Maths Learning Service: Revision

Straight Lines and Simultaneous Equations

Intro. to Fin. Maths I

The general equation of a straight line is

$$y = mx + k$$

where y is the vertical axis variable,
x is the horizontal axis variable,
m is the *slope/gradient/"rise over run"* of the line and
k is the $y-$intercept (ie. where the line hits the vertical axis).

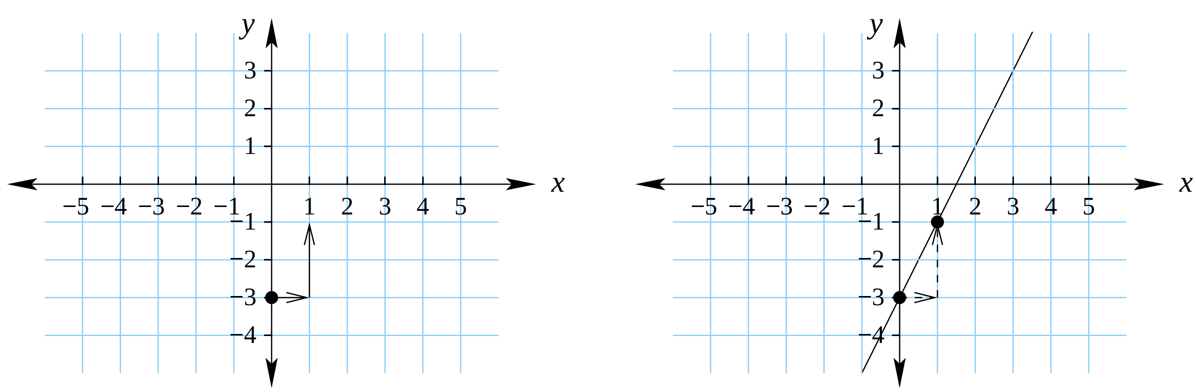
Sometimes the equation of a line appears in a re-arranged form such as

$$ax + by = c.$$

To sketch the graph of a line, you only need to locate two points on the line and then draw the line through them.

Example: Sketch $y = 2x - 3$.

Solution: This line is in the "$y = mx + k$" form so we know that it strikes the $y-$axis at $y = -3$ (that's one point on the line already). A slope of "2" says that, for every one step forward in the horizontal (x) direction, we go up *two* steps vertically (y direction). Applying this information, we get:



Note: For a more accurate sketch you can always take bigger steps forward. For example, two steps forward and *four* steps up also represents a slope of 2.

Example: Sketch $2x + y = 1$.

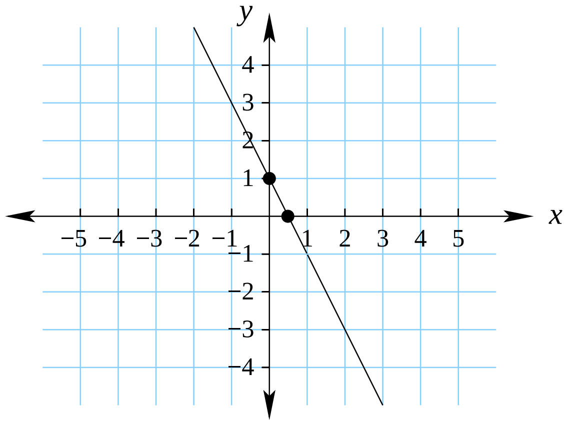
Solution: The slope and $y-$intercept aren't obvious but, if you like algebra, you can re-arrange the equation into "$y = mx + k$" form. Another method is to pick a value for x or y and work out what the other has to be. For example, picking $x = 0$ is an easy choice:

$$\begin{aligned} 2 \times 0 + y &= 1 \\ \Rightarrow \qquad y &= 1. \end{aligned}$$

Hence, the point $x = 0, y = 1$ is on this line (it is in fact the $y-$intercept). For a second point, try $y = 0$:

$$\begin{aligned} 2x + 0 &= 1 \\ \Rightarrow \qquad x &= \tfrac{1}{2} \quad \text{(or 0.5).} \end{aligned}$$

Hence, the point $x = \frac{1}{2}, y = 0$ is another point on this line. Plotting the points gives:



Note: Using the two points worked out above, we can see that for every *half* step forward, the line goes *down* one step. The slope of this line is thus -2.

Exercises

(1) Sketch the graphs of the following lines:

(a) $y = 2x + 1$

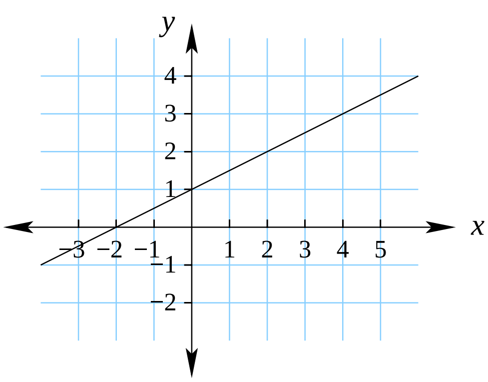
(b) $y = -x + 4$

(c) $2x + y = 2$

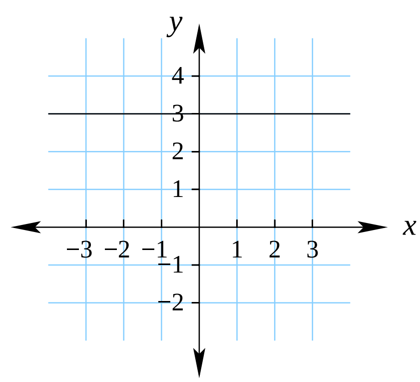
(d) $3x + 3y = -3$

(2) Find the equation of the following lines in "$y = mx + k$" form:

(a)



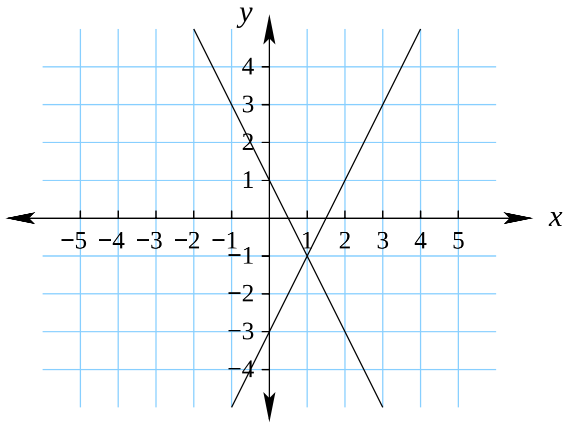
(b)



Systems of Equations (or Simultaneous Equations)

If two lines are drawn on the same axes, they must cross each other at some point (unless they are parallel to each other). Many mathematical problems involve locating these *intersections* because they represent input values (x and y) that satisfy two sets of conditions (equations) at once.

Using the two lines from the previous section:



we can see that the intersection is at $x = 1$, $y = -1$ but this is a tedious and inexact method (especially if the lines don't intersect at whole number co-ordinates, eg. see Exercise (1)(a) and (c)). We need an algebraic method.

The best way is to "line up" the two equations *term by term* as shown:

$$\begin{array}{rcrcl} & & y & = & 2x - 3 \\ 2x & + & y & = & 1 \end{array}$$

$$\Rightarrow \begin{array}{rcrcl} -2x & + & y & = & -3 \\ 2x & + & y & = & 1 \end{array}$$

Since we are looking for the x and y values which satisfy both equations at once, we can add or subtract the two equations, term by term to create a new equation with only one variable left. For example, subtracting the second equation (term by term) from the first gives:

$$\begin{array}{rrcrcl} & -2x & + & y & = & -3 \\ - & 2x & + & y & = & 1 \\ \hline & -4x & + & 0 & = & -4 \\ \Rightarrow & & & x & = & 1 \end{array}$$

Substituting $x = 1$ into either of the original equations reveals that $y = -1$ as expected.

Note: In this case, if we had added the two equations together, we would have got an equation with just y in it and the final solution would have been the same.

Example: Solve the system of equations:

$$\begin{array}{rcrcl} 2x & + & y & = & 2 \\ 3x & + & 3y & = & -3 \end{array}$$

Before adding or subtracting equations, we need to make the coefficients of one variable the same. Multiplying the first equation through by 3 is the quickest way to do this:

$$\begin{array}{rcrcl} \mathbf{3} \times (2x & + & y) & = & 2 \times \mathbf{3} \\ 3x & + & 3y & = & -3 \end{array}$$

$$\Rightarrow \begin{array}{rcrcl} 6x & + & 3y & = & 6 \\ 3x & + & 3y & = & -3 \end{array}$$

Subtracting the two equations gives:

$$\begin{array}{rrcrcr} & 6x & + & 3y & = & 6 \\ - & 3x & + & 3y & = & -3 \\ \hline & 3x & + & 0 & = & 9 \\ \Rightarrow & & & x & = & 3 \end{array}$$

Substituting this into, say, $2x+y=2$ gives $2\times 3+y=2$, so $y=-4$. (The reader may verify that $x=3$, $y=-4$ is the intersection of the two lines by checking the answers to Exercises (1)(c) and (d)).

Example: Solve the system of equations:

$$\begin{array}{rcrcr} x & + & y & = & 2 \\ x & + & y & = & 1 \end{array}$$

Subtracting the two equations gives:

$$\begin{array}{rrcrcr} & x & + & y & = & 2 \\ - & x & + & y & = & 1 \\ \hline & 0 & + & 0 & = & 1 \end{array}$$

Huh? This is impossible and indicates that the two lines do not intersect and must be parallel. (The reader may verify that this is the case by drawing the graphs of the two equations). The equations in Exercise (1)(b) and (d) are also parallel.

Exercises

(3) Solve the following systems of equations (if possible):

(a) $\begin{array}{rcrcr} x & + & y & = & 1 \\ x & - & y & = & -1 \end{array}$

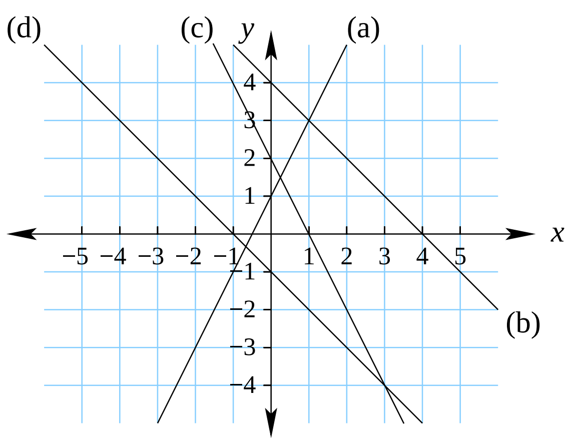
(b) $\begin{array}{rcrcr} -x & + & 2y & = & 4 \\ 3x & + & y & = & 9 \end{array}$

(c) $\begin{array}{rcrcr} 2x & - & y & = & -4 \\ 3x & - & 3y & = & -9 \end{array}$

(d) $\begin{array}{rcrcr} 2x & + & y & = & 1 \\ & & 2y & = & 6-4x \end{array}$

Answers to Exercises

(1)



(2) (a) $y=\frac{1}{2}x+1$ (b) $y=0x+3$ (or just $y=3$)

(3) (a) $x=0$, $y=1$ (b) $x=2$, $y=3$ (c) $x=-1$, $y=2$ (d) parallel