MathsTrack

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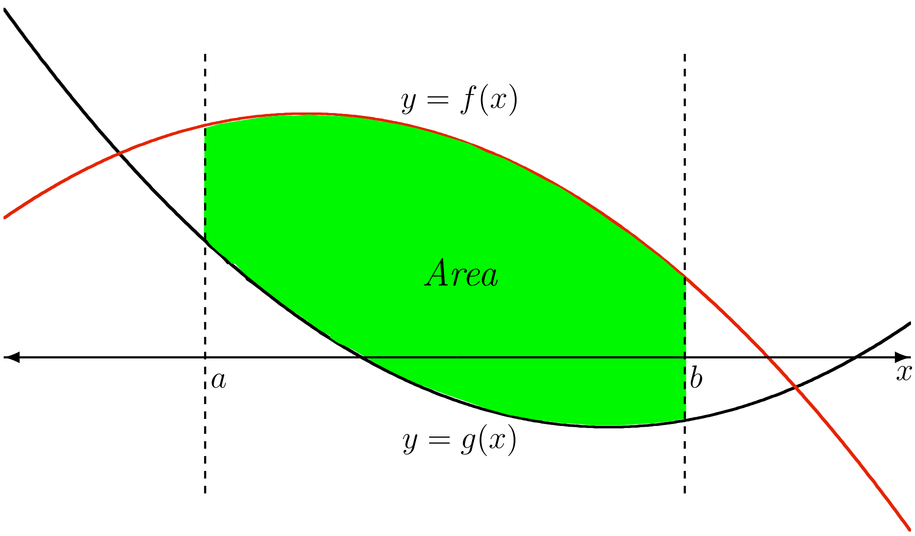
New books will be created during 2013 and 2014)

Topic 9

Integration



$$Area = \int_a^b f(x) - g(x)\ dx = [F(x) - G(x)]_a^b$$

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This Topic ...

The topic has 2 chapters:

Chapter 1 introduces integration which together with differentiation are parts of the branch of mathematics called Calculus.

The chapter begins by asking the question: *Given the rate of change of a quantity, how can we find the quantity*? This question is related to the problem of finding the area of the region between a curve and the horizontal axis. Upper and lower rectangles are used to approximate this region.

The definite integral is introduced together with its properties.

Chapter 2 introduces the Fundamental Theorem of Calculus. This central theorem links integration to differentiation and enables integrals to be evaluated by 'reverse' differentiation.

Antiderivatives and indefinite integrals are introduced. Standard integrals are used to used to integrate more complex functions. The substitution method is used to simplify the integration of composite functions.

Selected applications include calculation of the exact area between two curves and of net change in quantities.

The topic uses the standard derivatives and methods of differentiation introduced in Topic 6.

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January 2008

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Contents

Chapter 1

Integrals

1.1 Introduction

Differentiation was concerned with the question:

Given a quantity, how can we find its rate of change?

If we know the volume of water entering a dam as a function of time, we can use differentiation to find the rate at which the dam is filled.

This topic introduces *integration*, which is concerned with the question:

Given the rate of change of a quantity, how can we find the quantity?

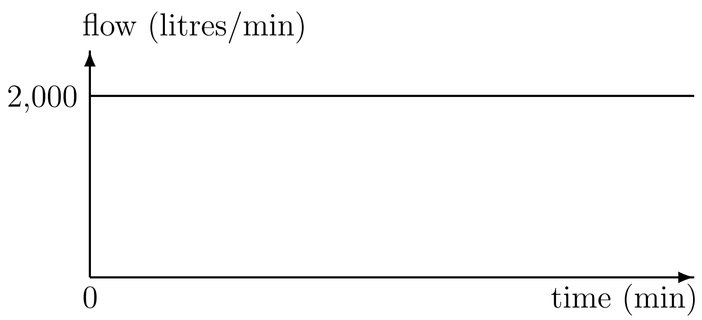
If we know the rate at which a dam is filled, we use integration to find the volume of water entering the dam as a function of time.

Differentiation and integration are parts of *Calculus*.[i]

Example

constant flow

A dam is filled from a creek with a constant flow of 2000 litres/min. This is shown on the *flow-time* graph below.

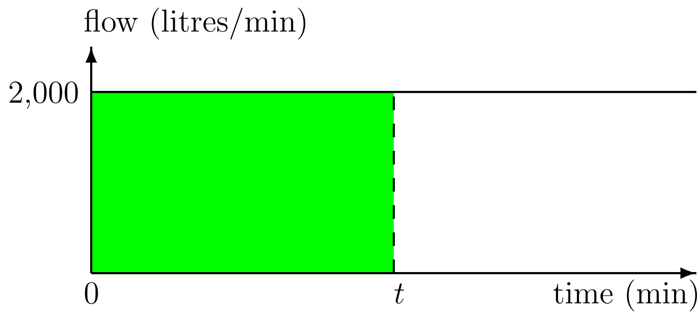


[i]Calculus is a branch of mathematics that includes the study of limits, differentiation, integration and infinite series, and has widespread applications in science and engineering. The word *calculus* was introduced in the mid 17th century from Latin, and means 'a small stone used for counting'.

As the flow is constant, the volume of water flowing into the dam after t minutes is:

$$\begin{aligned} \text{volume} &= \text{flow} \times \text{time} \\ &= 2000 \times t \\ &= 2000\,t \text{ litres} \end{aligned}$$

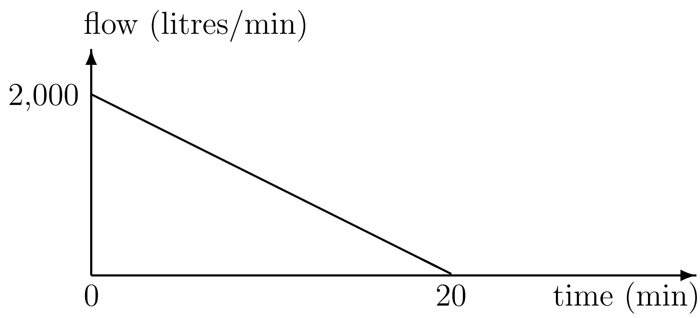
You can see that the volume of water entering the dam is the *area under the flow-time graph from* 0 *to* t *minutes:*



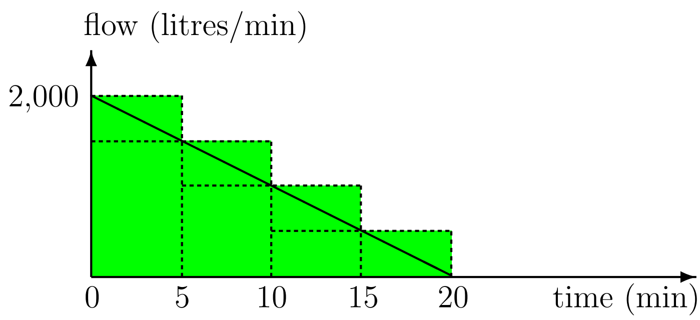
Example

changing flow

Suppose instead that the flow of water entering the dam changed at the rate $2000 - 100t$ litres/min after t minutes. The *flow-time* graph would then be:



We can *estimate* the volume of water entering the dam in 20 minutes by subdividing the interval [0, 20] on the x-axis into four equal parts:



You can see that the area of the *upper rectangle* above interval $[\,0,5\,]$

$$\begin{aligned}\text{height} \times \text{width} &= 2000 \times 5\\ &= 100,000\end{aligned}$$

is an upper estimate of the volume of water flowing into the dam for $0 \le t \le 5$, and that the area of the *lower rectangle*

$$\begin{aligned}\text{height} \times \text{width} &= (2000 - 100 \times 5) \times 5\\ &= 75,000\end{aligned}$$

is a lower estimate of the volume of water flowing into the dam for $0 \le t \le 5$.

The sum of the areas of the four upper rectangles gives an **upper** estimate of the total volume water flowing into the dam for the 20 minutes:

$$2000 \times 5 + (2000 - 100 \times 5) \times 5 + (2000 - 100 \times 10) \times 5 + (2000 - 100 \times 15) \times 5$$

$$= 25,000 \text{ litres}$$

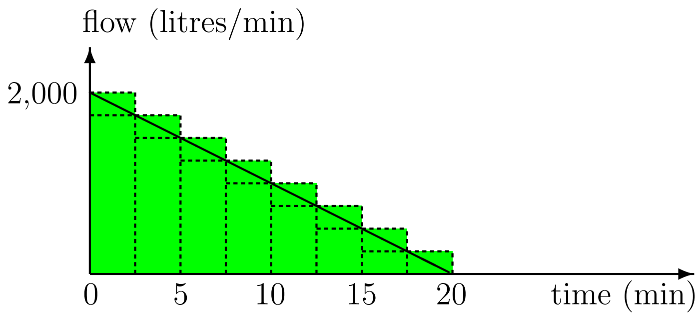
and the sum of the areas of the lower rectangles give a lower estimate of the total volume:

$$(2000 - 100 \times 5) \times 5 + (2000 - 100 \times 10) \times 5 + (2000 - 100 \times 15) \times 5 + 0$$

$$= 15,000 \text{ litres}$$

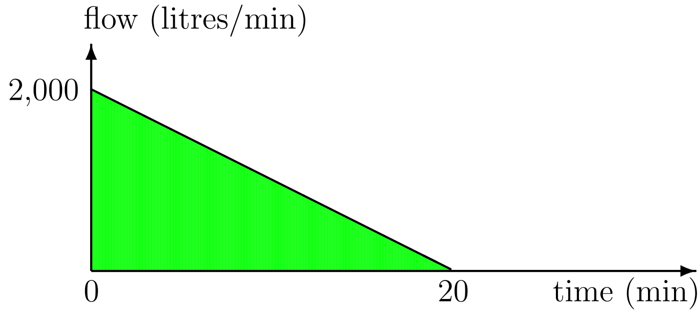
$\ldots$ *the volume of water entering the dam is between* $15,000$ *and* $25,000$ *litres.*

This estimate can be improved by taking finer subdivisions of interval $[\,0,20\,]$.



With eight subdivisions, the upper and lower estimates are $17,500$ and $22,500$.

As smaller subdivisions are taken, you can see that the sum of the areas of the upper and lower rectangles become closer to each other and that both sums become very close to the area under the flow-time curve between $t = 0$ *and* $t = 20$.



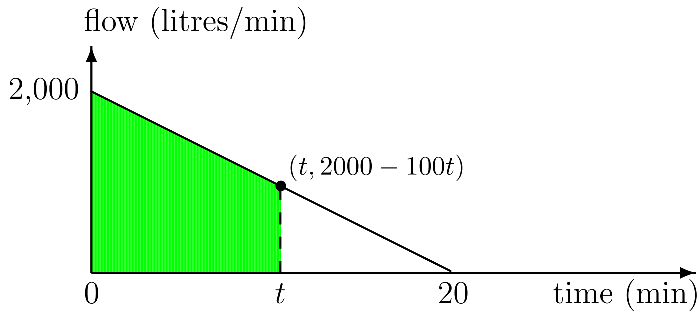
... in this case the total volume of water entering the dam (ie. the shaded area under the flow-time graph) is easy to find: $\frac{1}{2} \times 20 \times 2000 = 20,000$ *litres.*

Example

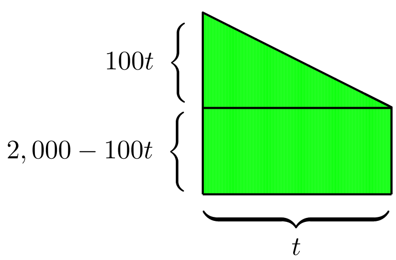
rate of change

total change

The reasoning in the previous example can be used to show that if water flows into the dam at the rate of $2000 - 100t$ litres/min, then the change in the volume of water in the dam after t minutes is equal to the area under the *flow-time* graph from 0 to t, when $t \leq 20$.



This area can be found by adding the areas of the rectangle and triangle below.



... the volume of water in the dam increases by

$$t \times (2000 - 100t) + \frac{1}{2} \times t \times 100t = 2000t - 50t^2 \textit{ litres}$$

after t minutes.[ii]

[ii] Differentiating this result confirms that the rate of change of in volume of water in the dam is

$$\frac{dV}{dt} = 2000 - 100t.$$

Exercise 1.1

1. Rainwater flowed into a 1,000 litre tank at a constant rate of 10 litres/min.

   (a) Draw a graph of the constant flow $f(t)$ into the tank for $t \geq 0$ minutes.

   (b) Calculate the total volume of the water flowing into the tank after t minutes.

   (c) Draw a graph of the volume $V(t)$ of water in the tank for $0 \leq t \leq 60$, if the tank contained 100 litres before the rain began.

   (d) Interprete the gradient of the line in (c).

2. At time $t = 0$, a car began travelling up a hill with a velocity[iii] of

$$v(t) = 30 - 0.5t \text{ m/s},$$

   where t is measured in seconds.

   (a) Draw a graph of the velocity $v(t)$ of the car for $0 \leq t \leq 60$ seconds.

   (b) Calculate the distance travelled by the car after 60 seconds.

   (c) Calculate the distance $s(t)$ travelled by the car after t seconds.

3. Sketch the parabola $y = 1 - x^2$ for $0 \leq x \leq 1$.

   Estimate the area between the parabola and the x-axis for $0 \leq x \leq 1$ by:

   i. subdividing the interval $[0, 1]$ into five equal parts.

   ii. constructing upper and lower rectangles on each subinterval to obtain upper and lower estimates of the area under the parabola and above each subinterval.

   iii. summing of the areas of the upper and lower rectangles.

[iii] velocity = rate of change of distance with time.

1.2 Area under a curve

In the previous section we discovered that the answer to the question:

Given the rate of change of a quantity, how can we find the quantity?

was found by examining the area under the rate of change graph.

We can calculate the exact area under some curves, but for others we cannot and in these cases we need to estimate the area.[iv]

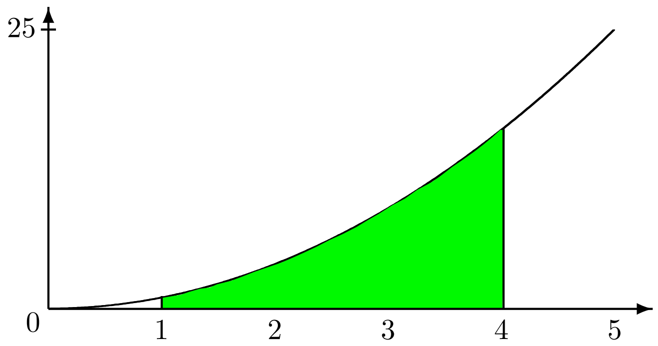
If $f(x)$ is a positive continuous function on $[a, b]$, then we can estimate the area between the curve and the x-axis from $x = a$ to $x = b$ by using *upper* and *lower rectangles.*

upper rectangles

lower rectangles

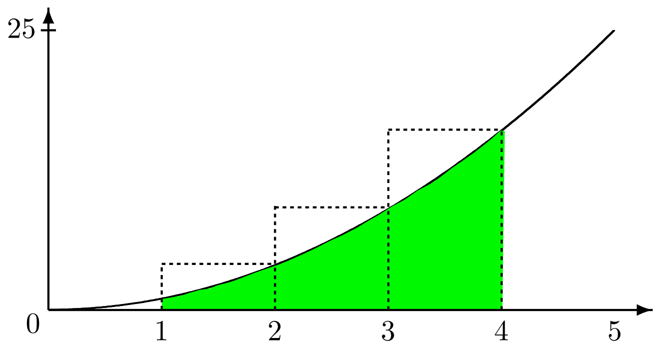
Example

Consider the function $f(x) = x^2$ and the region between the graph of $f(x)$ and the x-axis, bounded by the vertical lines $x = 1$ and $x = 4$.



We can estimate the area A of this region by subdividing the interval $[1, 4]$ into three equal intervals of length 1, and then using *upper rectangles* on each subinterval to estimate the area under the curve.

Upper rectangles are rectangles with height equal to the maximum value of a function on a subinterval.



[iv]This will be discussed in more detail later.

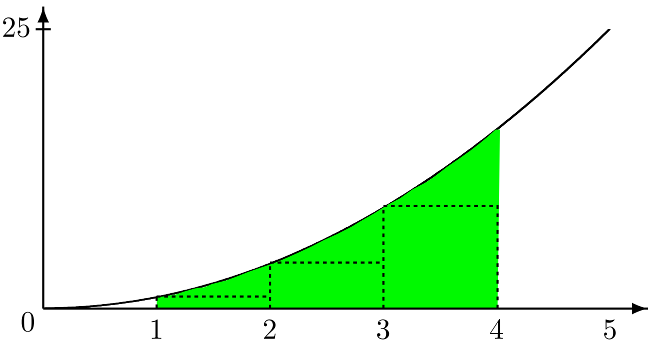
The sum A_U of the areas of the upper rectangles

$$\begin{aligned} A_U &= 1\times f(2)+1\times f(3)+1\times f(4) \\ &= 1\times 4+1\times 9+1\times 16 \\ &= 29 \end{aligned}$$

is an upper estimate of area A.

We can find a lower estimate for A by using *lower rectangles*.

Lower rectangles are rectangles with height equal to the minimum value of a function on a subinterval.



The sum A_L of the areas of the lower rectangles

$$\begin{aligned} A_L &= 1\times f(1)+1\times f(2)+1\times f(3) \\ &= 1\times 1+1\times 2+1\times 9 \\ &= 12 \end{aligned}$$

is an lower estimate of area A.

This shows that area A is between 12 and 29 unit^2.

We can get a better estimate of A by taking smaller subintervals.

For example, if $[1,4]$ was divided into 6 equal parts of length 0.5, then the sum of the areas of the upper rectangles would be:

$$\begin{aligned} A_U &= 0.5\times f(1.5)+0.5\times f(2)+0.5\times f(2.5)+0.5\times f(3) \\ &\quad +0.5\times f(3.5)+0.5\times f(4) \\ &= 0.5\times 1.5^2+0.5\times 2^2+0.5\times 2.5^2+0.5\times 3^2+0.5\times 3.5^2+0.5\times 4^2 \\ &= 24.875\ , \end{aligned}$$

and the sum of the areas of the lower rectangles would be:

$$\begin{aligned} A_L &= 0.5\times f(1)+0.5\times f(1.5)+0.5\times f(2)+0.5\times f(2.5) \\ &\quad +0.5\times f(3)+0.5\times f(3.5) \\ &= 0.5\times 1^2+0.5\times 1.5^2+0.5\times 2^2+0.5\times 2.5^2+0.5\times 3^2+0.5\times 3.5^2 \\ &= 17.375\ . \end{aligned}$$

...showing $17.375 \le A \le 24.875$.

As further subdivisions are taken, the difference between A_U and A_L becomes smaller and each become closer to the area A ($= 21$ unit^2).

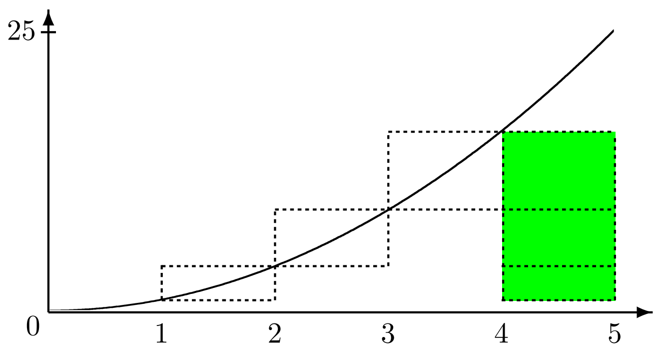
It is easy to estimate the difference $A_U - A_L$ when $f(x)$ is either an increasing function or a decreasing function.

Example (continued)

$A_U - A_L$

As $f(x) = x^2$ is an increasing function, the diagram below shows that the difference in areas $A_U - A_L$ is equal to the area of a rectangle with:

- *width* = width of subinterval
- *height* = height of largest upper rectangle
  − height of smallest lower rectangle



This observation can be used to calculate how small subintervals need be in order to estimate area A with a predetermined precision.

For example, if we wish to estimate A to within 0.1 unit^2, then we need to use subintervals of width w where

$$w \times (16 - 1) \le 0.1$$

Exercise 1.2

1. Sketch the parabola $y = 1 - x^2$ for $0 \leq x \leq 1$.

   (a) Estimate the area between the parabola and the x-axis for $0 \leq x \leq 1$ by:

      i. subdividing the interval $[\,0, 1\,]$ into *two* equal parts.

      ii. constructing upper and lower rectangles for each subinterval.

      iii. summing the areas of the upper and lower rectangles.

   (b) How many subintervals do you need to estimate the area to within 0.1 unit^2 ?

2. Sketch the parabola $y = 1 - x^2$ for $-1 \leq x \leq 1$.

   Estimate the area between the parabola and the x-axis by:

   i. subdividing the interval $[\,-1, 1\,]$ into *four* equal parts.

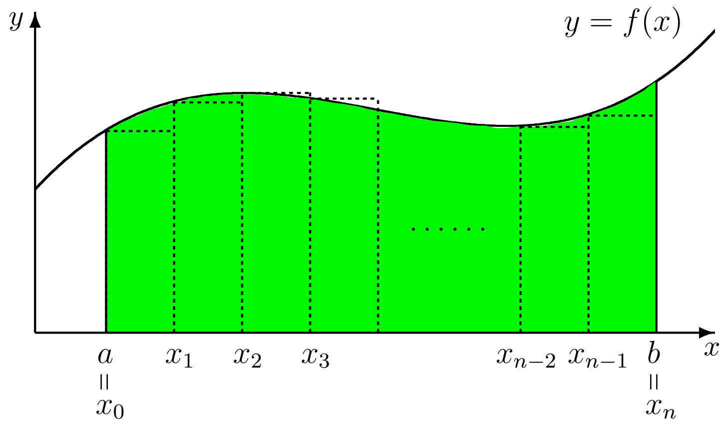
   ii. constructing upper and lower rectangles for each subinterval.

   iii. summing the areas of the upper and lower rectangles.

1.3 The definite integral

Suppose that $f(x)$ is a positive continuous function for $a \leq x \leq b$, and that the interval $[\,a, b\,]$ is divided into n equal parts by the points $x_0, x_1, \ldots, x_n$, with $a = x_0$ and $b = x_n$.

The area A between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ can be estimated by constructing rectangles of heights $f(x_0)$, $f(x_1)$, $\ldots$, $f(x_{n-1})$ on each of the intervals $[\,x_0, x_1\,]$, $[\,x_1, x_2\,]$, $\ldots$, $[\,x_{n-1}, x_n\,]$ as in the diagram below.



The sum of the areas of these rectangles is equal to [v]

$$f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$.

As each rectangle is between the upper and lower rectangles on the same subinterval, you can see that

$$\sum_{i=0}^{n-1} f(x_i)\Delta x \to A \text{ as } n \to \infty\,.$$

The limit

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x)\,\Delta x$$

is represented by

$$\int_a^b f(x)\,dx$$

which is read aloud as ‘the integral from a to b of $f(x)$ dee x’.[vi]

---

[v] See Appendix A.

[vi] The verb *to integrate* means *to form into one whole*, and *integral* is the *whole* obtained after integration.

In this form :

- the limit replaces the '$\sum$' with an elongated S, '$\int$' , called the *integral symbol.*
- the Δx is replaced by dx
- the values at the top and bottom of the integral symbol are the boundaries of the region between the curve $y = f(x)$ and the x-axis. They are called the *upper* and *lower limits* of the integral.

This integral is called a *definite integral* as its upper and lower limits are given. We will consider *indefinite integrals* in the next chapter.

If $f(x)$ is a positive continuous function for $a \le x \le b$, then the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is represented by the definite integral:

$$\int_a^b f(x)\,dx$$

Note: It is easier to work with a sum like

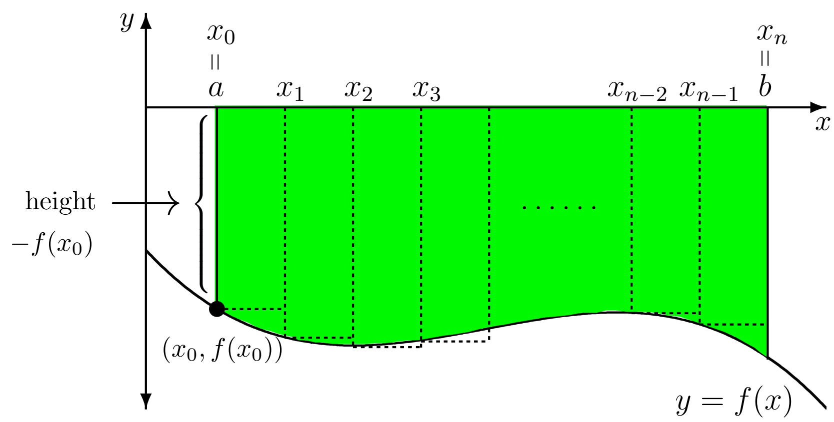
$$f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x \ ,$$

than it is with sums of areas of upper and lower rectangles, as the heights of the rectangles have a clear pattern.

We need to investigate the definite integral further . . . .

Suppose that $f(x)$ is a **negative** continuous function for $a \le x \le b$, and that the interval $[\,a, b\,]$ is divided into n equal parts by the points $x_0, x_1, \ldots, x_n$, with $a = x_0$ and $b = x_n$.

The area A between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ can be estimated using rectangles of heights $-f(x_0)$, $-f(x_1)$, . . . , $-f(x_{n-1})$ on each of the intervals $[\,x_0, x_1\,]$, $[\,x_1, x_2\,]$, . . . , $[\,x_{n-1}, x_n\,]$, as in the diagram below.



The sum of the areas of these rectangles is equal to

$$-f(x_0)\Delta x - f(x_1)\Delta x - f(x_2)\Delta x - \cdots - f(x_{n-1})\Delta x = -\sum_{i=0}^{n-1} f(x_i)\Delta x$$

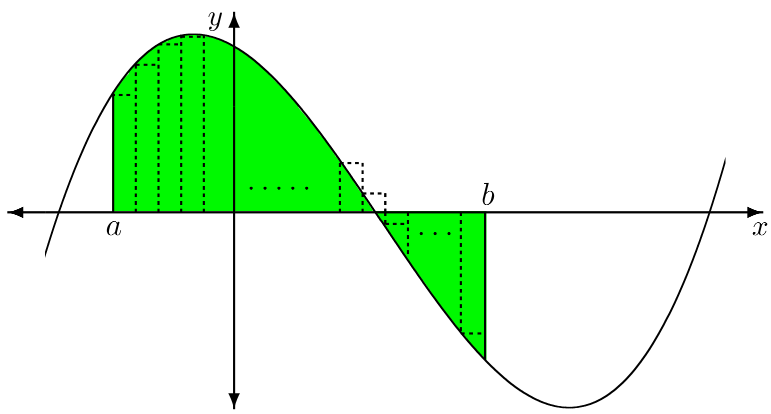
where $\Delta x = \dfrac{b-a}{n}$. As $n \to \infty$, you can see that

> If $f(x)$ is a **negative** continuous function for $a \le x \le b$, the integral
>
> $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$$
>
> is equal to the **negative** of the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$.

... and what if $f(x)$ is a continuous function on $[\,a, b\,]$ with sections above and below the x-axis? How should we interpret the integral $\int_a^b f(x)\,dx$ in this case?

Once more we subdivide the interval $[\,a, b\,]$ into n equal parts using the points $x_0, x_1, \ldots, x_n$, with $a = x_0$ and $b = x_n$, and then consider the limit

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x \;\left(= \int_a^b f(x)\,dx\right).$$



Using the diagram above as a guide, you can see that as $n \to \infty$,

- the area of each rectangle $\to 0$, and
- $\displaystyle\sum_{i=0}^{n-1} f(x_i)\Delta x \to$ (area below curve and above x-axis *minus* area above curve and below x-axis)

In general,

If $f(x)$ is a continuous function for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

is equal to the *difference between*

(a) the sum of the areas under $f(x)$ and above the x-axis and

(b) the sum of the areas above $f(x)$ and below the x-axis

for $a \leq x \leq b$.

Exercise 1.3

1. It is known that

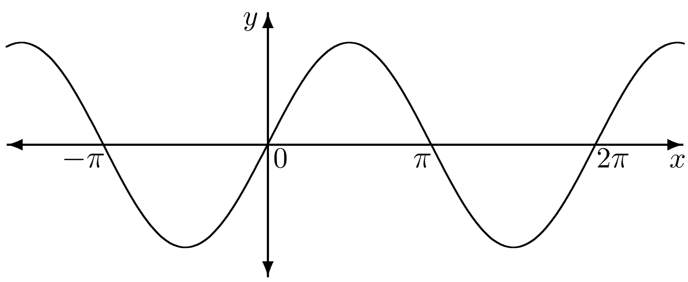
$$\int_0^{\pi/2} \sin(x)\,dx = 1$$

Use the graph of $\sin x$ below to evaluate

(a) $\displaystyle\int_0^{\pi} \sin(x)\,dx$

(b) $\displaystyle\int_{-\pi}^{0} \sin(x)\,dx$

(c) $\displaystyle\int_0^{3\pi/2} \sin(x)\,dx$



2. Draw an appropriate graph and use it to evaluate

$$\int_0^{\pi} (1+\sin(x))\,dx$$

1.4 Properties of the definite integral

1.4.1 Additive Properties

The following properties are useful when evaluating integrals.

Property 1 (Additivity)

If $f(x)$ is continuous on $[\,a, b\,]$ and $a < c < b$, then

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

This property is clearly true when $f(x)$ is a positive continuous function on $[\,a, b\,]$, as the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is equal to the sum of areas from $x = a$ to $x = c$, and from $x = c$ to $x = b$. It can also be confirmed directly when $f(x)$ takes negative values on $[\,a, b\,]$.

Property 2

$$\int_a^a f(x)\,dx \stackrel{\text{def}}{=} 0$$

The original definition of the integral $\int_a^b f(x)\,dx$ assumed that $a < b$. Property 2 is an extension of this definition to the case $a = b$.[vii] *It is intuitively valid as a rectangle with zero width has zero area.*

Property 3

If $f(x)$ is continuous on $[\,a, b\,]$, then

$$\int_b^a f(x)\,dx \stackrel{\text{def}}{=} -\int_a^b f(x)\,dx$$

The original definition of the integral $\int_a^b f(x)\,dx$ assumed that $a < b$. Property 3 is an extension of this definition to the case $a > b$. It is consistent with properties 1 and 2 as

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$$

Observe that properties 2 and 3 show that there is no restriction on the numbers that can be used as upper and lower limits in integrals.

[vii] The symbol $\stackrel{\text{def}}{=}$ means *is defined as*.

1.4.2 Linear Properties

The following properties are used to rewrite integrals of complex functions in terms of integrals of simpler functions.

Property 1

If $f(x)$ is continuous on $[\,a, b\,]$ and k is a constant, then

$$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$

This follows directly from the definition of an integral as

$$\sum_{i=0}^{n-1} k f(x)\Delta x = k\left(\sum_{i=0}^{n-1} f(x)\Delta x\right)$$

Property 2

If $f(x)$ and $g(x)$ are continuous on $[\,a, b\,]$, then

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

This follows directly from the definition of an integral as

$$\sum_{i=0}^{n-1}(f(x)+g(x))\Delta x = \sum_{i=0}^{n-1} f(x)\Delta x + \sum_{i=0}^{n-1} g(x)\Delta x$$

Note: Properties 1 and 2 can be extended to any linear combination[viii] of functions. If

- $f(x)$, $g(x)$, $h(x)$, $\ldots$ are continuous on $[\,a, b\,]$ and
- h, k, l, $\ldots$ are constants,

then

$$\int_a^b (hf(x)+kg(x)+lh(x)+\ldots)\,dx = h\int_a^b f(x)\,dx + k\int_a^b g(x)\,dx + l\int_a^b h(x)\,dx + \ldots$$

[viii] A linear combination of functions is a sum of multiples of the functions. Many mathematical functions are constructed from linear combinations of simpler functions. For example, polynomials are linear combinations of powers.

Exercise 1.4

1. It is known that
$$\int_0^1 x^n\,dx = \frac{1}{n+1}$$
for integers $n \geq 0$. Use this to evaluate

   (a) $\displaystyle\int_0^1 100x^4 dx$

   (b) $\displaystyle\int_0^1 (x^2 + x + 1)\,dx$

   (c) $\displaystyle\int_0^1 (x+2)(x-1)\,dx$

2. If $\displaystyle\int_0^1 f(x)\,dx = a$, $\displaystyle\int_2^3 f(x)\,dx = b$ and $\displaystyle\int_0^3 f(x)\,dx = c$, find
$$\int_1^2 f(x)\,dx.$$

Chapter 2

Integration

2.1 Fundamental Theorem of Calculus

The most important idea in calculus is that it is possible to calculate a definite integral without needing to use limits or to evaluate the area under a curve. This is called the *Fundamental Theorem of Calculus* and was discovered by Newton and Leibniz.[i]

Fundamental Theorem of Calculus[ii]

Let $f(x)$ be a continuous function on the interval $[\,a, b\,]$. If $F(x)$ is a solution of $F'(x) = f(x)$, then

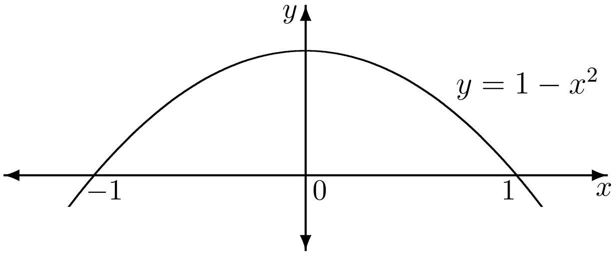
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The difference $F(b) - F(a)$ is written as $[F(x)]_a^b$.

Example

region above x-axis

What is the area of the region enclosed by the parabola $y = 1 - x^2$ and the x-axis from $x = -1$ to $x = 1$?



[i]Sir Isaac Newton (1643 - 1727), Gottfried Wilhelm von Leibniz (1646 - 1716)
[ii]See Appendix B for a justification of the fundamental theorem.

Answer

As $1 - x^2 \geq 0$ when $-1 \leq x \leq 1$, the enclosed area is $\int_{-1}^{1} (1 - x^2)\, dx$.

One solution of $F'(x) = 1 - x^2$ is $F(x) = x - \frac{1}{3}x^3$, so

$$\begin{aligned} \int_{-1}^{1} (1 - x^2)\, dx &= \left[x - \frac{x^3}{3} \right]_{-1}^{1} \\ &= \frac{2}{3} - (-\frac{2}{3}) \\ &= \frac{4}{3} \text{ unit}^2 \end{aligned}$$

Example

region straddles x-axis

What is the area of the region enclosed by the parabola $y = 1 - x^2$ and the x-axis from $x = 0$ to $x = 1.5$?

Answer

The region can be split into two parts:

- $0 \leq x \leq 1$, where $1 - x^2 \geq 0$.
- $1 \leq x \leq 1.5$, where $1 - x^2 \leq 0$.

The area of the region is

$$\int_{0}^{1} (1 - x^2)\, dx - \int_{1}^{1.5} (1 - x^2)\, dx \ .$$

One solution of $F'(x) = 1 - x^2$ is $F(x) = x - \frac{1}{3}x^3$, so

$$\begin{aligned} \int_{0}^{1} (1 - x^2)\, dx - \int_{1}^{1.5} (1 - x^2)\, dx &= \left[x - \frac{x^3}{3} \right]_{0}^{1} - \left[x - \frac{x^3}{3} \right]_{1}^{1.5} \\ &= \left(\frac{2}{3} - 0 \right) - \left(1.5 - \frac{(1.5)^3}{3} - \frac{2}{3} \right) \\ &= 0.958\dot{3} \text{ unit}^2 \end{aligned}$$

Integration was initially described as being concerned with the question:

Given the rate of change of a quantity, how can we find the quantity?

This question is central to the Fundamental Theorem of Calculus. In order to evaluate the definite integral $\int_a^b f(x)\, dx$, we need to answer:

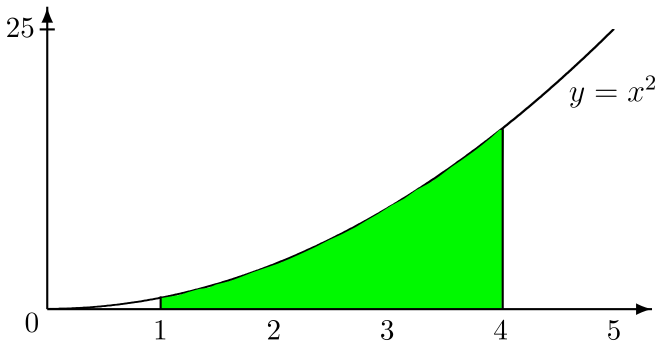
Given the rate of change of a quantity $f(x)$, how do we find the quantity $F(x)$?

This is explored in *Section 2.2*.

Exercise 2.1

1. $F(x) = x - \frac{1}{3}x^3 + 100$ is a solution of $F'(x) = 1 - x^2$.
   Rework the first example using this solution of $F'(x) = 1 - x^2$.

2. What is the area of the region between the parabola $y = x^2$ and the x-axis, bounded by the vertical lines $x = 1$ and $x = 4$?



3. Let $f(x)$ be a continuous function on the interval $[\,a, b\,]$. If

$$G(t) = \int_a^t f(x)\,dx$$

for $a \le t \le b$, show that $G'(t) = f(t)$.

2.2 Integration

2.2.1 Antiderivatives and indefinite integrals

Let $f(x)$ be a given function. In order to use the fundamental theorem of calculus we need to find a function $F(x)$ for which $F'(x) = f(x)$. The function $F(x)$ is called an *antiderivative* of $f(x)$.[iii]

We often want to find the most general solution for $F'(x) = f(x)$, or a *family of functions* whose derivative is $f(x)$. This can sometimes be done by a process of systematic guessing.

Example

systematic guessing

Consider the function $f(x) = x^2$.

We know that the way to get a power of x through differentiation is to differentiate another power of x and, as differentiation reduces the power of x by 1, it is natural to consider $F(x) = x^3$. This is the first guess.

If we differentiate $F(x) = x^3$, then we get $F'(x) = 3x^2$. This gives an x^2, but it is multiplied by 3. If we try $F(x) = \frac{1}{3}x^3$ instead, then will get $F'(x) = x^2$.

A better answer is $F(x) = \frac{1}{3}x^3 + C$, where C is a constant, as the constant differentiates to zero.

We write $F(x) = \frac{1}{3}x^3 + C$ as the antiderivative of $f(x) = x^2$. We can think of it as representing a family of solutions, one for each specific value of C.

It is useful to have a compact notation for an antiderivative.

We use the same notation with the definite integral but without the limits. Instead of saying "the antiderivative of $f(x)$ is $F(x) + C$", we write

$$\int f(x)\,dx = F(x) + C \ .$$

The left side is read aloud as "the integral of $f(x)$ dee x", $f(x)$ is referred to as the *integrand* and C is called *the constant of integration* or *an arbitrary constant.*[iv]

The process of finding an integral is called *integration.*

Integration is typically carried out by systematic guessing and checking the guess using differentiation. This cannot always be done as some ordinary looking functions[v] *have very complex integrals which are impossible to express in terms of common functions let alone guess!*

[iii] *antiderivative* = reverse of differentiation

[iv] The phrase "arbitrary constant" is commonly used to indicate that the constant is yet to be specified and can potentially be given any value.

[v] For example e^{x^2}.

In following example the constants are represented by different letters. This is because they may not all have the same value.

Example

indefinite integral

arbitrary constant

(a) $\displaystyle\int x^2\ dx = \frac{x^3}{3} + C$ Check: $\displaystyle\frac{d}{dx}(\frac{x^3}{3}) = \frac{3x^2}{3} = x^2$

(b) $\displaystyle\int 1 - x^2\ dx = x - \frac{x^3}{3} + D$ Check: $\displaystyle\frac{d}{dx}(x - \frac{x^3}{3}) = 1 - \frac{3x^2}{3} = 1 - x^2$

(c) $\displaystyle\int e^{2t}\ dt = \frac{e^{2t}}{2} + E$ Check: $\displaystyle\frac{d}{dt}(\frac{e^{2t}}{2}) = \frac{2e^{2t}}{2} = e^{2t}$

Notes

1. In (a), $\frac{1}{3}x^3 + 100$ also has derivative x^2. Using this instead of $\frac{1}{3}x^3$ leads to
$$\int x^2\ dx = \frac{1}{3}x^3 + 100 + C\ .$$
However indefinite integrals are traditionally written compactly with a single constant, so the right side should be rewritten as
$$\int x^2\ dx = \frac{1}{3}x^3 + D$$
where $D = 100 + C$. *As C can be any constant, D also can be any constant.*

2. In (c), the integral is taken with respect to the variable t. The variable used in the integrand is the independent variable that the function is expressed in terms of.

Example

definite integral

arbitrary constant

Calculate $\displaystyle\int_0^1 e^{2t}\ dt$

Answer

The function e^{2t} is continuous on $[0, 1]$ and
$$\int e^{2t}\ dt = \frac{e^{2t}}{2} + C$$
By the fundamental theorem
$$\begin{aligned}\int_0^1 e^{2t}\ dt &= \left[\frac{e^{2t}}{2} + C\right]_0^1 \\ &= \left[\frac{e^2}{2} + C\right] - \left[\frac{e^0}{2} + C\right] \\ &= \frac{1}{2}(e^2 - 1)\end{aligned}$$

Notes

1. The example above shows that it doesn't matter which value of the arbitary constant is used when evaluating $[F(x)]_a^b$ as the constant term always cancels out.
2. For simplicity, some texts just use $C = 0$ when evaluating $[F(x)]_a^b$.

Exercise 2.2.1

1. Find the general antiderivative of
   (a) x^3 by differentiating x^4
   (b) $10x^4$ by differentiating x^5
   (c) $7x^2$ by differentiating x^3
   (d) $x^2 + 2x + 1$ by differentiating x^3, x^2 and x
   (e) $4x^{-1/2}$ by differentiating $x^{1/2}$
   (f) $100e^{5t}$ by differentiating e^{5t}
2. Calculate each of the following indefinite integrals.
   (a) $\displaystyle\int x^3 \, dx$ (b) $\displaystyle\int 3x^7 \, dx$
   (c) $\displaystyle\int x^2 + 2x + 3 \, dx$ (d) $\displaystyle\int 4r^{-3} \, dr$
   (e) $\displaystyle\int t^{1/2} \, dt$ (f) $\displaystyle\int w^{-1/2} \, dw$
3. Calculate each of the following definite integrals.
   (a) $\displaystyle\int_0^1 x^2 \, dx$ (b) $\displaystyle\int_1^2 12x^5 \, dx$
   (c) $\displaystyle\int_{-1}^1 x^2 + 2 \, dx$ (d) $\displaystyle\int_{1/2}^1 4u^{-2} \, du$
   (e) $\displaystyle\int_4^{16} 2v^{1/2} \, dv$ (f) $\displaystyle\int_1^2 8w^{-1/2} \, dw$

2.2.2 Methods

(A) Standard Integrals (Part 1)

The standard integrals covered in this topic are:

$f(x)$	$\int f(x)\ dx$	
k	$kx + C$	(k is a constant)
$x^n, n \neq -1$	$\frac{1}{n+1}\, x^{n+1} + C$	
e^x	$e^x + C$	
$\frac{1}{x}$	$\ln\|x\| + C$	

The first three integrals can be checked directly by differentiation (*do check them*). The fourth is less obvious and has this form because logarithms are only defined for positive numbers.

Checking the integral $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + C$:

- *If* $x > 0$, *then* $\ln|x| = \ln x$ *and*
$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$$
- *If* $x < 0$, *then* $\ln|x| = \ln(-x)$ *and*
$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$$

So, for $x \neq 0$,
$$\int \frac{1}{x}\ dx = \ln|x| + C.$$

Example

square root

Calculate $\displaystyle\int x\sqrt{x}\ dx$.

Answer

$$\begin{aligned}
\int x\sqrt{x}\ dx &= \int x^{3/2}\ dx && \leftarrow \textit{write as standard integral} \\
&= \frac{2}{5}x^{5/2} + C && \\
&= \frac{2}{5}x^2\sqrt{x} + C &&
\end{aligned}$$

final answer presented in the same form as the integrand

Example

reciprocal power

Calculate $\displaystyle\int \frac{1}{x^4}\, dx$.

Answer

$$\begin{aligned}
\int \frac{1}{x^4}\, dx &= \int x^{-4}\, dx && \leftarrow \textit{write as standard integral} \\
&= \frac{1}{-3}x^{-3} + C && \\
&= -\frac{1}{3x^3} + C && \left.\right\} \textit{final answer presented in the same form as the integrand}
\end{aligned}$$

Example

logarithm

Calculate $\displaystyle\int_1^2 \frac{1}{x}\, dx$ and $\displaystyle\int_{-2}^{-1} \frac{1}{x}\, dx$

Answer

The function $1/x$ is continuous on $[\,1,2\,]$ and $[\,-2,-1\,]$, and

$$\int \frac{1}{x}\, dt = \ln|x| + C.$$

By the fundamental theorem

$$\begin{aligned}
\int_1^2 \frac{1}{x}\, dt &= \big[\ln|x| + C\big]_1^2 \\
&= (\ln 2 + C) - (\ln 1 + C) \\
&= \ln 2
\end{aligned}$$

and

$$\begin{aligned}
\int_{-2}^{-1} \frac{1}{x}\, dt &= \big[\ln|x| + D\big]_{-2}^{-1} \\
&= (\ln|-1| + D) - (\ln|-2| + D) \\
&= \ln 1 - \ln 2 \\
&= -\ln 2
\end{aligned}$$

Exercise 2.2.2

1. Calculate each of the following integrals.

(a) $\displaystyle\int 10\ dx$ (b) $\displaystyle\int -10\ dx$

(c) $\displaystyle\int \sqrt{x}\ dx$ (d) $\displaystyle\int \frac{1}{\sqrt{x}}\ dx$

(e) $\displaystyle\int r^2\sqrt{r}\ dr$ (f) $\displaystyle\int \frac{1}{s^2\sqrt{s}}\ ds$

(g) $\displaystyle\int t^3\ dt$ (h) $\displaystyle\int \frac{1}{u^3}\ du$

2. Use the Fundamental Theorem of Calculus to calculate the following integrals.

(a) $\displaystyle\int_0^1 e^x\ dx$ (b) $\displaystyle\int_{-1}^0 e^x\ dx$

(c) $\displaystyle\int_1^2 \frac{1}{x}\ dx$ (d) $\displaystyle\int_{-2}^{-1} \frac{1}{x}\ dx$

(e) $\displaystyle\int_0^T e^x\ dx$ (f) $\displaystyle\int_1^S \frac{1}{x}\ dx,\ S > 1$

(B) Linear Combinations

Many mathematical functions are constructed from linear combinations of simpler functions.[vi] For example, polynomials are linear combinations of powers.

We may be able to integrate these functions by systematically guessing the integral (or antiderivative) of each term in the linear combination.

This is called integrating *term-by-term*.

Example

integrating term-by-term

1. If $f(x) = 100x^2$, then
$$\begin{aligned}\int 100x^2\ dx &= 100 \times \frac{1}{3}x^3 + C \\ &= \frac{100}{3}x^3 + C\end{aligned}$$
2. If $f(x) = x^2 + 2x + 3$, then
$$\begin{aligned}\int x^2 + 2x + 3\ dx &= \frac{1}{3}x^3 + 2 \times \frac{1}{2}x^2 + 3 \times x + D \\ &= \frac{1}{3}x^3 + x^2 + 3x + D\end{aligned}$$
3. If $f(x) = (x+3)(x-7)$, then (expanding the brackets first)
$$\begin{aligned}\int (x+3)(x-7)\ dx &= \int x^2 - 4x - 21\ dx \\ &= \frac{1}{3}x^3 - 4 \times \frac{1}{2}x^2 - 21 \times x + E \\ &= \frac{1}{3}x^3 - 2x^2 - 21x + E\end{aligned}$$

We can summarise this method by the following rules:

Rule 1 (multiples)

The integral of a constant multiple is the multiple of the integral.

$$\int cf(x)\ dx = c\int f(x)\ dx$$

Rule 2 (sums of terms)

The integral of a sum of terms is the sum of their integrals.

$$\int f(x) + g(x) + \dots\ dx = \int f(x)\ dx + \int g(x)\ dx + \dots$$

[vi] A linear combination of the functions $f(x), g(x), h(x)\dots$ is sum of multiples of the functions, e.g. $af(x) + bg(x) + ch(x) + \cdots$ for constants $a, b, c\ \dots$ For example, $x^2 - 4x - 21$ is a linear combination of x^2, x and 1 with constants $1, -4$ and -21. Its terms are $x^2, -4x$ and -21.

Exercise 2.2.2

3. Calculate each of the following integrals.

   (a) $\displaystyle\int x^2 + 4x + 8 \; dx$

   (b) $\displaystyle\int 25 - 16x^3 \; dx$

   (c) $\displaystyle\int \sqrt{x} - \frac{4}{\sqrt{x}} \; dx$

   (d) $\displaystyle\int x - \frac{2}{x} \; dx$

   (e) $\displaystyle\int (t+1)(t+2) \; dt$

   (f) $\displaystyle\int (2-u)^2 \; du$

   (g) $\displaystyle\int \frac{(v+1)^2}{v} \; dv$

   (h) $\displaystyle\int \frac{e^w - e^{-w}}{2} \; dw$

4. What is the area of the region enclosed by $y = (x+1)(x-3)$ and the x-axis?

(C) Standard Integrals (Part 2)

The standard integrals introduced in (A) can be extended to include:

$f(x)$	$\int f(x)\ dx$
$(ax+b)^n, n \neq -1$	$\dfrac{1}{a(n+1)}(ax+b)^{n+1} + C$
e^{ax+b}	$\dfrac{1}{a}\, e^{ax+b} + C$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a} \ln \lvert ax+b \rvert + C$

for constants $a \neq 0$ and b.

Each integral in the table can be checked directly by differentiating, using the chain rule. (*Do this.*)

Example

square root

Calculate $\displaystyle\int 12\sqrt{2x+3}\ dx$.

Answer

$$\begin{aligned} \int 12\sqrt{2x+3}\ dx &= 12\int (2x+3)^{1/2}\ dx \\ &= 12 \times \frac{1}{2 \times \frac{3}{2}}(2x+3)^{3/2} + C \qquad \leftarrow \textit{check here}^{\text{vii}} \\ &= 4(2x+3)\sqrt{2x+3} + C \end{aligned}$$

Example

logarithm

Calculate $\displaystyle\int \frac{2}{3x+5}\ dx$.

Answer

$$\begin{aligned} \int \frac{2}{3x+5}\ dx &= 2\int \frac{1}{3x+5}\ dx \\ &= 2 \times \frac{1}{3} \ln|3x+5| + C \qquad \leftarrow \textit{check here} \\ &= \frac{2}{3} \ln|3x+5| + C \end{aligned}$$

[vii]See the note on arbitary constants on page 21.

Some functions may need to be rewritten as a function in the form $f(ax+b)$ before they can be integrated.

Example

square root

Calculate $\displaystyle\int 12x\sqrt{2x+3}\ dx$.

Answer

As

$$\begin{aligned} 12x\sqrt{2x+3} &= (12x+18-18)\sqrt{2x+3} \\ &= 6(2x+3)\sqrt{2x+3} - 18\sqrt{2x+3} \end{aligned}$$

we have ...

$$\begin{aligned} \int 12x\sqrt{2x+3}\ dx &= \int 6(2x+3)^{3/2} - 18(2x+3)^{1/2}\ dx \\ &= 6\times\frac{1}{2\times\frac{5}{2}}(2x+3)^{5/2} - 18\times\frac{1}{2\times\frac{3}{2}}(2x+3)^{3/2} + C \\ &= \frac{6}{5}(2x+3)^{5/2} - 6(2x+3)^{3/2} + C \\ &= \left(\frac{6}{5}(2x+3) - 6\right)(2x+3)^{3/2} + C \\ &= \frac{6}{5}\left((2x+3)-5\right)(2x+3)^{3/2} + C \\ &= \frac{12}{5}(x-1)(2x+3)\sqrt{2x+3} + C \end{aligned}$$

Exercise 2.2.2

5. Calculate the following integrals.

   (a) $\int (3x+1)^{11}\,dx$

   (b) $\int 16(1-2x)^3\,dx$

   (c) $\int 2\sqrt{x+5}\,dx$

   (d) $\int \frac{2}{\sqrt{p+5}}\,dp$

   (e) $\int \frac{12}{3q+7}\,dq$

   (f) $\int \frac{e^{w/2}-e^{-w/2}}{2}\,dw$

6. Rewrite each integrand in an appropriate form and then calculate the integral.

   (a) $\int 3x(3x+1)^{11}\,dx$

   (b) $\int 16x(1-2x)^3\,dx$

   (c) $\int 2x\sqrt{x+5}\,dx$

   (d) $\int \frac{12q}{3q+7}\,dq$

(D) Composite Functions

The standard integrals on page 28 were obtained by applying the chain rule to simple functions of the form $f(ax + b)$.

We can extend our integrating skills by making use of the general chain rule for composite functions:

The derivative of a composite function is the derivative of the outside function multiplied by the derivative of the inside function.

In symbols ...

> *The Chain Rule*
>
> If $f(u)$ and $g(x)$ are given functions, then
>
> $$F(x) = f(g(x)) \Longrightarrow F'(x) = f'(g(x))g'(x)$$

... giving the indefinite integral:

$$\int f'(g(x))g'(x)\ dx = f(g(x)) + C$$

Integration involves systematic guessing followed by checking using differentiation. The most efficient way to decide if an integral has the form

$$\int f'(g(x))g'(x)\ dx$$

is to (a) guess which function is the inside function $g(x)$

(b) confirm the presence of $g'(x)$

(c) verify that the integrand has the form $f'(g(x))g'(x)$.

Example

first $g(x)$
then $g'(x)$

Calculate $\int x\sqrt{x^2+1}\ dx$.

Answer

If $g(x) = x^2 + 1$, then $g'(x) = 2x$. The integrand has x as a factor rather than $g'(x) = 2x$, but this shouldn't be a problem as $x = \frac{1}{2} \times 2x$.

$$\begin{aligned}
\int x\sqrt{x^2+1}\ dx &= \frac{1}{2}\int \sqrt{x^2+1}\ 2x\ dx \\
&= \frac{1}{2} \times \frac{1}{3/2}(x^2+1)^{3/2} + C \\
&= \frac{1}{3}(x^2+1)\sqrt{x^2+1} + C
\end{aligned}$$

Example

first $g(x)$
then $g'(x)$

Calculate $\displaystyle\int \frac{x^3}{x^4+10}\,dx$.

Answer

If $g(x) = x^4 + 10$, then $g'(x) = 4x^3$. The integrand has x^3 as a factor rather than $g'(x) = 4x^3$, but this isn't a problem as it is a constant multiple of $g'(x)$.

$$\begin{aligned}\int \frac{x^3}{x^4+10}\,dx &= \frac{1}{4}\int \frac{1}{x^4+10}\,4x^3\,dx \\ &= \frac{1}{4}\times \ln|x^4+10| + C\end{aligned}$$

It was difficult to jump from

$$\int \frac{1}{x^4+10}\,4x^3 \quad \textit{to} \quad \ln|x^4+10| + C$$

in this example. The substitution method below makes this easier.

The *substitution* or *change of variable* method makes the integral

$$\int f'(g(x))g'(x)\,dx$$

easier and more straightforward to calculate.

The idea is to simplify the integral by using the new variable u instead of x, where $u = g(x)$.[viii]

This is done by replacing

- $g(x)$ by u ... as $u = g(x)$
- $g'(x)\,dx$ by du ... as $\dfrac{du}{dx} = g'(x)$ [ix]

When this is done

$$\int f'(g(x))g'(x)\,dx \quad \text{is transformed to} \quad \int f'(u)\,du$$

with integral $f(u) + C = f(g(x)) + C$.

Observe the difference when the substitution method is applied to the previous two examples (next page).

[viii] Any letter can be used to represent a new variable, not just u.

[ix] While it doesn't make sense to separate the top and bottom parts of the symbol $\dfrac{du}{dx}$, the procedure always leads to a correct outcome. It's best to think of this as working with the notation in a suggestive way.

Example

first guess $g(x)$

Calculate $\int x\sqrt{x^2+1}\ dx$.

Answer

If $g(x) = x^2 + 1$, then $g'(x) = 2x$. The integrand has x as a factor rather than $g'(x) = 2x$, but this shouldn't be a problem as $x = \frac{1}{2} \times 2x$.

Put $u = x^2 + 1$, then $du = 2x\ dx$, and

$$\begin{aligned}\int x\sqrt{x^2+1}\ dx &= \frac{1}{2}\int \sqrt{x^2+1}\ 2x\ dx \\ &= \frac{1}{2}\int \sqrt{u}\ du \\ &= \frac{1}{2} \times \frac{1}{3/2} u^{3/2} + C \\ &= \frac{1}{3}\, u\sqrt{u} + C\end{aligned}$$

Rewriting the answer in terms of x gives

$$\int x\sqrt{x^2+1}\ dx = \frac{1}{3}(x^2+1)\sqrt{x^2+1} + C$$

Example

first $g(x)$ then $g'(x)$

Calculate $\int \frac{x^3}{x^4+10}\ dx$.

Answer

If $g(x) = x^4 + 10$, then $g'(x) = 4x^3$. The integrand has x^3 as a factor rather than $g'(x) = 4x^3$, but this isn't a problem as it is a constant multiple of $g'(x)$.

Put $u = x^4 + 10$, then $du = 4x^3\ dx$, and

$$\begin{aligned}\int \frac{x^3}{x^4+10}\ dx &= \frac{1}{4}\int \frac{1}{x^4+10}\ 4x^3\ dx \\ &= \frac{1}{4}\int \frac{1}{u}\ du \\ &= \frac{1}{4}\ \ln|u| + C\end{aligned}$$

Rewriting the answer in terms of x gives

$$\int \frac{x^3}{x^4+10}\ dx = \frac{1}{4}\ \ln|x^4+10| + C$$

Exercise 2.2.2

7. Calculate each of the following integrals.

(a) $\displaystyle\int 2x(x^2+1)^3\,dx$ (b) $\displaystyle\int (2x+4)(x^2+4x)^4\,dx$

(c) $\displaystyle\int (x^2+4)(x^3+12x+1)^5\,dx$ (d) $\displaystyle\int 2x\sqrt{x^2+1}\,dx$

(e) $\displaystyle\int (2t+1)\sqrt{t^2+t}\,dt$ (f) $\displaystyle\int \frac{u^3+1}{\sqrt{u^4+4u}}\,du$

(g) $\displaystyle\int \frac{2v}{(v^2+1)^3}\,dv$ (h) $\displaystyle\int \frac{w+2}{(w^2+4w+3)^5}\,dw$

8. Calculate each of the following integrals.

(a) $\displaystyle\int 2x\,e^{x^2-2}\,dx$ (b) $\displaystyle\int x^2\,e^{x^3+10}\,dx$

(c) $\displaystyle\int (x+1)e^{x^2+2x-3}\,dx$ (d) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

(e) $\displaystyle\int e^t(e^t+10)^5\,dt$ (f) $\displaystyle\int e^{2u}\sqrt{e^{2u}+1}\,du$

(g) $\displaystyle\int \frac{10e^v}{(e^v+1)^3}\,dv$ (h) $\displaystyle\int \frac{e^{2w}+2e^w}{(e^{2w}+4e^w+3)^5}\,dw$

9. Calculate each of the following integrals.

(a) $\displaystyle\int \frac{3x^2}{x^3+1}\,dx$ (b) $\displaystyle\int \frac{2x+4}{x^2+4x}\,dx$

(c) $\displaystyle\int \frac{x^2+4}{x^3+12x+1}\,dx$ (d) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

(e) $\displaystyle\int \frac{e^{2t}+3e^t}{e^{2t}+6e^t+10}\,dt$ (f) $\displaystyle\int \frac{e^u-e^{-u}}{e^u+e^{-u}}\,du$

(E) Numerical Methods (optional)

There are many antiderivatives that can not be found explicitly, for example

$$\int e^{x^2}\ dx.$$

In these cases we use numerical methods to estimate their definite integrals. The method of using upper and lower methods to estimate an integral is one such method, but there are other methods that provide closer closer approximations and are easier to program.

See

http://en.wikipedia.org/wiki/Numerical_integration

http://en.wikibooks.org/wiki/Numerical_Methods/Numerical_Integration

and

http://people.hofstra.edu/Stefan_Waner/realworld/integral/numint.html

http://people.hofstra.edu/Stefan_waner/realworld/integral/integral.html

[All pages accessed 18/3/08]

2.2.3 Selected Applications

Integration has widespread applications in science and engineering including :

- calculation of areas
- calculation of volumes
- calculation of centres of mass of solids
- calculation of the work done by a force
- problems concerned with rate of change
- problems in economics
- statistical problems

This topic only considers problems associated with areas and change in quantities over time.

(I) Calculation of the area between two curves

We have seen in section 1.3 that if $f(x)$ is a continuous function for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx$$

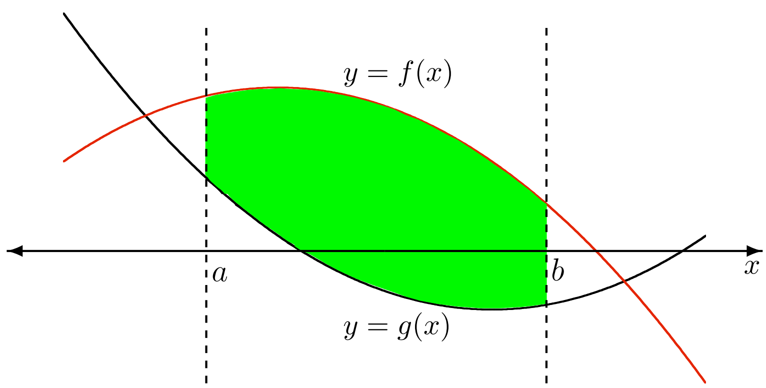
is equal to the the difference between

- the sum of the areas under $f(x)$ and above the x-axis
- the sum of the areas above $f(x)$ and below the x-axis

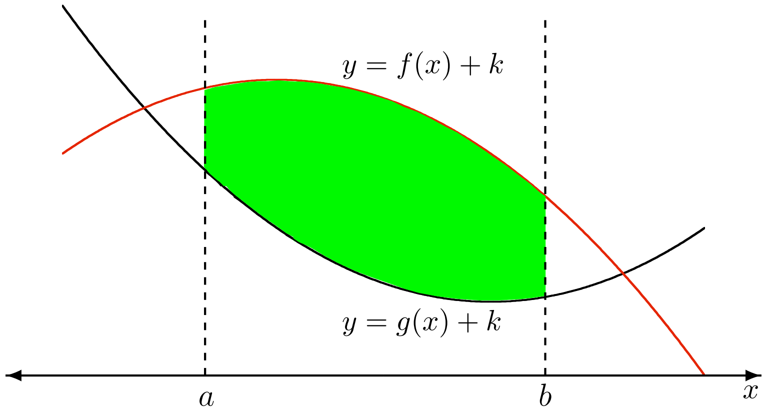
for $a \leq x \leq b$.

Let $f(x)$ and $g(x)$ be positive continuous functions for $a \leq x \leq b$. Suppose also that $f(x) \geq g(x)$ on $[a, b]$. This is shown in the diagram below.

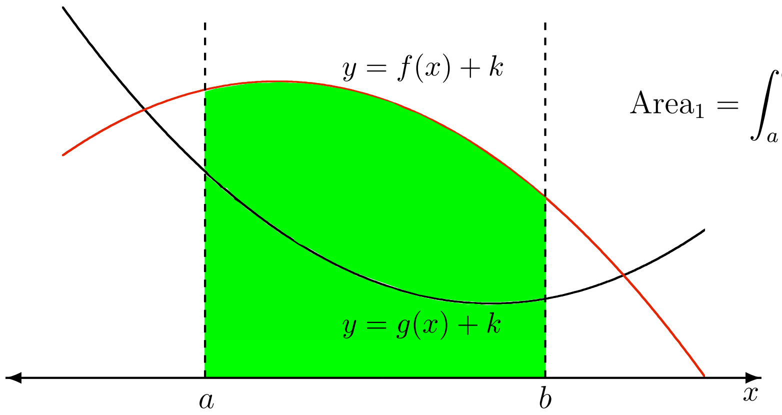
What is the area of the shaded region between $f(x)$ and $g(x)$ that is bounded by the lines $x = a$ and $x = b$?



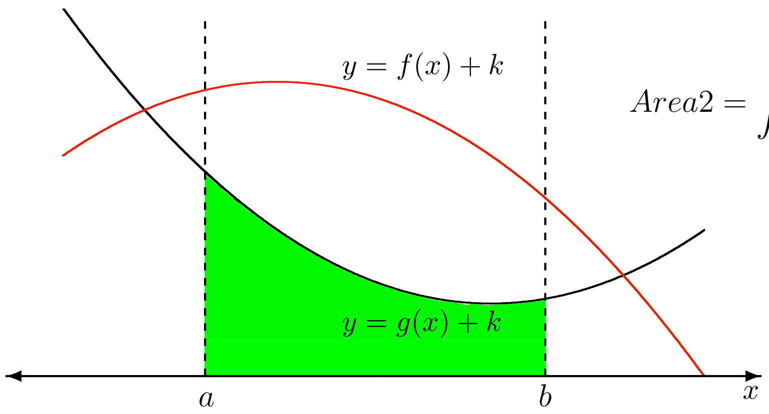
The area remains the same if each curve is translated vertically by the same distance k, with k taken so that both curves are above the x-axis.



You can now be seen that the area of the shaded region is the difference between the areas below:



$$\text{Area}_1 = \int_a^b (f(x) + k)\ dx$$



$$Area2 = \int_a^b (g(x) + k)\ dx$$

...and that the difference in areas is

$$A = \int_a^b (f(x) + k) - (g(x) + k)\ dx = \int_a^b f(x) - g(x)\ dx$$

In general,

If $f(x)$ and $g(x)$ are continuous functions for $a \leq x \leq b$, then

$$\int_a^b f(x) - g(x)\, dx$$

is equal to the the difference between

(a) the area under $f(x)$ and above $g(x)$ when $f(x) \geq g(x)$

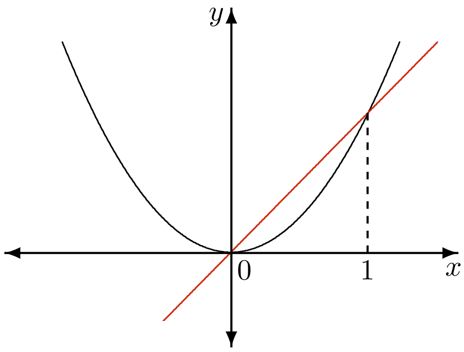
(b) the area under $g(x)$ and above $f(x)$ when $g(x) \geq f(x)$

for $a \leq x \leq b$.

Example

area between curves

What is the area of the region enclosed by the parabola $y = x^2$ and the line $y = x$?



Answer

The parabola and line intersect when

$$\begin{aligned} x^2 &= x \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ x &= 0\ \&\ 1 \end{aligned}$$

The enclosed area is

$$\begin{aligned}
\int_0^1 x - x^2 \ dx &= \left[\frac{x^2}{2} - \frac{x^3}{3} + C\right]_0^1 \\
&= \left[\frac{1}{2} - \frac{1}{3}\right] - 0 \\
&= \frac{1}{6}
\end{aligned}$$

Exercise 2.2.3

1. Calculate the area between the curves below over the given interval

   (a) $f(x) = x^2 - 4$ and $g(x) = -x^2 + 9$ $\quad -1 \le x \le 1$

   (b) $f(x) = x^2$ and $g(x) = x^3$ $\quad 0 \le x \le 1$

   (c) $f(x) = 2x$ and $g(x) = -x^2 + 3$ $\quad -3 \le x \le 1$

   (e) $f(x) = e^x$ and $g(x) = x$ $\quad 0 \le x \le 3$

2. Calculate the area enclosed by the curves below

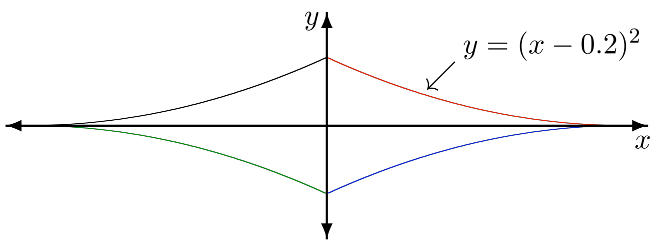
   (a) $f(x) = x$ and $g(x) = x^2$

   (b) $f(x) = \sqrt{x}$ and $g(x) = x^2$

   (c) $f(x) = x^4$ and $g(x) = 3x^2$

   (d) $f(x) = x^4$ and $g(x) = -2x^2 + 3$

3. The cable used to construct a new arts centre will have the cross-section shown below.



   (a) Express the cross-sectional area as an integral.

   (b) Calculate the area.

   (c) If 300 m of cable are requied, calculate the volume of the cable.

4. Consider the graphs of the form $y = x^n$ for positive integers n.

   (a) At what points do these graphs intersect?
   (b) Does the area between the graphs and the x-axis, from $x = 0$ to $x = 1$ increase or decrease as n increases?
   (c) What happens if the interval in (b) was changed to $[0, 2]$?
   (d) What happens if n is allowed to be a negative integer?

(II) Change in Quantities

We discovered in section 1.3 that if the rate of change of a quantity is known *and is positive*, then the net change in the quantity is equal to the area under the graph of the rate of change and above the horizontal axis.

The Fundamental Theorem in section 2.1 extends this discovery:

Net Change

If the rate of change $f(x)$ with respect to x is known for a quantity $F(x)$, then the net change in the quantity from $x = a$ to $x = b$ is

$$F(b) - F(a) = \int_a^b f(x)\ dx$$

Example

total change

The maintenance costs $M(x)$ for a building increase with age x years. Records for a certain building show that the rate of increase in costs is approximately

$$\frac{dM}{dx} = 60x^2 + 400 \text{ dollars/year.}$$

What is the total maintenance cost for

(a) the first 5 years?

(b) the first t years?

Answer

(a) The total cost for the first 5 years is

$$\begin{aligned} \int_0^5 60x^2 + 400\ dx &= \left[20x^3 + 400x + C\right]_0^5 \\ &= \left[20 \times 5^3 + 400 \times 5\right] - 0 \\ &= \$\ 4500 \end{aligned}$$

(b) The total cost for the first t years is

$$\begin{aligned} \int_0^t 60x^2 + 400\ dx &= \left[20x^3 + 400x + D\right]_0^t \\ &= 20t^3 + 400t \text{ dollars} \end{aligned}$$

Example

definite integral vs indefinite integral

The *marginal cost*[x] of manufacturing x radios per week is

$$\frac{dC}{dx} = 25 - 0.1x \text{ dollars/radio}$$

when $0 \leq x \leq 200$. The fixed costs per week before production begins are \$ 1000.[xi]

What is the total cost of producing 100 radios per week?

Answer (definite integral)

As

$$\text{Total Cost} = \text{Fixed Costs} + \text{Cost of Production}$$

we have

$$\begin{aligned}
\text{Total cost (100 radios)} &= 1000 + \int_0^{100} 25 - 0.1x \; dx \\
&= 1000 + \left[25x - 0.05x^2 + C\right]_0^{100} \\
&= 1000 + \left[25 \times 100 - 0.05 \times 100^2\right] - 0 \\
&= \$\ 3000
\end{aligned}$$

Answer (indefinite integral)

As the marginal cost is $\dfrac{dC}{dx} = 25 - 0.1x$, the cost of production is

$$\begin{aligned}
C(x) &= \int 25 - 0.1x \; dx \\
&= 25x - 0.05x^2 + C
\end{aligned}$$

for some constant C and, as $C(0) = 0 \Rightarrow C = 0$,

$$= 25x - 0.05x^2$$

So

$$\begin{aligned}
\text{Total cost (100 radios)} &= 1000 + C(100) \\
&= 1000 + (25 \times 100 - 0.05 \times 100^2) \\
&= \$\ 3000
\end{aligned}$$

[x] The *marginal cost* of production is the *rate of change of cost of production relative to output.*

[xi] The *fixed costs* are necessary costs that are independent of the number of radios produced. They might include lease, wages, insurance, etc.

When an object travels along a straight line, its *displacement* $s(t)$ is its position from the origin. A positive displacement corresponds to a position on the right of the origin, and a negative displacement corresponds to a position on the left of the origin. If the object moves 1m away from the origin then returns to the origin its displacement will be zero even though the distance travelled is 2m.

Velocity $v(t)$ is the rate of change of *displacement*, $s'(t)$. An object travelling in a positive direction has a positive velocity. When it returns towards the origin its velocity is negative.

Example

velocity
displacement
distance

A object P moves in a straight line with velocity

$$v(t) = \frac{ds}{dt} = 2 - 2t \text{ m/s}.$$

for $t \geq 0$ seconds. What is

(a) its change in displacement between $t = 0$ and $t = 1$?

(b) the distance it travelled between $t = 0$ and $t = 1$?

(c) its change in displacement between $t = 0$ and $t = 2$?

(d) the distance it travelled between $t = 0$ and $t = 2$?

Answer

(a) The change in displacement between $t = 0$ and $t = 1$ is

$$\begin{aligned} s(1) - s(0) &= \int_0^1 2 - 2t \; dt \\ &= \left[2t - t^2 + C\right]_0^1 \\ &= 1 \text{ m} \end{aligned}$$

(b) As $v(t) \geq 0$ for $0 \leq t \leq 1$,

$$\begin{aligned} \text{distance travelled} &= \int_0^1 2 - 2t \; dt \\ &= \left[2t - t^2 + C\right]_0^1 \\ &= 1 \text{ m} \end{aligned}$$

(c) The change in displacement between $t = 0$ and $t = 2$ is

$$\begin{aligned} s(2) - s(0) &= \int_0^2 2 - 2t \; dt \\ &= \left[2t - t^2 + C\right]_0^2 \\ &= 0 \text{ m} \end{aligned}$$

(d) As $v(t) \geq 0$ for $0 \leq t \leq 1$, and $v(t) \leq 0$ for $1 \leq t \leq 2$

$$\begin{aligned}
\text{distance travelled} &= \int_0^1 2 - 2t \; dt - \int_1^2 2 - 2t \; dt \\
&= \left[2t - t^2 + C\right]_0^1 - \left[2t - t^2 + D\right]_1^2 \\
&= 1 - (-1) \\
&= 2 \text{ m}
\end{aligned}$$

Exercise 2.2.3

5. The marginal cost of manufacturing and storing x cardboard cartons per week is
$$C'(x) = 3.2 + 0.0004x \text{ dollars per item,}$$
and the fixed costs are \$500 per week.

   What is the total cost of manufacturing and storing 1000 cartons per week?

6. A new suburb is estimated to grow at the rate of
$$\frac{dP}{dt} = 1500 + 300\sqrt{t}$$
people per year. If the current population is 1000, estimate
   (a) the population in 25 years
   (b) the population $P(t)$ after t years

7. The area $A(t)$ covered by an ulcer changes at a rate
$$\frac{dA}{dt} = -\frac{4}{(t+1)^2} \text{ cm}^2/\text{day}$$
as it heals, where t is in days. If the area is initially $A(0) = 4$ cm^2, what is
   (a) the area of the wound in 10 days
   (b) the area $A(t)$ sfter t days

8. A object P moves in a straight line with velocity
$$v(t) = \frac{ds}{dt} = 2 - t^2 \text{ m/s.}$$
for $t \geq 0$ seconds. What is the
   (a) change in displacement between $t = 0$ and $t = 1$?
   (b) distance it travelled between $t = 0$ and $t = 1$?
   (c) change in displacement between $t = 0$ and $t = 2$?
   (d) distance it travelled between $t = 0$ and $t = 2$?

9. A oil tanker hit a reef and is producing a circular oil slick that is expanding at an approximate rate of
$$\frac{dr}{dt} = \frac{20}{\sqrt{t+1}}$$
where r metres is the radius of the slick after t minutes. If $r(0) = 20$, estimate
   (a) the radius of the slick after 30 minutes
   (b) the radius of the slick after t minutes

(c) the rate at which the area of the slick is changing

10. A circular plastic tube, with internal diameter 4 cm and external diameter 4.5 cm, carries water at a constant temperature of $70°C$. The temperature inside the tube drops off at a rate of

$$\frac{dT}{dx} = -\frac{5}{x}$$

where x is the distance from the centre of the tube and $4 \leq x \leq 4.5$. What is the temperature on the outside of the tube?

Appendix A

Summation Notation

Summation notation[i] is used in situations where we need to write down the sum of many numbers or terms. We could write *the sum of the squares of the numbers from 1 to 100* as

$$1^2 + 2^2 + 3^2 + \ \dots \ + 100^2 \,,$$

leaving it to the reader to guess the pattern of numbers, but summation notation can be used to express this sum concisely and without ambiguity as

$$\sum_{i=1}^{100} i^2 \ .$$

If $f(i)$ represents an expression involving i, then

$$\sum_{i=1}^{n} f(i)$$

has the following meaning:

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + f(3) + \ \dots \ + f(n) \ .$$

This notation has a number of parts:

end

summation sign $\quad \sum_{i=1}^{n} f(i) \quad$ general term

index start

[i]Sometimes *sigma notation* because the Greek letter sigma Σ is used.

- The summation sign Σ

  Σ is the Greek upper case letter corresponding to "S". It tells us to add the terms in a sum, where the general term is given to the right of the summation sign.

- The index variable i

  This variable is used to number or label each term in the sum. The index is often represented by i. Other common possibilities include j and k.

- The start

  The "$i =$" part beneath the summation sign tells shows which index number to begin with. It is usually either zero or one but it can be anything. *You always increase the index by one at each successive step in the sum.*

- The end

  The number on top of the summation sign is the final index number.

Example

sum of indexed terms

(a) $\sum_{i=0}^{50} i^2 = 0^2 + 1^2 + 2^2 + \ \ldots \ + 50^2$

(b) $\sum_{j=1}^{50} j^2 = 1^2 + 2^2 + 3^2 + \ \ldots \ + 50^2$

(c) $\sum_{k=3}^{15} \frac{1}{k} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ \ldots \ + \frac{1}{15}$

(d) $\sum_{t=0}^{8} (-1)^i \frac{2}{1+i} = \frac{2}{1+0} - \frac{2}{1+1} + \frac{2}{1+2} - \ \ldots \ + \frac{2}{1+8}$

(e) $\sum_{t=1}^{n} 3\sqrt{t} = 3\sqrt{1} + 3\sqrt{2} + 3\sqrt{3} + \ \ldots \ + 3\sqrt{n}$

Appendix B

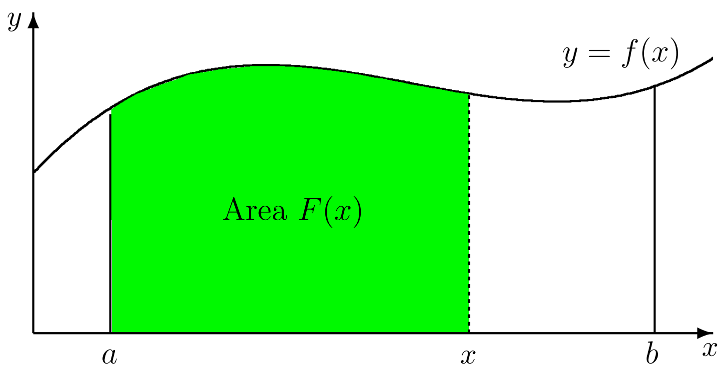
Justification for the Fundamental Theorem

Let $f(x)$ be a positive continuous function on the interval $[\,a, b\,]$.

The area between $f(x)$ and the x-axis from $x = a$ to $x = b$ is equal to the definite integral

$$\int_a^b f(x)\ dx.$$

Define the *area function* $F(x)$ for $a \leq x \leq b$ to be the area between $f(x)$ and the x-axis on the interval $[\,0, x\,]$ as shown in the diagram below.



The area between $f(x)$ and the x-axis from $x = a$ to $x = b$ is equal to $F(b) - F(a)$ so

$$\int_a^b f(x)\ dx = F(b) - F(a)\ .$$

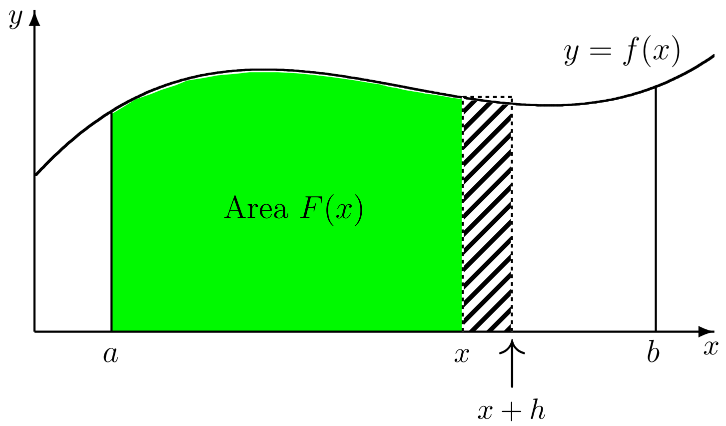
We now show that $F(x)$ is a solution of the equation

$$F'(x) = f(x)\ .$$

The area between $f(x)$ and the x-axis on the interval from 0 to $x+h$ is equal to $F(x+h)$.

It can be seen from the diagram below that when h is very small this area is closely approximated by adding

- the area between $f(x)$ and the x-axis from 0 to x
- the area of the rectangle with height $f(x)$ and base the interval $[x, x+h]$.



In other words

$$F(x+h) \approx F(x) + hf(x)$$

and

$$\frac{F(x+h) - F(x)}{h} \approx f(x)\ .$$

We know that as h becomes very small,

$$\frac{F(x+h) - F(x)}{h} \to F'(x)$$

which implies

$$F'(x) = f(x)\ .$$

So

$$\int_a^b f(x)\ dx = F(b) - F(a)\ ,$$

where $F(x)$ is a solution of the equation

$$F'(x) = f(x)\ .$$

The final step in the justification is to show that this is true for *any* function that satisfies $F'(x) = f(x)$ ...

If $G(x)$ is another function with $G'(x) = f(x)$, then

$$\frac{d}{dx}(G(x) - F(x)) = G'(x) - F'(x) = 0 \ .$$

This implies that $G(x) - F(x) = k$ for some constant k.

As

$$G(b) - G(a) = (F(b) + k) - (F(a) + k) = F(b) - F(a)$$

you can see that

$$\int_a^b f(x)\ dx = F(b) - F(a) = G(b) - G(a) \ ,$$

... so it doesn't matter which solution of $F'(x) = f(x)$ is chosen.

Appendix C

Answers

Exercise 1.1

1(a) Horizontal line with vertical intercept $(0, 1000)$.

1(b) From graph, $V(t) = 10t$.

1(c) Line with gradient 10 and vertical intercept $(0, 100)$.

1(d) Rate of change of volume per minute.

2(a) Straight line with intercepts $(60, 0)$ and $(0, 30)$.

2(b) 900 m.

2(c) $30t - 0.25t^2$

3(iii) Upper estimate = 0.76, Lower estimate = 0.56

Exercise 1.2

1(a) Upper estimate = 0.875 , Lower estimate = 0.375.

1(b) Ten equal intervals of width 0.1.

2(a) Upper estimate = 1.75 , Lower estimate = 0.75.

Exercise 1.3

1(i) 2   1(ii) -2 1(iii) 1

2. The graph of $1 + \sin(x)$ is obtained by translating the graph of $\sin(x)$ vertically by one unit. The area is $2 + 1 \times \pi = 2 + \pi$

Exercise 1.4

1(i) 20   1(ii) $\frac{11}{6}$ 1(iii) $-\frac{7}{6}$

2(b) $c - a - b$

Exercise 2.1

2. 21

Exercise 2.2.1

1(a) $\frac{1}{4}x^4 + C$ 1(b) $2x^5 + D$ 1(c) $\frac{7}{3}x^3 + E$

1(d) $\frac{1}{3}x^3 + x^2 + x + F$ 1(e) $8x^{1/2} + G$ 1(f) $20e^{5t} + H$

2(a) $\frac{1}{4}x^4 + C$ 2(b) $\frac{3}{8}x^8 + D$ 2(c) $\frac{1}{3}x^3 + x^2 + 3x + E$

2(d) $-2r^{-2} + F$ 2(e) $\frac{2}{3}t^{3/2} + G$ 2(f) $2w^{1/2} + H$

3(a) $\frac{1}{3}$ 3(b) 126   3(c) $\frac{14}{3}$

3(d) 4   3(e) $\frac{224}{3}$ 3(f) $16(\sqrt{2} - 1)$

Exercise 2.2.2

1(a) $10x + C$ 1(b) $-10x + D$ 1(c) $\frac{2}{3}x\sqrt{x} + E$

1(d) $2\sqrt{x} + F$ 1(e) $\frac{2}{7}r^3\sqrt{r} + G$ 1(f) $-\frac{2}{3s\sqrt{s}} + H$

1(g) $\frac{1}{4}t^4 + I$ 1(e) $-\frac{1}{2}u^{-2} + J$

2(a) $e - 1$ 2(b) $1 - e^{-1}$ 2(c) $\ln 2$

2(d) $-\ln 2$ 2(e) $e^T - 1$ 2(f) $\ln S$

3(a) $\frac{1}{3}x^3 + 2x^2 + 8x + C$ 3(b) $25x - 4x^4 + D$

3(c) $\frac{2}{3}x\sqrt{x} - 8\sqrt{x} + E$ 3(d) $\frac{1}{2}x^2 - 2\ln|x| + F$

3(e) $\frac{1}{3}t^3 + \frac{3}{2}t^2 + 2t + G$ 3(f) $4u - 2u^2 + \frac{1}{3}u^3 + H$

3(g) $\frac{1}{2}v^2 + 2v + \ln v + I$ 3(h) $\frac{e^w + e^{-w}}{2} + J$

4. $\dfrac{32}{3}$

5(a) $\dfrac{1}{36}(3x+1)^{12}+C$ 5(b) $-2(1-2x)^4$+D

5(c) $\dfrac{4}{3}(x+5)\sqrt{x+5}+E$ 5(d) $4\sqrt{p+2}+F$

5(e) $4\ln|3q+7|+G$ 5(f) $e^{w/2}+e^{-w/2}+H$

6(a) $\displaystyle\int (3x+1)^{12}-(3x+1)^{11}\ dx = \frac{1}{39}(3x+1)^{13}-\frac{1}{36}(3x+1)^{12}+C$

6(b) $\displaystyle\int -8(1-2x)^4+8(1-2x)^3\ dx = \frac{4}{5}(1-2x)^5-(1-2x)^4+D$

6(c) $\displaystyle\int 2(x+5)^{3/2}-10(x+5)^{1/2}\ dx \Rightarrow \frac{4}{5}(x+5)^2\sqrt{x+5}-\frac{20}{3}(x+5)\sqrt{x+5}+E$

6(d) $\displaystyle\int 4-\frac{28}{3q+7}\ dq = 4q-\frac{28}{3}\ln|3q+7|+F$

7(a) $\dfrac{1}{4}(x^2+1)^4+C$ 7(b) $\dfrac{1}{5}(x^2+4x)^5+D$

7(c) $\dfrac{1}{18}(x^3+12x+1)^6+E$ 7(d) $\dfrac{2}{3}(x^2+1)\sqrt{x^2+1}+F$

7(e) $\dfrac{2}{3}(t^2+t)\sqrt{t^2+t}+G$ 7(f) $\dfrac{1}{2}\sqrt{u^4+4u}+H$

7(g) $-\dfrac{1}{2(v^2+1)^2}+I$ 7(h) $-\dfrac{1}{8(w^2+4w+3)^4}+J$

8(a) $e^{x^2-2}+C$ 8(b) $\dfrac{1}{3}e^{x^3+10}+D$

8(c) $\dfrac{1}{2}e^{x^2+2x-3}+E$ 8(d) $-e^{1/x}+F$

8(e) $\dfrac{1}{6}(e^t+10)^6+G$ 8(f) $\dfrac{1}{3}(e^{2u}+1)^{3/2}+H$

8(g) $-\dfrac{5}{(e^v+1)^2}+I$ 8(h) $-\dfrac{1}{8(e^{2w}+4e^w+3)^4}+J$

9(a) $\ln|x^3+1|+C$ 9(b) $\ln|x^2+4x|+D$

9(c) $\dfrac{1}{3}\ln|x^3+12x+1|+E$ 9(d) $\ln|e^x+1|+F$

9(e) $\dfrac{1}{2}\ln|e^{2t}+6e^t+10|+G$ 9(f) $\ln|e^u+e^{-u}|+H$

Exercise 2.2.3

1(a) $\dfrac{76}{3}$ 1(b) $\dfrac{1}{12}$ 1(c) $\dfrac{32}{3}$ 1(d) $e^3 - \dfrac{11}{2}$

2(a) $\dfrac{1}{6}$ 2(b) $\dfrac{1}{3}$ 2(c) $\dfrac{12\sqrt{3}}{5}$ 2(d) $\dfrac{64}{15}$

3(a) $4\displaystyle\int_0^{0.2} (x - 0.2)^2 \, dx$ 3(b) $\dfrac{4}{375}$ m^2 3(c) 3.2 m^3

4(a) They all intersect at $(0, 0)$ and $(1, 0)$.

4(b) The area between $y = x^n$ and x-axis from $x = 0$ to $x = 1$ is

$$\frac{1}{n+1},$$

which decreases as n increases.

4(c) The area would then be

$$\frac{2^{n+1}}{n+1},$$

which increases as n increases. You can check this last statement by considering

$$\frac{2^{n+1}}{n+1} - \frac{2^n}{n} = 2^n \left(\frac{2}{n+1} - \frac{1}{n} \right) = 2^n \left(\frac{n-1}{n(n+1)} \right) \geq 0$$

4(d) The area is infinite when $n \leq -1$. You can check this by calculating the area from $x = h$ to $x = 1$ for $h > 0$.

For example, for $n = -1$ it is

$$\int_h^1 x^{-1} \, dx = [\ln |x|]_h^1 = -\ln h,$$

which becomes infinite as $h \to 0$.

Note. We couldn't calculate the integral directly for the interval $[\,0, 1\,]$ *as* x^{-1} *is not defined for* $x = 0$.

Check out what happens with $x = \dfrac{1}{2}$!

5. Total Cost $= 500 + \displaystyle\int_0^{1000} 3.2 + 0.0004x \, dx = \$3900$

6(a) $P(25) = 1000 + \displaystyle\int_0^{25} 1500 + 300\sqrt{t} \, dt = 63,500$

6(b) $P(t) = 1000 + \displaystyle\int_0^{t} 1500 + 300\sqrt{t} \, dt = 1000 + 1500t + 200t\sqrt{t}$

7(a) $A(10) = 4 + \int_0^{10} -\frac{4}{(t+1)^2}\ dt = \frac{4}{11}\ \mathrm{cm}^2$

7(b) $A(t) = 4 + \int_0^{t} -\frac{4}{(t+1)^2}\ dt = \frac{4}{t+1}\ \mathrm{cm}^2$

8(a) $s(1) - s(0) = \int_0^1 2 - t^2\ dt = \frac{5}{3}\ \mathrm{cm}$

8(b) distance travelled $= \int_0^1 2 - t^2\ dt = \frac{5}{3}\ \mathrm{cm}$

8(c) $s(1) - s(0) = \int_0^2 2 - t^2\ dt = \frac{4}{3}\ \mathrm{cm}$

8(d) distance travelled $= \int_0^{\sqrt{2}} 2 - t^2\ dt - \int_{\sqrt{2}}^2 2 - t^2\ dt = \frac{8\sqrt{2}-4}{3}\ \mathrm{cm}$

9(a) $r(30) = r(0) + \int_0^{30} \frac{20}{\sqrt{t+1}}\ dt = 10 + 10\sqrt{31}\ \mathrm{cm}$

9(b) $r(t) = r(0) + \int_0^{t} \frac{20}{\sqrt{t+1}}\ dt = 10 + 10\sqrt{t+1}\ \mathrm{m}$

9(c) $A(t) = \pi(10 + 10\sqrt{t+1})^2 = 100\pi(2 + 2\sqrt{t+1} + t)\ \mathrm{m}^2$

$$\Rightarrow A'(t) = 100\pi\left(1 + \frac{1}{\sqrt{t+1}}\right)\ \mathrm{m}^2/\mathrm{min}$$

10. $T(4.5) = T(4) + \int_4^{4.5} -\frac{5}{x}\ dx = 70 + 5\ln(4/4.5)\ ^\circ C$