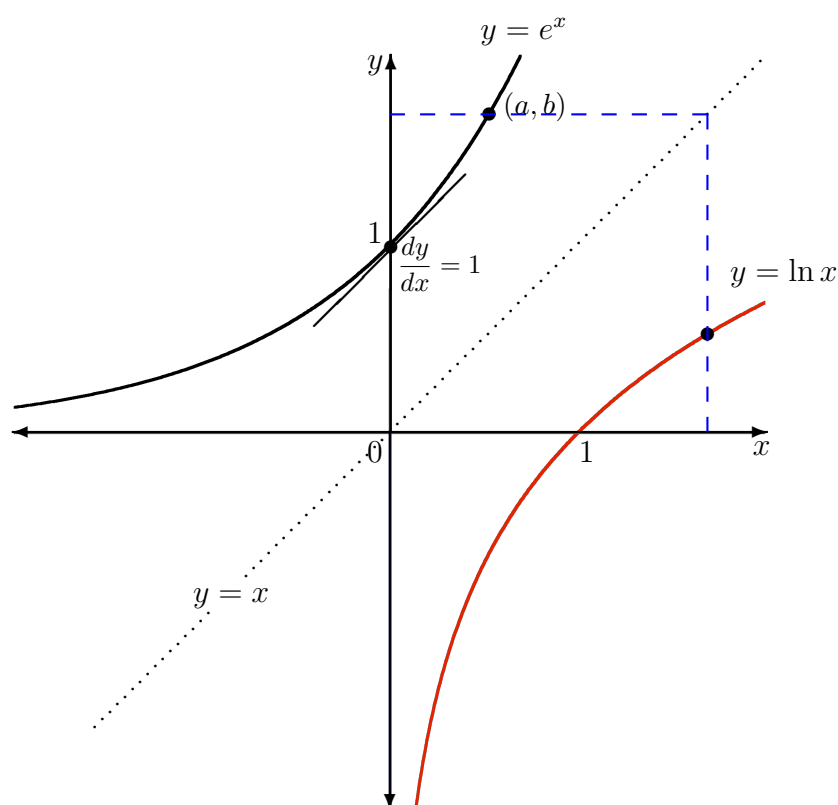


(NOTE Feb 2013: This is the old version of MathsTrack.
New books will be created during 2013 and 2014)

Module 8

Exponential and Logarithm Functions



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This Topic . . .

The topic has 2 chapters:

Chapter 1 reviews the family of exponential functions, and then introduces the natural exponential function and its derivative.

The natural exponential functions is used in numerous mathematical models, and differentiation provides information on the rate of change of the quantities studied in these models.

Chapter 2 reviews the algebraic properties of the natural logarithm function and introduces its derivative.

The natural logarithm is used to solve exponential equations of the form $e^x = a$ and is found along with the exponential function in many areas of mathematics.

Chapter 3 considers the use of exponential and logarithm functions in common mathematical models. These include exponential and logistic growth models, excretion of medications and heat transfer models.

The module builds upon the concepts introduced in MathsStart Topics 7 & 8.

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Chapter 1

Exponential functions

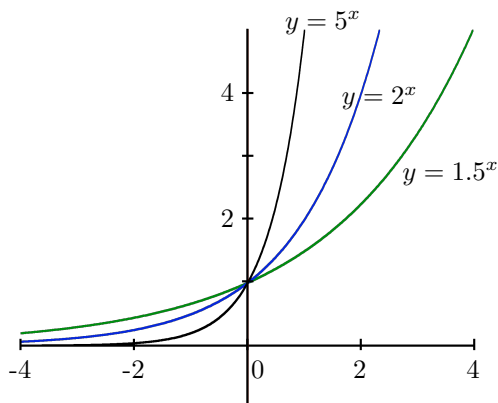
1.1 Introduction

A general exponential function with base a has the form $f(x) = a^x$ where a is a positive constant, and $a \neq 1$.

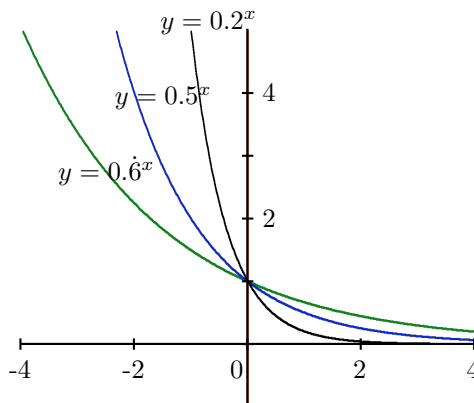
Each member of the exponential family¹

- has domain \mathbb{R}
- is positive
- is concave up
- passes through the point $(0, 1)$
- is either (A) increasing for $a > 1$ or (B) decreasing for $0 < a < 1$

(A) $y = a^x, a > 1$



(B) $y = a^x, a < 1$



¹We call the general exponential functions a *family* of functions when studying similarities between the members for different values of a .

1.2 Derivative of the exponential function

The most important exponential function is e^x to base $e = 2.2.71828\dots$ ² It is called the *natural* exponential function or simply *the* exponential function. The reason why this particular exponential function is important is that its derivative is equal to the function itself.

Let us find the derivative of the general exponential $f(x) = a^x$ using first principles.

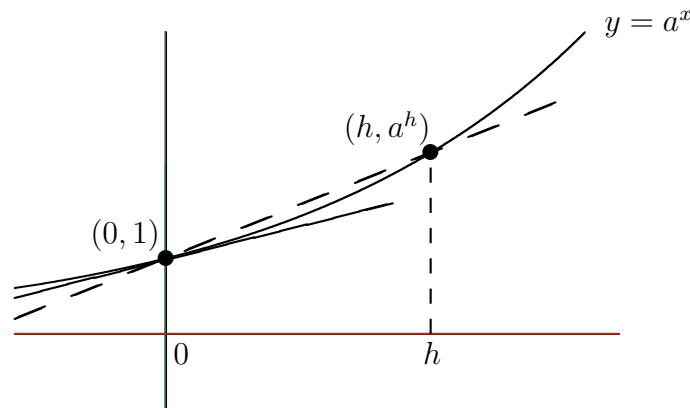
As in Topic 6 (§ 2.1), the derivative is given by the limit

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{(x+h) - x} \\ &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{(x+h) - x} \\ &= \lim_{h \rightarrow 0} \left(\frac{a^h - 1}{h} \right) a^x \\ &= C a^x, \end{aligned}$$

where

$$C = \lim_{h \rightarrow 0} \left(\frac{a^h - 1}{h} \right).$$

The diagram below shows that C is equal to the gradient of the curve $y = a^x$ at $(0, 1)$.



The value of a for which $C = 1$ is $e = 2.71828\dots$ so the derivative of e^x is e^x ! *This makes differentiating easy.*

Example

*composite
functions*

- (i) The function e^{hx} is a composite function and can be differentiated by using the chain rule.

²The notation $\exp(x)$ is used instead of e^x in programming and spreadsheet calculations.

If $y = e^{hx}$, then $y = e^u$ where $u = hx$ and

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} && \dots \text{by the chain rule} \\ &= e^u \times h \\ &= he^{hx}\end{aligned}$$

(ii) The function $e^{f(x)}$ can be differentiated similarly.

If $y = e^{f(x)}$, then $y = e^u$ where $u = f(x)$ and

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} && \dots \text{by the chain rule} \\ &= e^u \times f'(x) \\ &= f'(x)e^{f(x)}\end{aligned}$$

Summarizing ...

<i>function</i>	<i>derivative</i>
e^x	e^x
e^{hx}	he^{hx}
$e^{f(x)}$	$f'(x)e^{f(x)}$

Example

*product
rule*

Differentiate $f(x) = x^5 e^{2x}$.

Answer

By the product rule,

$$\begin{aligned}f'(x) &= 5x^4 e^{2x} + x^5 \times 2e^{2x} \\ &= (2x + 5)x^4 e^{2x}\end{aligned}$$

Example

*quotient
rule*

Differentiate $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Answer

By the quotient rule,

$$\begin{aligned}y' &= \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x + e^{-x})^2} \\ &= -\frac{4}{(e^x + e^{-x})^2}\end{aligned}$$

Example*general
exponential*

To differentiate an exponential function like $g(x) = 2^{3x}$, we first need to express it to the base e .

$$\begin{aligned} g(x) &= 2^{3x} \\ &= (e^{\ln 2})^{3x} \\ &= e^{(3 \ln 2)x} \\ g'(x) &= (3 \ln 2) e^{3 \ln 2x} \\ &= (3 \ln 2) (e^{\ln 2})^{3x} \\ &= (3 \ln 2) 2^{3x} \end{aligned}$$

Example*turning
points*

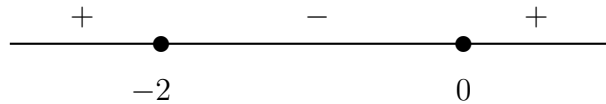
Find the location of any turning points on $y = x^2 e^x$. Use a sign diagram to decide whether they correspond to local maxima or minima.

Answer

Differentiating using the product rule:

$$\begin{aligned} y &= x^2 e^x \\ y' &= 2x e^x + x^2 e^x \\ &= x(x+2)e^x \end{aligned}$$

The sign diagram for $y' = x(x+2)e^x$ is:



This shows that $(0, 0)$ is a local minimum, and $(-2, 4e^{-2})$ is a local maximum.

As $x(2+x)e^x \geq 0$ for all $x \in \mathbb{R}$, $(0, 0)$ is in fact a global minimum.

Exercise 1.2

1. Differentiate the following for $f(x)$ equal to:

- (a) $5e^x$ (b) e^{7x} (c) $20e^{-5x}$
(d) $\exp(-2x)$ (e) $2e^{5x} - 10$ (f) $4x^3 + 10x - e^x$
(g) $3e^{4x} + 2x^2 + e^3$ (h) $5(e^x + e^{-x})$ (i) $3e^{2x}(e^x + 1)$
(j) $(e^x + 1)(e^{-2x} + 1)$ (k) $\frac{2e^x + 3e^{-x}}{e^x}$ (l) $\frac{5e^x(e^x + 1)}{e^{2x}}$

2. Use the product or quotient rule to differentiate:

- (a) xe^x (b) x^2e^{-x} (c) $\sqrt{x}e^{2x}$
(d) $\frac{2e^x}{x}$ (e) $\frac{1}{1 - e^{-x}}$ (f) $\frac{e^x - 1}{e^x + 1}$

3. Use the chain rule to differentiate:

- (a) $(e^{2x} + 1)^3$ (b) $\sqrt{1 + e^{-x}}$ (c) $\frac{1}{\sqrt{1 + e^{2x}}}$
(d) $x\sqrt{1 + e^x}$ (e) $e^{(x+1)^2}$ (f) $e^{\sqrt{x^2+1}}$

4. What is the gradient of the tangent line to $y = 10 \times 3^{2x}$ at $(0, 10)$?

5. If $y = ae^{hx}$, where a and h are constants, show that

- (a) $\frac{dy}{dx} = hy$
(b) $\frac{d^2y}{dx^2} = h^2y$

6. Find the location of any turning points on the following curves. Use a sign diagram to decide whether they are local maxima or minima.

- (a) $y = (x - 1)e^x$
(b) $y = xe^{2x}$
(c) $y = (x + 1)e^{-x}$
(d) $y = \frac{e^x}{x}, x \neq 0$
-

Chapter 2

The natural logarithm

2.1 Introduction

The natural logarithm is used to solve equations of the form $e^x = a$.^{1,2}

$$e^x = a \Leftrightarrow x = \ln a$$

Example

*population
growth*

The population of a region is anticipated to grow according to the exponential model $P(t) = 10\,000e^{0.02t}$ after t years.

How long will it take the population to reach 15 000?

Answer

We need to solve

$$10\,000e^{0.02t} = 15\,000$$

First, divide both sides by 10 000 to obtain a power of e on the left side

$$e^{0.02t} = 1.5$$

‘unpack’ the exponent by using the natural logarithm

$$0.02t = \ln 1.5$$

then solve for t

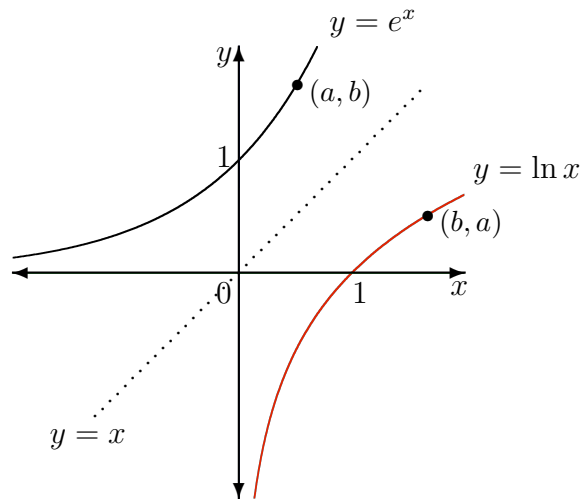
¹The symbol $\ln a$ in the box below represents the natural logarithm of a . The symbol $\log_e a$ can also be used.

²The symbol \Leftrightarrow in the box is read aloud as “if and only if”. It means: if $e^x = a$, then $x = \ln a$ and, if $x = \ln a$, then $e^x = a$.

$$\begin{aligned} t &= \frac{\ln 1.5}{0.02} \\ &= 20.27 \text{ years} \end{aligned}$$

If (a, b) is a point on the graph of $y = e^x$, then $b = e^a$ and $a = \ln b$. So (b, a) is a point on the graph of $y = \ln x$.

This means that the graph of $y = \ln x$ can be obtained from the graph of $y = e^x$ by interchanging the x and y coordinates, as seen in the diagram below. *The two graphs are reflections about the line $y = x$.*



Observe that

- $\ln x$ has domain \mathbb{R}^+
- the graph of $y = \ln x$ is concave down
- $y = \ln x$ passes through the point $(1, 0)$

Other important properties of the logarithm function are

- $\ln UV = \ln U + \ln V$ for all $U, V > 0$
- $\ln(U/V) = \ln U - \ln V$ for all $U, V > 0$
- $\ln U^h = h \ln U$ for all $U > 0$

These properties enable us to solve any exponential equation of the form $a^{hx} = b$.

Example

*exponential
equation*

Solve $2^{3x} = 5$

Answer

Take logarithms of both sides, then solve for x .

$$\begin{aligned}2^{3x} &= 5 \\ \ln(2^{3x}) &= \ln 5 \\ 3x \ln 2 &= \ln 5 \\ x &= \frac{\ln 5}{3 \ln 2} = 0.7740\end{aligned}$$

Exercise 2.1

1. Solve the following equations if possible.

(a) $e^x = 5$

(b) $e^x = -3$

(c) $e^{2x} = 7$

(d) $20e^{-x} = 2$

(e) $e^{2x} + 3 = 19$

(f) $e^x(e^x - 2) = 0$

(g) $e^{2x} - 3e^x + 2 = 0$

(h) $e^x = \frac{2}{e^x - 1}$

(i) $e^x + e^{-x} = 2$

2. Solve the following equations if possible.

(a) $\ln x + \ln(x - 1) = \ln 2$

(b) $\ln x - \ln(x - 3) = \ln 2$

2.2 Derivative of the natural logarithm

The derivative of the $\ln x$ can be found directly from its relationship with e^x .

If $y = \ln x$, then $x = e^y$. Differentiating both sides of $x = e^y$ by x , gives

$$\begin{aligned} x &= e^y \\ 1 &= e^y \times \frac{dy}{dx} && \dots \text{by the chain rule} \\ \text{so } \frac{dy}{dx} &= \frac{1}{e^y} \\ &= \frac{1}{x} \end{aligned}$$

The derivative of $\ln x$ is the function x^{-1} ... this fills the gap in our knowledge on the functions whose derivatives are powers. See the table below.

<i>function</i>	<i>derivative</i>
$x^n, n \neq 0$	nx^{n-1}
$\ln x$	x^{-1}

Example

*composite
functions*

The function $\ln(f(x))$ is a composite function and can be differentiated by using the chain rule.

If $y = \ln(f(x))$, then $y = \ln u$ where $u = f(x)$ and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} && \dots \text{by the chain rule} \\ &= \frac{1}{u} \times f'(x) \\ &= \frac{f'(x)}{f(x)} \end{aligned}$$

Summarizing ...

<i>function</i>	<i>derivative</i>
$\ln x$	$1/x$
$\ln(f(x))$	$f'(x)/f(x)$

The properties of logarithms enable us to differentiate any exponential function of the form a^{hx} .

Example

*other
exponential
functions*

Differentiate $f(x) = 2^{3x}$

Answer

Rewrite 2^{3x} as a power to the base e , then differentiate.

$$\begin{aligned} f(x) &= 2^{3x} \\ &= (e^{\ln 2})^{3x} \\ &= e^{3 \ln 2x} \\ f'(x) &= 3 \ln 2 e^{3 \ln 2x} \\ &= (3 \ln 2) 2^{3x} \end{aligned}$$

If we want to differentiate the logarithm of a function, then we must use the chain rule (*or else remember the formula on the previous page*).

Example

*composite
function*

Differentiate $f(x) = \ln(3x^2 + 7)$

Answer

If $y = \ln(3x^2 + 7)$, then $y = \ln u$ where $u = 3x^2 + 7$ and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} && \dots \text{by the chain rule} \\ &= \frac{1}{u} \times 6x \\ &= \frac{6x}{3x^2 + 7} \end{aligned}$$

The properties of logarithms can be used to simplify differentiation.

Example

*properties
of logarithms*

Differentiate $f(x) = \ln \left(\frac{x^2}{(x+1)(x-2)} \right)$

Answer

As $f(x) = 2 \ln x - \ln(x + 1) - \ln(x - 2)$,

$$f'(x) = \frac{2}{x} - \frac{1}{x+1} - \frac{1}{x-2}.$$

Exercise 2.2

1. Differentiate the following for $f(x)$ equal to:

(a) $5 \ln x$

(b) $\ln(7x)$

(c) $20 \ln(5x)$

(d) $\log_e(2x)$

(e) $2 \ln(5x) - 10$

(f) $4x^3 + 10x - \ln x$

2. Use the product or quotient rule to differentiate:

(a) $x \ln x$

(b) $x^2 \ln x$

(c) $\sqrt{x} \ln(2x)$

(d) $e^x \ln x$

(e) $\frac{2 \ln x}{x}$

(f) $\frac{x}{2 \ln x}$

3. Use the chain rule to differentiate:

(a) $(\ln x)^3$

(b) $\sqrt{\ln x}$

(c) $\ln(x + 1)$

(d) $\ln(x^2 + 1)$

(e) $\ln x(x^2 + 1)$

(f) $\ln[(x^2 + 1)(x^2 + 2)(x^2 + 3)]$

4. Show that the minimum value of $f(x) = x^2 \ln x$ is $\frac{1}{\sqrt{e}}$.

Chapter 3

Mathematical models

3.1 Introduction

A mathematical model is an equation which is intended to match or model the behavior of some natural quantities.

Exponential functions are found in many mathematical models. Exponential, surge and logistic models make use of exponential functions and are described in sections 3.2 to 3.4.

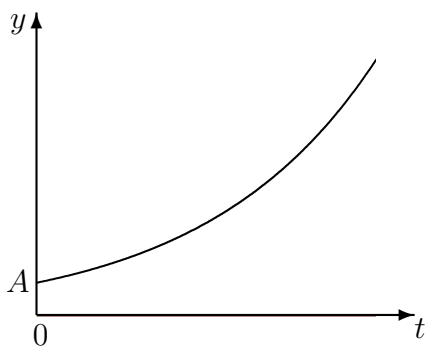
3.2 Exponential models

Exponential growth and decay models have the form

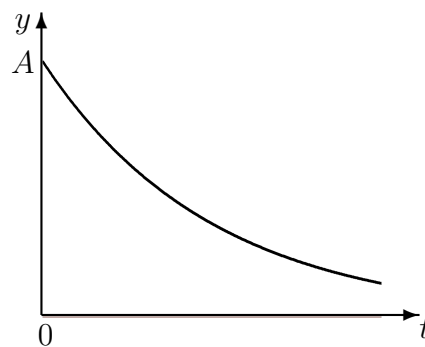
$$y = A e^{bt}, t \geq 0$$

for constants A and b , where independent variable t usually represents *time*.

(a) *Growth Model: $b > 0$*



(b) *Decay Model: $b < 0$*



Exponential growth models¹ are typically used to model populations that have a

¹Also known as Malthusian models

constant percentage growth rate due to an unchanging environment.² Populations can range from micro-organisms to people.

Exponential decay models are typically used to model the loss of matter that has a constant percentage decay rate.³ Examples include herbicides, radioactive materials and the elimination of medicines from the body.

Example

population
growth

The population of a rabbit colony grows according to the exponential growth model

$$P(t) = 60 e^{1.6t}$$

where time t is given in years.

This model shows that ...

- the initial population was

$$P(0) = 60 e^{1.6 \times 0} = 60 \text{ rabbits}$$

- at t years, the population grew at the rate

$$\begin{aligned} \frac{dP}{dt} &= 60 \times 1.6 e^{1.6t} \\ &= 96 e^{1.6t} \text{ rabbits per year} \end{aligned}$$

- the constant growth rate per head of population was

$$\begin{aligned} \frac{dP}{dt} \div P &= \frac{60 \times 1.6 e^{1.6t}}{60 e^{1.6t}} \\ &= 1.6 \text{ rabbits per year per head of population} \end{aligned}$$

... a growth rate of 160% per year.

The model can also be used for predictions:

- (a) After 5 years there will be

$$P(5) = 60 e^{1.6 \times 5} \approx 7291 \text{ rabbits}$$

- (b) The time taken for the population to reach 10,000 can be found from solving the equation $60 e^{1.6t} = 10000$.

$$\begin{aligned} 60 e^{1.6t} &= 10000 \\ e^{1.6t} &= 10000/60 \\ 1.6t &= \ln(10000/60) \\ t &= \frac{\ln(10000/60)}{1.6} \\ &= 3.2 \text{ years} \end{aligned}$$

²percentage growth rate = growth rate per head of population \times 100 %

³percentage decay rate = decay rate per amount of material \times 100 %

Example*bacterial
growth*

The amount of live bacteria in a Petri dish is modelled by the formula

$$M(t) = 50 e^{0.18t} \text{ gm}$$

after t days.

You can see that

- the initial amount of live bacteria was

$$M(0) = 50 e^{0.18 \times 0} = 50 \text{ gm}$$

- the bacterial grew at the rate

$$\begin{aligned} \frac{dM}{dt} &= 50 \times 0.18 e^{0.18t} \\ &= 9 e^{0.18t} \text{ gm/day} \end{aligned}$$

- the constant growth rate per gram was

$$\begin{aligned} \frac{dM}{dt} \div M &= \frac{50 \times 0.18 e^{0.18t}}{50 e^{0.18t}} \\ &= 0.18 \text{ gm/day per gram} \end{aligned}$$

... a growth rate of 18% per day

Example*decay
model*

In laboratory conditions, the mass $M(t)$ of a pesticide decayed according to the exponential decay model.

$$M(t) = 10 e^{-0.15t} \text{ gm}$$

after t days.

The model shows that

- the pesticide decayed at the rate

$$\begin{aligned} \frac{dM}{dt} &= 10 \times (-0.15) e^{-0.15t} \\ &= -1.5 e^{-0.15t} \text{ gm/day} \end{aligned}$$

- the constant decay rate per gram was

$$\begin{aligned} \frac{dM}{dt} \div M &= \frac{10 \times (-0.15) e^{-0.15t}}{10 e^{-0.15t}} \\ &= -0.15 \text{ gm/day per gram} \end{aligned}$$

... a decay rate of 15% per day

Newton's *Law of Cooling* models how the temperature $T(t)$ of an object changes⁴ from an initial temperature of $T(0)$ when it is placed in an environment having temperature T_{env} .

$$T(t) = T_{env} + (T(0) - T_{env})e^{-kt}$$

Example

*heat
transfer*

A turkey is cooking in a convection oven which is at a baking temperature of $200^\circ C$. The turkey starts at a temperature of $20^\circ C$ and after a half hour has warmed to $30^\circ C$. How long will it take to warm to a well-done temperature of $80^\circ C$?

Answer

We need to find k first of all. As turkey took 30 min to heat from $20^\circ C$ to $30^\circ C$, we have

$$\begin{aligned} 30 &= 200 + (20 - 200)e^{-0.5k} \\ e^{-0.5k} &= \frac{170}{180} \\ k &= 0.1143 \end{aligned}$$

To find the time taken to heat to $80^\circ C$, solve

$$\begin{aligned} 80 &= 200 + (20 - 200)e^{-0.1143t} \\ e^{-0.1143t} &= \frac{120}{180} \\ t &= 3.5 \text{ hours} \end{aligned}$$

Exercise 3.2

1. A population of bacteria is given by $P(t) = 5\,000 e^{0.18t}$ after t hours.
 - (a) What is the population at
 - (i) $t = 0$ hours
 - (ii) $t = 30$ minutes
 - (iii) $t = 2$ hours?
 - (b) How long would it take for the population to reach 15 000?
 - (c) What is the rate of increase of the population at
 - (i) $t = 0$
 - (ii) $t = 30$ min?

⁴The model can be used to for general *heat transfer* problems, not just *cooling*! It is generally a very good approximation, though there are exceptions when the heat transfer is primarily through radiation, like the transfer of heat from the sun to the earth, or from the heating element in an oven. One of the best applications is for home heating. How much heat is lost through the walls of a house during winter? How much fuel is saved by adding insulation in the walls?

2. The mass $M(t)$ of a radioactive isotope remaining after t years is given by $M(t) = 5e^{-0.005t}$ grams.
- What is the mass remaining after
 - $t = 0$ hours
 - $t = 6$ months?
 - How long would it take for the mass to decay to 1 gram?
 - What is the rate of radioactive decay at
 - $t = 1$ year
 - $t = 100$ years?
 - Show that $M'(t) = 0.005 \times M(t)$
3. Diabetics with type 1 diabetes are unable to produce insulin, which is needed to process glucose. These diabetics must injection medications containing insulin that are designed to release insulin slowly. The insulin itself breaks down quickly.

The decay rate varies greatly between individuals, but the following model shows a typical pattern of insulin breakdown. Here I represents the units of insulin in the bloodstream, and t is the time since the insulin entered the bloodstream in minutes.

$$I = 10e^{-0.05t}$$

- explain what the value 10 tells about the amount of insulin in the bloodstream.
 - What is the rate of breakdown in insulin in the bloodstream at time t ?
4. A population grows according to the model $P(t) = P(0)e^{rt}$ where time t is in years.
- Show that the growth rate $\frac{dP}{dt}$ is proportional to $P(t)$.⁵
 - Show that the growth rate *per head of population* is r .
5. Show that Newton's Law of Cooling implies that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the temperature of the environment, that is

$$\frac{dT}{dt} \propto (T(t) - T_{env}).$$

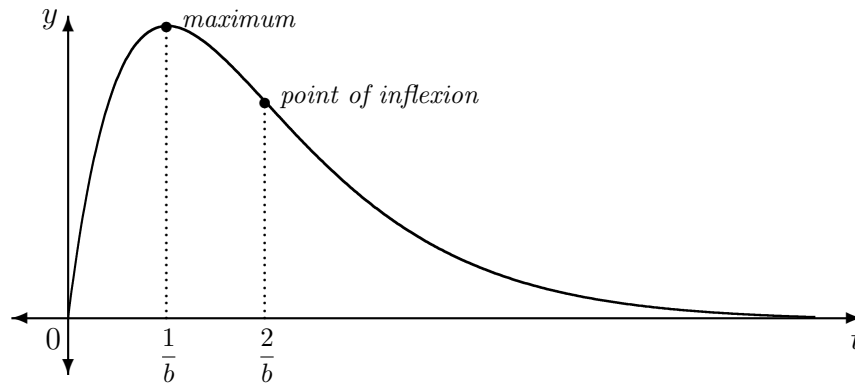
⁵Two quantities, Y and X are said to be proportional, in symbols $Y \propto X$, if Y is equal to a constant multiple of X . The constant is called the *constant of proportionality*.

3.3 Surge models

Surge models have the form

$$y = At e^{-bt}, t \geq 0$$

for constants A and b , where independent variable t usually represents *time*.⁶



Surge models are used in Pharmacokinetics to model the uptake of medication. There is a rapid increase in concentration in the bloodstream after introduction by ingestion, injection, or other means, then a slow elimination through excretion or metabolism.

Exercise 3.3

1. After an aspirin tablet is ingested, the amount entering the bloodstream is modelled by $M(t) = 100te^{-0.5t}$ mg, t hours after its absorption into the bloodstream has begun.
 - (a) How much aspirin is in the bloodstream after
 - (i) $t = 0$ hour
 - (ii) $t = 1$ hour
 - (iii) $t = 2$ hours?
 - (b) When is the amount of aspirin in the bloodstream a maximum, and what is the maximum?
 - (c) What is the point of inflexion of the graph of $M(t) = 100te^{-0.5t}$. What is the significance of this point?
2. The amount of aspirin entering the bloodstream is modelled closely by

$$M(t) = Ate^{-bt} \text{ mg,}$$

t hours after initial absorption into the bloodstream, where A and b can be varied according to the type of tablet and amount of aspirin used.

⁶This is a special case of the general Makoid-Banakar model in which the amount of dissolved drug at time t is given by

$$d(t) = At^n e^{-bt}, \text{ where } A, n, b > 0.$$

What should the values of A and b be if the the maximum amount of aspirin in the blood was 120 gm at $t = 2$ hours?

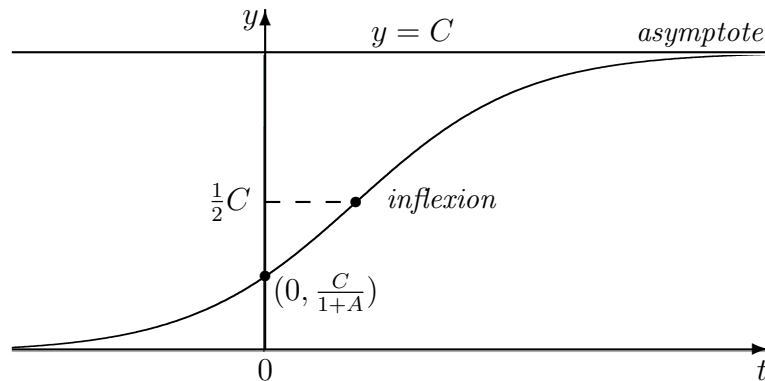
3. What is the turning point of the curve $y = Ax^n e^{-bx}$, where $A, n, b, x > 0$?
-

3.4 Logistic models

Logistic models have the form

$$y = \frac{C}{1 + Ae^{-bt}}, \quad t \geq 0$$

for constants A , b and C , where independent variable t usually represents *time*.



Logistic models are used to model self-limiting populations where growth is restricted by competition for limited resources.⁷ The number C is called the *carrying capacity* of the population.

Exercise 3.4

- The population of a new colony of bees after t months is given by

$$P(t) = \frac{50\,000}{1 + 1000e^{-0.5t}}$$

- What is the initial population of the colony?
 - What is the carrying capacity of the colony.
 - How long will it take the population to reach 40 000?
 - Show that $P'(t) \geq 0$ for all $t \geq 0$, and interpret this.
 - Find when the population growth rate is greatest.
- Show that the logistic function

$$y = \frac{C}{1 + Ae^{-bt}}, \quad t \geq 0$$

has a point of inflexion at $\left(\frac{\ln A}{b}, \frac{C}{2}\right)$.

⁷The logistic function was discovered by Pierre F. Verhulst in 1838, and is also called the Verhulst equation. The shape of the graph is sometimes referred to as *S-curve* or a *Sigmoid curve*.

Appendix A

Answers

Exercise 1.2

1(a) $5e^x$

1(b) $7e^{7x}$

1(c) $-100e^{-5x}$

1(d) $-2 \exp(-2x)$

1(e) $10e^{5x}$

1(f) $12x^2 + 10 - e^x$

1(g) $12e^{4x} + 4x$

1(h) $5(e^x - e^{-x})$

1(i) $9e^{3x} + 6e^{2x}$

1(j) $-e^{-x} + e^x - 2e^{-2x}$

1(k) $-6e^{-2x}$

1(l) $-5e^{-x}$

2(a) $(x + 1)e^x$

2(b) $(2x - x^2)e^{-x}$

2(c) $\frac{1 + 4x}{2\sqrt{x}} e^{2x}$

2(d) $\frac{2(x - 1)}{x^2} e^x$

2(e) $\frac{-e^{-x}}{(1 - e^{-x})^2}$

2(f) $\frac{2e^x}{(e^x + 1)^2}$

3(a) $6e^{2x}(e^{2x} + 1)^2$

3(b) $\frac{-e^{-x}}{2\sqrt{1 + e^{-x}}}$

3(c) $\frac{-e^{2x}}{(1 + e^{2x})^{3/2}}$

3(d) $\frac{2 + 2e^x + xe^x}{2\sqrt{1 + e^x}}$

3(e) $2(x + 1)e^{(x+1)^2}$

3(f) $\frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

4. $y'(0) = 20 \ln 3$

6(a) turning point $(0, 0)$; global minimum

6(b) turning point $(-1/2, -1/2 e^{-1})$; global minimum

6(c) turning point $(0, 0)$; global maximum

6(d) turning point $(1, 2e^{-2})$; local minimum

Exercise 2.1

- | | | |
|-----------------------------|--------------------------------------|--------------------------|
| 1(a) $\ln 5$ | 1(b) <i>not possible</i> | 1(c) $\frac{1}{2} \ln 7$ |
| 1(d) $-\ln 0.1$ or $\ln 10$ | 1(e) $\frac{1}{2} \ln 16$ or $\ln 4$ | 1(f) $\ln 2$ |
| 1(g) 0 or $\ln 2$ | 1(h) $\ln 2$ | 1(i) 0 |
| 2(a) -1 or 2 | 2(b) 6 | |

Exercise 2.2

- | | | |
|-----------------------------------|---|---|
| 1(a) $\frac{5}{x}$ | 1(b) $\frac{1}{x}$ | 1(c) $\frac{20}{x}$ |
| 1(d) $\frac{1}{x}$ | 1(e) $\frac{2}{x}$ | 1(f) $12x^2 + x - \frac{1}{x}$ |
| 2(a) $\ln x + 1$ | 2(b) $(2 \ln x + 1)x$ | 2(c) $\frac{\ln(2x) + 2}{2\sqrt{x}}$ |
| 2(d) $\frac{(x \ln x + 1)e^x}{x}$ | 2(e) $\frac{2(1 - \ln x)}{x^2}$ | 2(f) $\frac{\ln x - 1}{2 \ln^2 x}$ |
| 3(a) $\frac{3 \ln^2 x}{x}$ | 3(b) $\frac{1}{2x\sqrt{\ln x}}$ | 3(c) $\frac{1}{x + 1}$ |
| 3(d) $\frac{2x}{x^2 + 1}$ | 3(e) $\frac{1}{x} + \frac{2x}{x^2 + 1}$ | 3(f) $\frac{2x}{x^2 + 1} + \frac{2x}{x^2 + 2} + \frac{2x}{x^2 + 3}$ |

Exercise 3.2

- | | | |
|---------------------|--------------------------------|--------------------------|
| 1a(i) 5000 | 1a(ii) 5471 | 1a(iii) 7167 |
| 1(b) 6.1 hours | 1c(i) 900 bacteria/hour | 1c(ii) 985 bacteria/hour |
| 2a(i) 5 gm | 2a(ii) 4.99 gm | 2(b) 322 years |
| 2c(i) 0.025 gm/year | 2c(ii) 0.015 gm/year | |
| 3(a) initial amount | 3(b) $0.5e^{-0.05t}$ units/min | |

Exercise 3.3

1a(i) 0 mg

1a(ii) 60.7 gm

1a(iii) 73.6 gm

1b(i) 2 hours

1b(ii) $200/e$ mg

1c(i) 4 hours

1c(ii) rate of elimination is greatest

2(i) $b = 0.5$ 2(ii) $A = 60e$

3. $\left[\frac{n}{b}, A \left(\frac{n}{b} \right)^n e^{-n} \right]$

Exercise 3.4

1(a) 49 or 50

1(b) 50 000

1(c) ≈ 16.6 months

1d(i) $P'(t) = \frac{25000e^{-0.5t}}{(1 + 10e^{-0.5t})^2} > 0$

1d(ii) $P(t)$ is increasing1(e) $t = 13.8$ months