(x)}{g(x)} \Longrightarrow y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

quotients $\frac{f'g - fg'}{g^2}$

Example

(a) If $y = \frac{2x+1}{x+3}$, then

$$\begin{aligned} y' &= \frac{2 \times (x+3) - (2x+1) \times 1}{(x+3)^2} \\ &= \frac{5}{(x+3)^2} \end{aligned}$$

(b) If $f(x) = \frac{1}{x+3}$, then

$$\begin{aligned} f'(x) &= \frac{0 \times (x+3) - 1 \times 1}{(x+3)^2} \\ &= -\frac{1}{(x+3)^2} \end{aligned}$$

Exercise 2.2.3

2. Use the quotient rule to differentiate the following

   (a) $f(x) = \frac{x+2}{x-1}$ (b) $g(x) = \frac{x^2}{2x+1}$

   (c) $y = \frac{t}{t^2-3}$ (d) $y = \frac{u^2-1}{u^2+1}$

   (e) $x = \frac{u^2-u+1}{u^2+u+1}$ (f) $t = \frac{\sqrt{x}}{1-2x}$

   (g) $y = \frac{1}{x^2+1}$ (h) $y = \frac{1}{(x+1)^2}$

3. The density of algae in a water tank is equal to $\frac{n}{V}$, where n is the number of algae and V is the volume of the water in the tank. If n and V vary with time t according to the formulas $n = \sqrt{t}$ and $V = \sqrt{t} + 1$, calculate the rate of change of the density.

2.2.4 Composite functions and the chain rule

A function like $y = (x^2+1)^{50}$ can be thought of as being constructed from two simpler functions using substitution:

$$y = u^{50}, \text{ where } u = x^2 + 1.$$

This is an example of a *composite function* or a 'function of a function'. Composite functions are encountered frequently in mathematics, and can be differentiated using the *chain rule*.

Here are some more examples of composite functions ....

Example

composite functions

1. If $y = (2x-1)^3$, then

$$y = u^3, \text{ where } u = 2x - 1.$$

2. If $y = \sqrt{1-x^2}$, then

$$y = \sqrt{u}, \text{ where } u = 1 - x^2.$$

3. If $y = \dfrac{1}{x^2+4}$, then

$$y = \frac{1}{u}, \text{ where } u = x^2 + 1.$$

In general, the symbols

$$f((g(x)) \quad \text{or} \quad f \circ g(x)$$

are used represent the composite function of x obtained by substituting $u = g(x)$ into $f(u)$. Here $g(x)$ is called the inside function and $f(u)$ is called the outside function.

Example

'function of a function'

If $f(x) = x^7$ and $g(x) = 1 - x^2$, then

(a) $f(g(x)) = (1-x^2)^7$

(b) $g(f(x)) = 1 - (x^7)^2 = 1 - x^{14}$

Exercise 2.2.4

1. If $f(x) = x^2 - 3$ and $g(x) = x^5$, find
   (a) $f(g(x))$ or $f \circ g(x)$
   (b) $g(f(x))$ or $g \circ f(x)$
2. If $f(x) = 3x + 2$ and $g(x) = \sqrt{x}$, find
   (a) $f(g(x))$
   (b) $g(f(x))$
3. If $h(x) = 2x^2 + 1$ and $j(x) = x^3$, find
   (a) $h \circ j(x)$
   (b) $j \circ h(x)$
4. If $l(x) = x^2$ and $m(x) = \frac{1}{2}x$, find
   (a) $l(m(x))$
   (b) $m(l(x))$
5. Identify outside and inside functions for the composite functions below.[7]
   (a) $(x + 1)^5$
   (b) $\sqrt{x - 4}$
   (c) $(x^2 - 3x + 4)^2$
   (d) $(3x + \sqrt{x})^3$
6. If $f(x) = x^2$, $g(x) = x + 1$ and $h(x) = 2x$, find $f(g(h(x)))$ or $f \circ g \circ h(x)$.

[7] When there is more than one answer, choose the inside and outside functions that are simplest.

Rule 6 (the chain rule for composite functions)
The derivative of a composite function is the derivative of the outside function multiplied by the derivative of the inside function.[8]

$$\text{If } y = f(u) \text{ where } u = g(x), \text{ then } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

or alternatively

$$y = f(g(x)) \Longrightarrow y' = f'(g(x))g'(x)$$

Example

chain rule

Differentiate $y = (1 - x^2)^7$.

Method 1

Put $y = u^7$, where $u = 1 - x^2$, then

$$\frac{dy}{du} = 7u^6 \quad \text{and} \quad \frac{du}{dx} = -2x$$

so

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 7u^6 \times (-2x) \\ &= -14x(1 - x^2)^6 \end{aligned}$$

Method 2

Put $y = f(g(x))$, where $f(x) = x^7$ and $g(x) = 1 - x^2$, then

$$\begin{aligned} \frac{dy}{dx} &= f'(g(x)) \times g'(x) \\ &= 7(1 - x^2)^6 \times (-2x) \\ &= -14x(1 - x^2)6 \end{aligned}$$

Exercise 2.2.4

7. Use the chain rule to differentiate:

   (a) $(x + 1)^4$ (b) $(x - 1)^4$ (c) $(2 - x)^3$ (d) $(4 - x)^5$

   (e) $(x^2 + 1)^3$ (f) $(x^2 - 3x)^2$ (g) $(2x^2 - 3x + 1)^3$ (h) $(3x - x^3)^5$

8. Differentiate:

   (a) $\sqrt{x + 1}$ (b) $\sqrt{2x + 3}$ (c) $\sqrt{x^2 + 3x - 1}$ (d) $\sqrt{2x - x^3}$

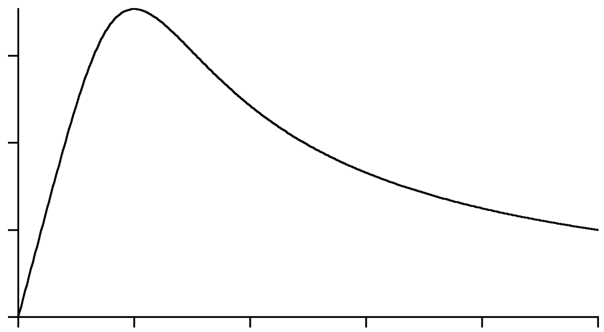
   (e) $\dfrac{1}{\sqrt{x - 1}}$ (f) $\dfrac{1}{\sqrt{x^2 + 3}}$ (g) $\dfrac{3}{\sqrt{3x^2 - 1}}$ (h) $\dfrac{5}{\sqrt{10 - x^2}}$

[8] This is called the *chain* rule because of the *chain* of derivatives. In the case of a 'function $f(x)$ of a function $g(x)$ of a function $h(x)$', the derivative is $f'(g(h(x)))g'(h(x))h'(x)$.

9. Use the product, quotient and chain rules to differentiate:

   (a) $x\sqrt{x+1}$ (b) $x^2\sqrt{2x+3}$ (c) $\dfrac{x}{\sqrt{3x+4}}$ (d) $\dfrac{x^2}{\sqrt{x^2+1}}$

10. The graph of $y = \dfrac{x}{\sqrt{x^4+1}}$ is shown below for $x \geq 0$.



   (a) Find $\dfrac{dy}{dx}$

   (b) Solve $\dfrac{dy}{dx} = 0$

   (c) Find the maximum value of $\dfrac{x}{\sqrt{x^4+1}}$ for $x \geq 0$.

2.2.5 Implicit Differentiation

When functions are *explicitly* defined in the form $y = f(x)$ they can be differentiated using the previous rules of differentiation. However functions can also be *implicitly* defined.

Example

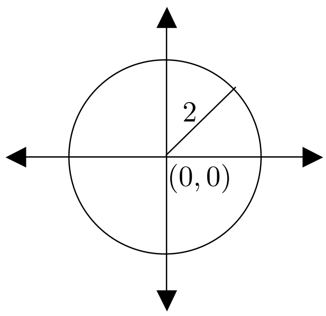
implicit relationship

Consider the equation of the circle

$$x^2 + y^2 = 4$$

with centre $(0, 0)$ and radius 2.

This is an example of an *implicit* relationship between x and y.



Solving this relationship for y gives y *explicitly* in terms of x, that is as the subject of a formula with x as the independent variable.

$$\begin{aligned} x^2 + y^2 &= 4 \\ y^2 &= 4 - x^2 \\ y &= \pm\sqrt{4 - x^2} \end{aligned}$$

The gradient of the tangent line to $x^2 + y^2 = 4$ at $(\sqrt{2}, \sqrt{2})$, can now be found by differentiating $y = \sqrt{4 - x^2}$, giving ...

$$\frac{dy}{dx} = \frac{-x}{\sqrt{4 - x^2}} = \frac{-\sqrt{2}}{\sqrt{4 - (\sqrt{2})^2}} = -1$$

It may be difficult or impossible to solve an implicit relationship between x and y in such a way as to make y the subject of a formula with x as the independent variable. In these cases we use the technique of *implicit differentiation* to find $\dfrac{dy}{dx}$.

Example

implicit differentiation

To find $\dfrac{dy}{dx}$ directly from the implicit relationship $x^2 + y^2 = 4$ ...

1. Assume that y is a function of x, writing it as $y(x)$.

2. Differentiate both sides of $x^2 + y^2 = 4$, e.g.

$$\begin{aligned}
x^2 + y^2 &= 4 \\
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(4) \\
2x + \frac{d}{dx}(y^2) &= 0 && \ldots \text{differentiating each term} \\
2x + 2y\frac{dy}{dx} &= 0 && \ldots \text{by the chain rule} \\
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= -\frac{x}{y} && \ldots \text{provided } y \neq 0
\end{aligned}$$

The derivative can now be used to find the gradient of the tangent line at any point on $x^2 + y^2 = 4$. For example, at the point $(\sqrt{2}, \sqrt{2})$:

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{\sqrt{2}}{\sqrt{2}} = -1$$

Example

tangent to an ellipse

Use implicit differentiation to find the gradient of the tangent line to the general point $P(x, y)$ on the ellipse $x^2 + xy + y^2 = 1$. Then find the equation of the tangent line at $(1, -1)$

Answer

1. Assume that y is a function of x, writing it as $y(x)$.

2. Differentiate both sides of $x^2 + xy + y^2 = 1$, e.g.

$$\begin{aligned}
x^2 + xy + y^2 &= 1 \\
\frac{d}{dx}(x^2 + xy + y^2) &= \frac{d}{dx}(1) \\
2x + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) &= 0 && \ldots \text{differentiating each term} \\
2x + (y + x\frac{dy}{dx}) + \frac{d}{dx}(y^2) &= 0 && \ldots \text{by the product rule} \\
2x + (y + x\frac{dy}{dx}) + 2y\frac{dy}{dx} &= 0 && \ldots \text{by the chain rule} \\
(2x + y) + (x + 2y)\frac{dy}{dx} &= 0 && \ldots \text{collecting like terms} \\
\frac{dy}{dx} &= -\frac{2x + y}{x + 2y} && \ldots \text{provided } x + 2y \neq 0
\end{aligned}$$

The gradient of the tangent line at $(1, -1)$ is:

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = -\frac{2 - 1}{1 - 2} = -1$$

The equation of the tangent line is:

$$y = -x + c, \text{ for some number } c.$$

Substituting $(1, -1)$ into this equation shows that $c = 2$, and that the equation of the tangent line at $(1, -1)$ is $y = -x + 2$.

A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.

If the gradient of the tangent line is $m_1 (\neq 0)$, and the gradient of the normal is m_2, then

$$m_1 m_2 = -1.$$

Example

normal to an ellipse

Use implicit differentiation to find the gradient of the tangent line at the general point $P(x, y)$ on the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$. Then find the equation of the normal line at $(\sqrt{2}, 2\sqrt{2})$.

Answer

1. Rewrite the equation of the ellipse as $4x^2 + y^2 = 16$.
2. Assume that y is a function of x, writing it as $y(x)$.
3. Differentiate both sides of $4x^2 + y^2 = 16$, e.g.

$$\begin{aligned}
4x^2 + y^2 &= 16 \\
\frac{d}{dx}(4x^2 + y^2) &= \frac{d}{dx}(16) \\
8x + \frac{d}{dx}(y^2) &= 0 \qquad \text{...differentiating each term} \\
8x + 2y\frac{dy}{dx} &= 0 \qquad \text{...by the chain rule} \\
\frac{dy}{dx} &= -\frac{4x}{y} \qquad \text{...provided } y \neq 0
\end{aligned}$$

The gradient of the tangent line at $P(x, y)$ is $-\dfrac{4x}{y}$.

The gradient of the normal at $P(x, y)$ is

$$-\frac{1}{\left(-\dfrac{4x}{y}\right)} = -1 \times \left(-\frac{y}{4x}\right) = \frac{y}{4x}$$

The equation of the normal at $(\sqrt{2}, 2\sqrt{2})$ is

$$y = \frac{1}{2}x + c, \text{ for some number } c.$$

Substituting $(\sqrt{2}, 2\sqrt{2})$ into this equation shows that

$$c = \frac{3\sqrt{2}}{2},$$

and that the equation of the tangent line at $(\sqrt{2}, 2\sqrt{2})$ is $y = 2x + \dfrac{3\sqrt{2}}{2}$.

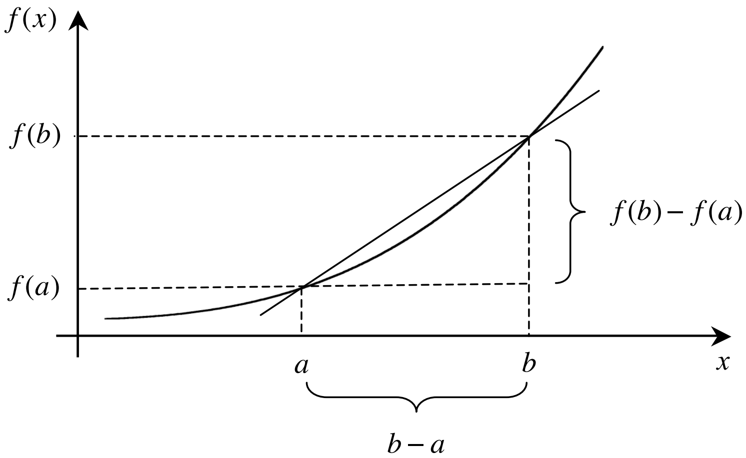
Exercise 2.2.5

1. Find $\dfrac{dy}{dx}$ if:
   (a) $x^2 + y^2 = 16$
   (b) $x^2 + 3y^2 = 9$
   (c) $x^2 - y^2 = 25$
   (d) $x^2 + xy + y^2 = 10$
   (e) $x^3 + 2x^2y + y^2 = 10$
2. Find the gradient of the tangent line to:
   (a) $x^2 + y^2 = 1$ at $(\sqrt{2}, \sqrt{2})$
   (b) $x^2 - xy + y^2 = 1$ at $(1, 1)$
   (c) $x + y = 2xy$ at $(1, 1)$
3. Find the equation of the normal to:
   (a) $x^2 + y^2 = 8$ at $(2, 2)$
   (b) $x^2 + \dfrac{y^2}{2} = 3$ at $(1, 2)$
4. Show that the normal to the circle $x^2 + y^2 = 1$ at the point (a, b) with $ab \neq 0$ always passes through the origin.

Appendix A

First Principles

The graph below shows how a function $f(x)$ might change between $x = a$ and $x = b$.



The gradient of the chord from $(a, f(a))$ to $(b, f(b)$ is

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}.$$

As the width of the interval $[a, b]$ decreases, the approximation

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

becomes closer to *the gradient of the tangent line* to the graph of $f(x)$ at $x = a$, and so to *the derivative* of $f(x)$ at $x = a$.

If we put $b = a + h$, then the derivative of $f(x)$ at $x = a$ is given by the limit

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$

Definition

The derivative of $y = f(x)$ at the point $(a, f(a))$ is given by the limit

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Example

first principles at $x = a$

Find the derivative of $y = x^2$ at $x = a$ using first principles

1. From the definition ...

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0} \frac{(a^2 + 2ah + h^2) - a^2}{h} \\ &= \lim_{h\to 0} \frac{2ah + h^2}{h} \\ &= \lim_{h\to 0} 2a + h \\ &= 2a\end{aligned}$$

The derivative at $x = a$ is $\dfrac{dy}{dx} = 2a$

Example

first principles without using $x = a$

Differentiate $y = x^2 + 4x + 2$ using first principles

1. From the definition ...

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{[(x+h)^2 + 4(x+h) + 2] - [x^2 + 4x + 2]}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0} \frac{[x^2 + (2h+4)x + (h^2 + 4h + 2)] - [x^2 + 4x + 2]}{h} \\ &= \lim_{h\to 0} \frac{2hx + (h^2 + 4h)}{h} \\ &= \lim_{h\to 0} 2x + h + 4 \\ &= 2x + 4\end{aligned}$$

The derivative is $\dfrac{dy}{dx} = 2x + 4$

Example

cubic function

Differentiate $y = x^3$ using first principles

1. From the definition ...

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\ &= 3x^2 \end{aligned}$$

The derivative is $\dfrac{dy}{dx} = 3x^2$

Example

rational function

Differentiate $f(x) = \dfrac{x+1}{x+2}$ using first principles

1. From the definition ...

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{(x+h)+1}{(x+h)+2} - \dfrac{x+1}{x+2}}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h+1)(x+2) - (x+1)(x+h+2)}{(x+h+2)(x+2)h} \\ &= \lim_{h\to 0} \frac{1}{(x+h+2)(x+2)} \\ &= \frac{1}{(x+2)(x+2)} \\ &= \frac{1}{(x+2)^2} \end{aligned}$$

The derivative is $f'(x) = \dfrac{1}{(x+2)^2}$.

Exercise A

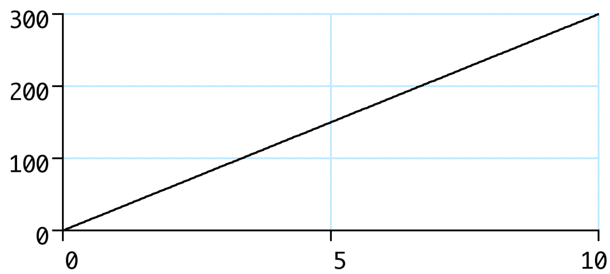
1. Use first principles to find the derivative of $y = x^2 - x$ at $x = a$.
2. Differentiate $y = x^4$ using first principles.
3. Differentiate $f(x) = \dfrac{x-1}{x-2}$ using first principles.

Appendix B

Answers

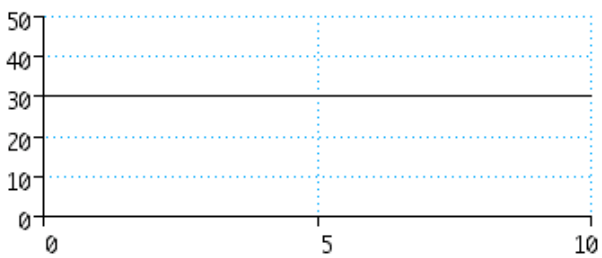
Exercise 1.1

1(a)



1(b) The constant velocity is 30 m/s.

1(c)



1(d) The constant acceleration is 0 m/s^2.

Exercise 1.2

1(a) (i) $m_{PQ} = \dfrac{1 - 0.25}{1 - 0.5} = 1.5$ (ii) $m_{PR} = \dfrac{2.25 - 1}{1.5 - 1} = 2.5$

1(b) This follows from $m_{PQ} < m_{tangent} < m_{PR}$.

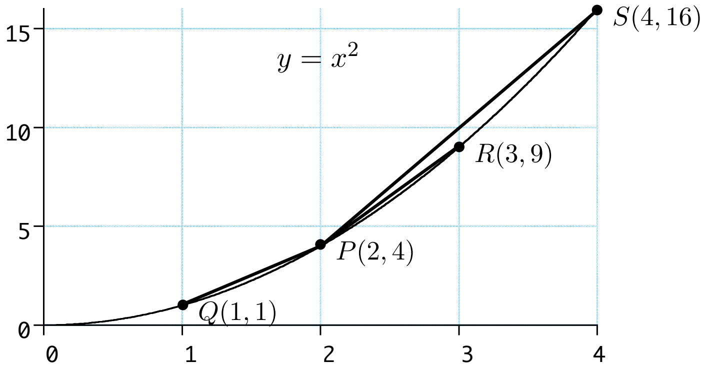
1(c) Using the points $L(0.9, 0.81)$ and $M(1.1, 1.21)$, $m_{tangent} \approx 2$ is an estimate to within ± 0.1 as

$$m_{PL} = 1.9 < m_{tangent} < m_{PM} = 2.1$$

2. (i) matches (a), (ii) matches (c), (iii) matches (d), (iv) matches (b)

Exercise 1.3

1.



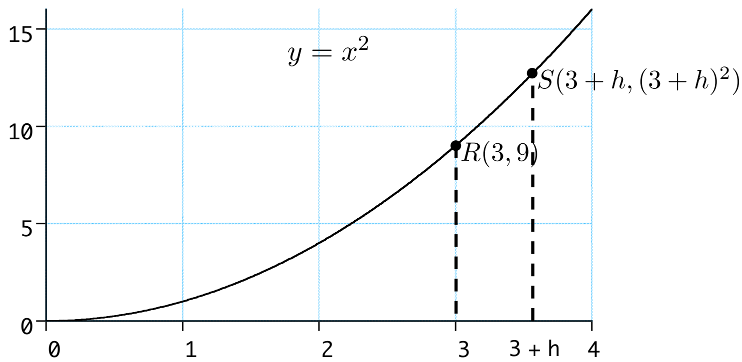
2(a) $m_{QP} = 3$ 2(b) $m_{PR} = 5$ 2(c) $m_{PS} = 6$
3. As $m_{QP} \le m_{tangent} \le m_{PR}$
4. Using the points $L(1.9, 0.81)$ and $M(2.1, 1.21)$, $m_{tangent} \approx 4$ is an estimate to within ± 0.1 as

$$m_{PL} = 3.9 < m_{tangent} < m_{PM} = 4.1$$

5. When the graph is shifted the new tangent line remains parallel to the old, and so has the same gradient.

Exercise 2.1

1(a) & 1(b)

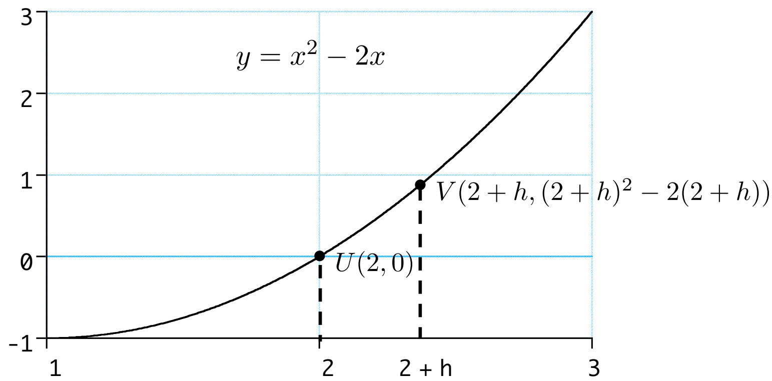


1(c) & 1(d)

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{(3+h)^2 - 9}{(3+h) - 3} \\ &= \frac{6h + h^2}{h} \\ &= 6 + h \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{h \to 0} 6 + h \\ &= 6 \end{aligned}$$

2(a) & 2(b)



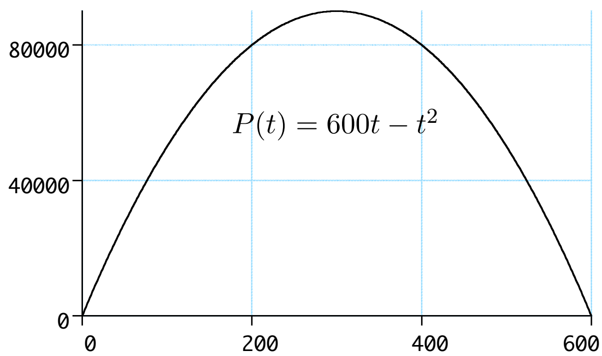
2(c) & 2(d)

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{[(2+h)^2 - 2(2+h)] - 0}{(2+h) - 2} \\
&= \frac{2h + h^2}{h} \\
&= 2 + h
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{h \to 0} 2 + h \\
&= 2
\end{aligned}$$

3(a) The y-intercept is $(0, 0)$. The gradient is $m = -2$.
3(b) The tangent line is $y = -2x$.

4(a)



4(b) The population growth rate is $P'(100) = 600 - 2 \times 100 = 400$.
4(c) Solving $P'(t) = 0$ gives $t = 300$ days.
4(d) This occurs when $P'(t) = 0$ at $t = 300$. The maximum population size is $P(300) = 90,000$.

Exercise 2.2.1

1(a) $\dfrac{dy}{dx} = 20x^{19}$ 1(b) $\dfrac{dy}{dx} = -\dfrac{2}{x^3}$ 1(c) $\dfrac{dv}{du} = 3u^2$ 1(d) $\dfrac{dv}{du} = -\dfrac{1}{2u\sqrt{u}}$

2. As $h'(t) = 4t^3$, $h'(2) = 32$.

3. The derivative is $\dfrac{dy}{dx} = 3x^2$, so the gradient of the curve is 3 at $(1, 1)$.

Exercise 2.2.2

1(a) $\dfrac{dy}{dx} = 9x^2 - 2$ 1(b) $\dfrac{dy}{dx} = 40x - 60$ 1(c) $\dfrac{dy}{dx} = 4x + 4$ 1(d) $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

2. As $\dfrac{dy}{dx} = 2x - 4$, the gradients are $m = -2$ and $m = 2$.

3(a) The vertical velocity is $\dfrac{dh}{dt} = 98 - 9.8t\ m/s$
3(b) At $t = 0$, the initial velocity is 98 m/s, and at $t = 2$, the velocity is 78.4 m/s
3(c) The maximum height is reached when $t = 10\ s$. This height is $h(10) = 490\ m$.

Exercise 2.2.3

1(a) $\dfrac{dy}{dx} = 6x^2 - 2x$ 1(b) $\dfrac{dy}{dx} = 4x^3 + 3x^2 + 3$

1(c) $\dfrac{dy}{dx} = 5x^4 + 24x^3 - 3x^2 - 12x$ 1(d) $\dfrac{du}{dx} = -40x + 11$

1(e) $\dfrac{du}{dx} = 320x^3 + 2400x^2 + 3520x + 560$ 1(f) $f'(x) = -6x^2 + 14x + 13$

1(g) $g'(t) = 15t^2 + 5 + \dfrac{1}{t^2} + \dfrac{3}{t^4}$ 1(h) $h'(x) = 24x^3 - 33x^2 + 18x - 11$

2(a) $f'(x) = \dfrac{-3}{(x-1)^2}$ 2(b) $g'(x) = \dfrac{2x(x+1)}{(2x+1)^2}$

2(c) $\dfrac{dy}{dt} = -\dfrac{t^3+3}{(t^2-3)^2}$ 2(d) $\dfrac{dy}{du} = \dfrac{4u}{(u^2+1)^2}$

2(e) $\dfrac{dx}{du} = \dfrac{2(u^2-1)}{(u^2+u+1)^2}$ 2(f) $\dfrac{dt}{dx} = \dfrac{1+2x}{2\sqrt{x}(1-2x)^2}$

2(g) $\dfrac{dy}{dx} = -\dfrac{2x}{(x^2+1)^2}$ 2(h) $\dfrac{dy}{dx} = -\dfrac{2}{(x+1)^3}$

3. Density is given by the function $D(t) = \dfrac{\sqrt{t}}{\sqrt{t}+1}$, so the rate of change of density is $\dfrac{dD}{dt} = \dfrac{1}{2\sqrt{t}(\sqrt{t}+1)^2}$.

Exercise 2.2.4

1(a) $f \circ g(x) = x^{10} - 3$

1(b) $g \circ f(x) = (x^2 - 3)^5$

2(a) $f(g(x)) = 3\sqrt{x} + 2$

2(b) $g(f(x)) = \sqrt{3x + 2}$

3(a) $h \circ j(x) = 2x^6 + 1$

3(b) $j \circ h(x) = (2x^2 + 1)^3$

4(a) $l(m(x)) = \frac{1}{4}x^2$

4(b) $m(l(x)) = \frac{1}{2}x^2$

5(a) $f(x) = x^5$ and $g(x) = x + 1$

5(b) $f(x) = \sqrt{x}$ and $g(x) = x - 4$

5(c) $f(x) = x^2$ and $g(x) = x^2 - 3x + 4$

5(d) $f(x) = x^3$ and $g(x) = 3x + \sqrt{x}$

6. $f(g(h(x))) = (2x + 1)^2$

Exercise 2.2.4 (cont.)

7(a) $4(x + 1)^3$

7(b) $4(x - 1)^3$

7(c) $-3(2 - x)^2$

7(d) $-5(4 - x)^4$

7(e) $6x(x^2 + 1)^2$

7(f) $2(2x - 3)(x^2 - 3x)$

7(g) $3(4x - 3)(2x^2 - 3x + 1)^2$

7(h) $15(1 - x^2)(3x - x^3)^4$

8(a) $\dfrac{1}{2\sqrt{x + 1}}$

8(b) $\dfrac{1}{\sqrt{2x + 3}}$

8(c) $\dfrac{2x + 3}{2\sqrt{x^2 + 3x - 1}}$

8(d) $\dfrac{2 - 3x^2}{2\sqrt{2x - x^3}}$

8(e) $-\dfrac{1}{2(x - 1)\sqrt{x - 1}}$

8(f) $-\dfrac{x}{2(x^2 + 3)\sqrt{x^2 + 3}}$

8(g) $-\dfrac{9x}{(3x^2 - 1)\sqrt{3x^2 - 1}}$

8(h) $\dfrac{5x}{(10 - x^2)\sqrt{10 - x^2}}$

9(a) $\dfrac{3x + 2}{2\sqrt{x + 1}}$

9(b) $\dfrac{5x^2 + 6x}{\sqrt{2x + 3}}$

9(c) $\dfrac{3x + 8}{2(3x + 4)\sqrt{3x + 4}}$

9(d) $\dfrac{x^3 + 2x}{(x^2 + 1)\sqrt{x^2 + 1}}$

10(a) $\dfrac{dy}{dx} = -\dfrac{x^4 - 1}{(x^4 + 1)\sqrt{x^4 + 1}}$

10(b) $x = 2^{1/4} \approx 1.19$

10(c) $max = \dfrac{2^{1/4}}{2 + 1} \approx 0.396.$

Exercise 2.2.5

1(a) $\dfrac{dy}{dx} = -\dfrac{x}{y}$

1(b) $\dfrac{dy}{dx} = -\dfrac{x}{3y}$

1(c) $\dfrac{dy}{dx} = \dfrac{x}{y}$ 1(d) $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$

1(e) $\dfrac{dy}{dx} = -\dfrac{3x^2+4xy}{2x^2+2y}$

2(a) $m = -1$ 2(b) $m = -1$ 2(c) $m = -1$

3(a) $y = x$ 3(b) $y = x + 1$

4. The gradient of the tangent to the circle through the point (a, b) is

$$m_{tangent} = -\frac{a}{b} \ ,$$

so the gradient of the normal is

$$m_{normal} = \frac{b}{a} \ ,$$

and the equation of the normal is $y = \dfrac{b}{a}x$. This shows the normal passes through $(0, 0)$.

Exercise A

1. $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{[(a+h)^2 - (a+h)] - [a^2 - a]}{(a+h) - a} = \lim\limits_{h\to 0} \dfrac{2ah + h^2 - h}{h} = 2a - 1$

2. $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{(x+h)^4 - x^4}{(x+h) - x} = \lim\limits_{h\to 0} \dfrac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = 4x^3$

3. $f'(x) = \lim\limits_{h\to 0} \dfrac{\dfrac{x+h-1}{x+h-2} - \dfrac{x-1}{x-2}}{(x+h) - x} = \lim\limits_{h\to 0} -\dfrac{h}{h(x+h-2)(x-2)} = -\dfrac{1}{(x-2)^2}$