SHAPE OF POLYNOMIAL GRAPHS & EQUATIONS

1. The direction of the ends

Polynomial graphs have squiggly bits in the middle and then eventually just "straighten out" to go on forever at the ends. The direction these ends point is to do with the highest power and the sign of its coefficient.

If the highest power is odd the ends will point in opposite directions:

eg: $y = x^3$ $\quad$ $y = 5 - 2x - x^3$ $\quad$ $y = 5x^5 - 2x^2 + 3$

If the highest power is even the ends will point in the same direction:

eg: $y = x^2 + 2x - 1$ $\quad$ $y = x^4 + x^3$ $\quad$ $y = 2 + x - 3x^6$

If the coefficient of the highest power is <u>positive</u> the right-hand end will point <u>up</u>.

<u>eg</u>

$y=x^3$ $\quad y=5x^5-2x^2+3$ $\quad y=x^2+2x-1$

If the coefficient of the highest power is <u>negative</u> the right-hand end will point <u>down</u>.

<u>eg</u>: $y=5-2x-x^3$ $\quad y=2+x-3x^6$

So the general shapes are:

		COEF OF HIGHEST POWER	
		+	−
HIGHEST POWER	odd		
	even		

2. The number of turns

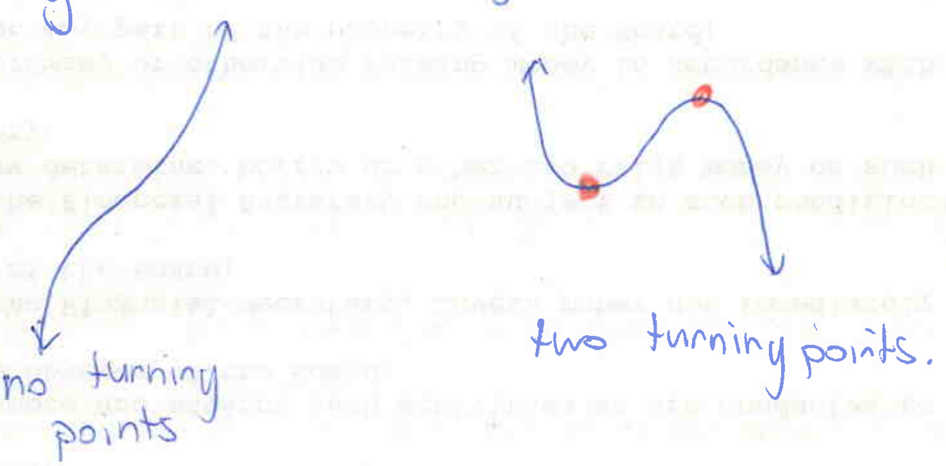
Polynomial graphs tend to have squiggly bits in the middle (though some are quite smooth).

The size of the highest power tells you how many turning points the graph might have.

If the highest power is n, the graph can have up to $n-1$ turning points.

eg A cubic (highest power 3) can have up to 2 turning points.

eg $y = x^3$ $y = 5 - 2x - x^3$

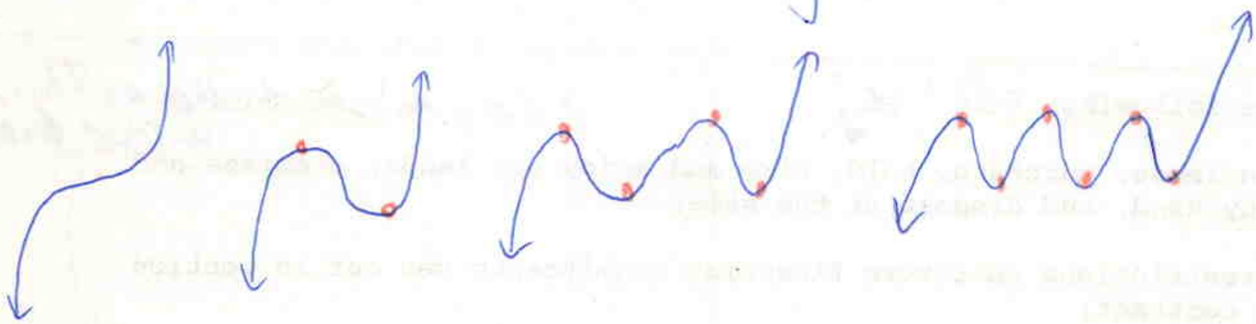


[note it can't have only one or otherwise the ends wouldn't point in opposite directions]

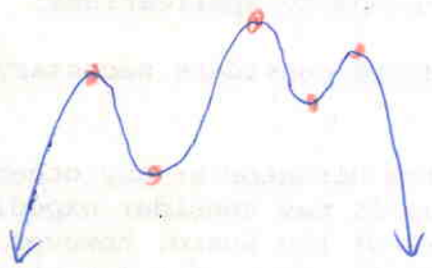
eg A quadratic (highest power 2) has exactly one turning point

– it can't have more than one, and the ends point in the same direction so it must have at least one.

eg If the polynomial has highest power 7, then it could have 0, 2, 4 or 6 turning points:



eg The following graph could be a polynomial of any even degree above 6 [since it has 5 turning points and both ends point in the same direction].



[did I mention that the highest power is called the degree? Well, it is — it makes it easier to talk about it to have a name!]

3. <u>where the graph crosses the x-axis</u>

[also known as the x-intercepts, or the zeros]

The x-intercepts are useful to figure out exactly where you should draw your graph on the xy plane, as well as helping to figure out exactly how many turning points there are.

In order to find the x-intercepts, you have to find out where the formula (that is, the y-value) comes out to zero, because the x-axis is the place where the y-coordinate is zero.

Let's do an example:

eg Let $y = x^2 - 3x + 2$

To find x-intercepts, find when $y=0$.

so $x^2 - 3x + 2 = 0$

[Now, to solve this, I need to write it as the product of two brackets, which you can see how to do in Maths Start 3]

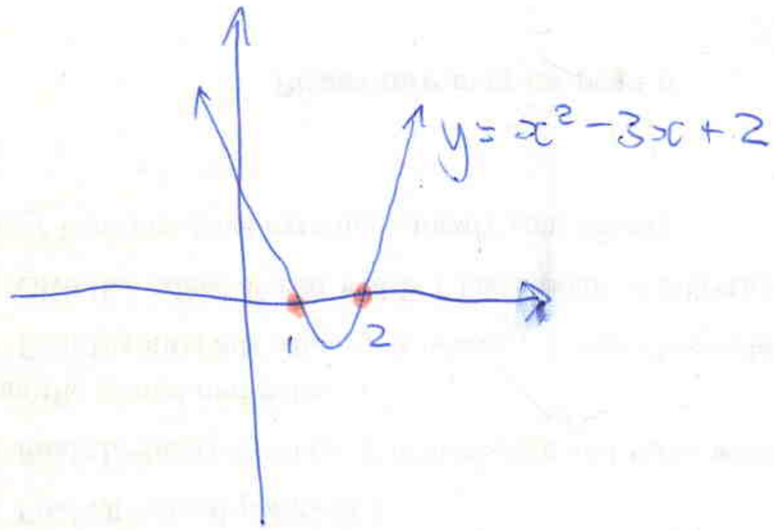
$$(x-1)(x-2)=0$$

so either $(x-1)=0$ OR $(x-2)=0$

$x=1$ OR $x=2$.

The x-intercepts are 1 & 2.

[We can draw it now because we know that a degree 2 has one turning point, and both ends point in the same direction, and this direction is up because the coefficient of x^2 is positive]



So, to find the zeros you must factorise

The degree tells you how many zeros to expect: you can have up to the degree. You might get less, but you can't get more because otherwise you'd have to multiply $(x-a)(x-b)$ etc and you'd get a higher degree when you expand it out.

This means you can get the formula from the zeros:

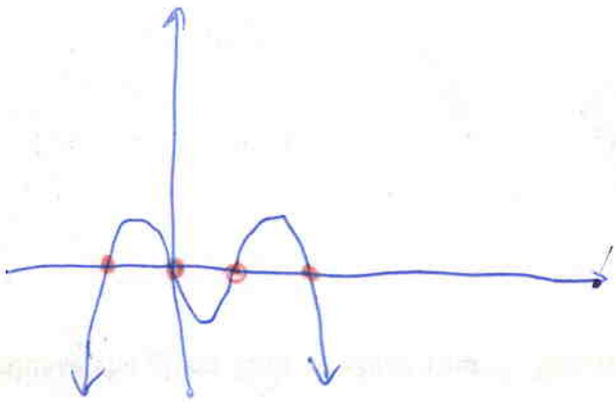
eg A polynomial of degree 4 has zeros 1, 2, −1, 0. Write down a possible formula.

$$y = (x-1)(x-2)(x-(-1))(x-0)$$
$$= (x-1)(x-2)(x+1)x.$$

[Note: you could have anything in front and it still be ok. for example, $y = -3(x-1)(x-2)(x+1)x$.

So $\boxed{\text{A degree } n \text{ polynomial has at most } n \text{ zeros}}$

[Note, we can sketch the polynomial from its zeros usually.
For example, if $y = -3(x-1)(x-2)(x+1)x$,
this means the graph crosses at $1, 2, -1, 0$.
Also, if you expand it out, the coefficient of x^4
would have to be -3, so the ends point
downwards. So it's something like this:



4. The shape of the graph at the x-intercepts

So when you factorise your polynomial, it tells
you the x-intercepts, but it tells you more than
this:

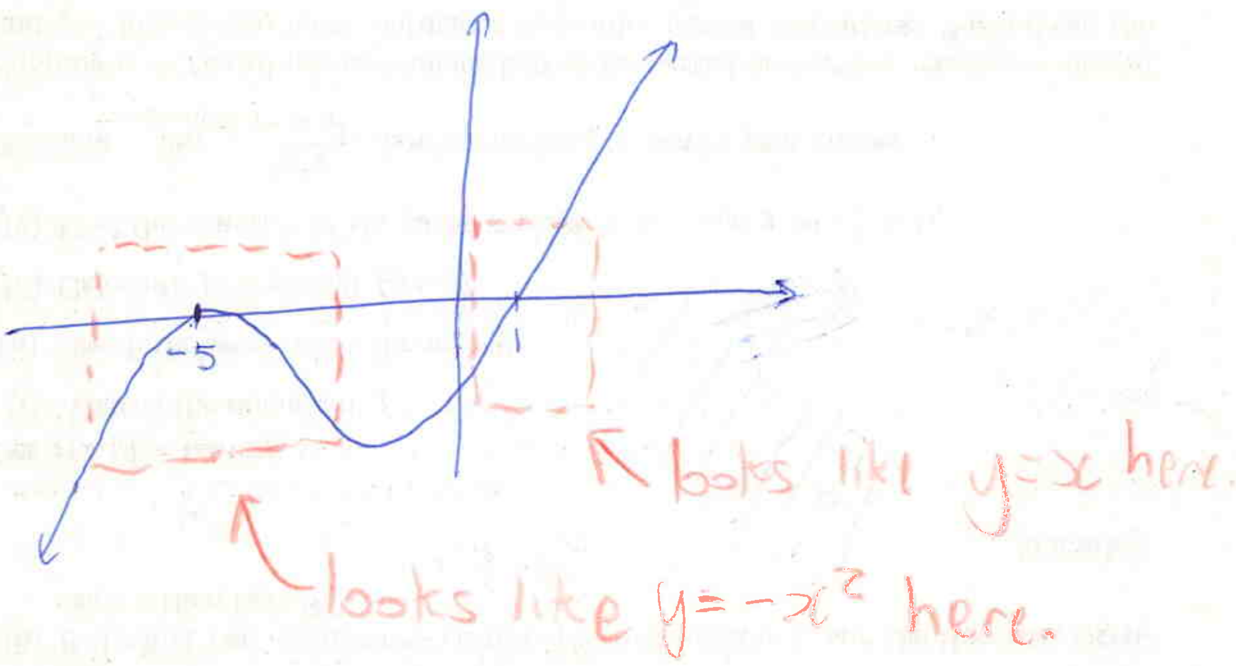
> If $(x-a)^n$ appears in the factorised polynomial
> then the graph looks like $y = \pm x^n$ at the x-intercept
> of a.

This makes most sense with some examples.

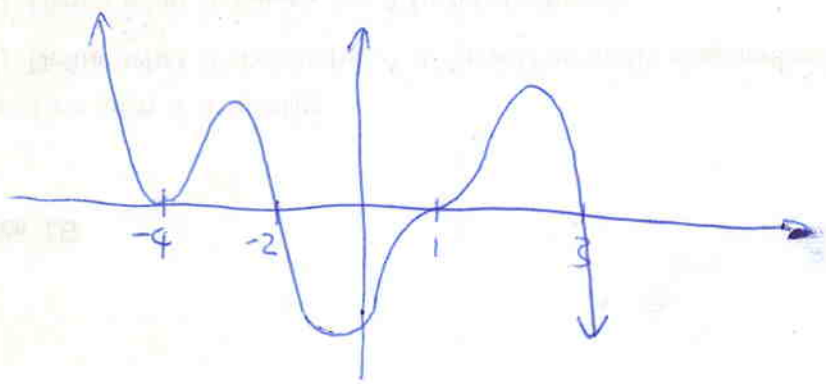
eg $y = 2(x+5)^2(x-1)$

[The graph will look like $y=x^2$ at the zero $x=-5$, and it will look like $y=x$ at the zero $x=1$.

If you multiply it out, it's a cubic with the coefficient of x^3 positive, so it generally looks like this ↗ or this ∿ so let's draw it:]



eg The following is the graph of a degree 7 polynomial. Write down a possible formula:



[ok, so at $x=-4$ it looks like $y=x^2$

so this means we have $(x+4)^2$.

At $x=-2$ it looks like $y=x$ so

we have $(x+2)$.

At $x=1$ it looks like $y=x^3$ so

we have $(x-1)^3$

At $x=3$ it looks like $y=x$ so

we have $(x-3)$

This gives seven powers of x in total, so

that's all the factors.

finally the right-hand end points down, so

the coefficient is negative.]

A possible formula is

$$y = -(x+4)^2(x+2)(x-1)^3(x-3)$$

5. The y-intercept (where it meets the y-axis)

There's one final thing we could put on our graph and that's the place where it meets the y-axis.

This is called the y-intercept and it happens when $x=0$.

To find the y-intercept, sub in $x=0$

eg What is the y-intercept of $y = x^2 + 2x - 3$?

ANSWER: Sub. in $x=0$

So $y = 0^2 + 2\times 0 - 3 = -3$.

So y-intercept is -3.

eg What is the y-intercept of $y = 3(x-2)^2(x+1)$?

ANSWER: Sub. in $x=0$

So $y = 3(0-2)^2(0+1)$

$= 3(-2)^2(1)$

$= 3\times 4\times 1$

$= 12$.

So y-intercept is 12.

eg A cubic polynomial has zeros 1, 3 and 5, and its y-intercept is 30. Write the formula for it.

[So we know $(x-1)$ $(x-3)$ and $(x-5)$ are factors, but there might be a number too, so...]

Let $y = a(x-1)(x-3)(x-5)$.

y-intercept occurs when $x=0$ so sub in $x=0$ and $y=30$.

$$30 = a(0-1)(0-3)(0-5)$$

$$30 = a(-1)(-3)(-5)$$

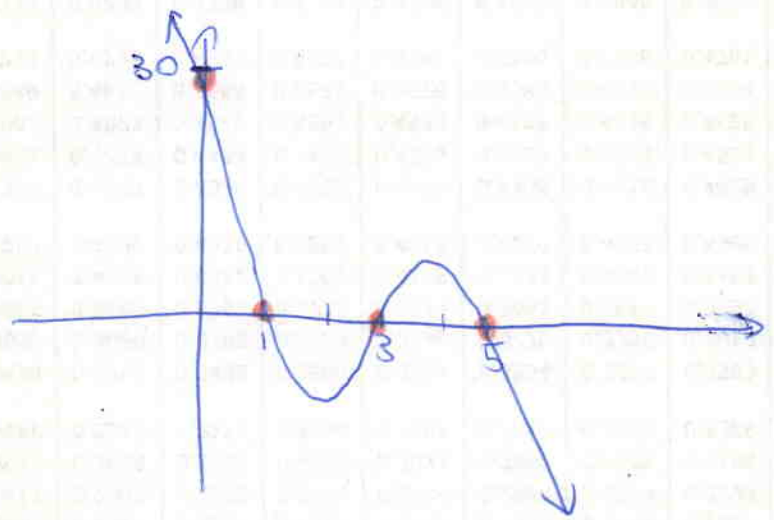
$$30 = -15a$$

$$\frac{30}{-15} = a$$

$$a = -2$$

So a formula is $y = -2(x-1)(x-3)(x-5)$.

[we can sketch this now:]



[See how we used the y-intercept to help us finish the formula? We substituted 0 for x and the y-intercept for y and it told us what the value of a was.

Technically if you knew any point on the graph other than the x-intercepts, you should be able to do the same trick to find the missing number.]