Maths Learning Service: Revision ***Mathematics IA***

Polynomials

A polynomial of degree n is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$. Quadratics, for example, are polynomials of degree two.

A **zero** of a polynomial is a value of x which makes the polynomial equal to zero.

The solutions to a polynomial equation of the form $P(x) = 0$ are called **roots** of the equation $P(x) = 0$.

Exercises

1. Find the roots of the following equations.

   (a) $x^3 - 5x = 0$ (b) $(2x+1)(x^2-3) = 0$ (c) $x^3 + 2x^2 = 4x$

2. Find a and b such that

$$x^4 - 10x^2 + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$$

   and hence solve the equation $x^4 + 1 = 10x^2$.

Polynomial Division

If an integer is divided by a smaller one, we can use long division to find the whole number part and the fractional part (or **remainder**). For example, dividing 313 by 6 gives:

$$
\begin{array}{r|l}
 & \,5 \\ \cline{2-2}
6 & 3\,1\,3
\end{array}
\quad\Rightarrow\quad
\begin{array}{rr|l}
 & & \,5 \\ \cline{3-3}
 & 6 & 3\,1\,3 \\
- & & 3\,0\,0 \\ \cline{3-3}
 & & \,1\,3
\end{array}
\quad\Rightarrow\quad
\begin{array}{rr|l}
 & & \,5\,2 \\ \cline{3-3}
 & 6 & 3\,1\,3 \\
- & & 3\,0\,0 \\ \cline{3-3}
 & & \,1\,3 \\
 & & -\,1\,2 \\ \cline{3-3}
 & & \,1
\end{array}
$$

The answer is then a whole number of 52 with a remainder of 1, or $52\frac{1}{6}$.

Similarly, a polynomial can be divided by one of *lower* degree.

Example: Divide $2x^2 - 5x - 3$ by $x + 2$.

$$
\begin{array}{r|l}
 \\ \cline{2-2}
x+2 & 2x^2 - 5x - 3
\end{array}
$$

At the first stage, the first term of the polynomial ($2x^2$) is divided by the first term of the divisor (x) to give $2x$ and this is put at the top.

$$\begin{array}{r|rrr} & 2x & & \\ \cline{2-4} x+2 & 2x^2 & -\ 5x & -\ 3 \end{array}$$

This is then multiplied by the entire divisor to produce the first row in the subtraction.

$$\begin{array}{r|rrr} & 2x & & \\ \cline{2-4} x+2 & 2x^2\ - & 5x & -\ 3 \\ - & 2x^2\ + & 4x & \\ \cline{2-3} & & -9x & -\ 3 \end{array}$$

The leading term of the remainder ($-9x$) is of the same degree as the divisor so the process is repeated.

$$\begin{array}{r|rrr} & 2x\ - & 9 & \\ \cline{2-4} x+2 & 2x^2\ - & 5x & -\ 3 \\ - & 2x^2\ + & 4x & \\ \cline{2-3} & & -9x & -\ 3 \\ & - & -9x & -\ 18 \\ \cline{3-4} & & & 15 \end{array}$$

Now the remainder (15) is of lower degree than the divisor so this represents the **remainder** and $2x - 9$ is the equivalent of the whole number part of integer division. The full answer is then

$$\frac{2x^2 - 5x - 3}{x+2} = 2x - 9 + \frac{15}{x+2}.$$

Exercise

3. Perform the following divisions.

(a) $\dfrac{x^2 + 2x - 3}{x+2}$ (b) $\dfrac{x^2 - 5x + 1}{x-1}$ (c) $\dfrac{2x^3 + 6x^2 - 4x + 3}{x-2}$

(d) $\dfrac{2x^3 + 3x^2 - 3x - 2}{2x+1}$ (e) $\dfrac{3x^3 + 11x^2 + 8x + 7}{3x-1}$ (f) $\dfrac{2x^4 - x^3 - x^2 + 7x + 4}{2x+3}$

If a is a zero of $P(x)$, then $(x-a)$ must be a factor of $P(x)$. Dividing $P(x)$ by $x - a$ leaves a remainder of 0. (This is the Factor Theorem.)

Example: We know that $P(x) = 2x^2 - 5x - 3 = (2x+1)(x-3)$, so $(x-3)$ is a factor of $P(x)$. Dividing this into $P(x)$ we get

$$
\begin{array}{rrrrrr}
 & 2x & + & 1 & & \\
\cline{2-6}
x-3 \; \big) & 2x^2 & - & 5x & - & 3 \\
- & 2x^2 & - & 6x & & \\
\cline{2-4}
 & & & x & - & 3 \\
 & & - & x & - & 3 \\
\cline{4-6}
 & & & & & 0
\end{array}
$$

Exercise

4. Show that $z^2 + 4z + 8$ is a factor of $z^4 + 64$ and hence find its other quadratic factor.

5. $x + 3$ and $x - 1$ are factors of $x^4 + 3x^3 + ax^2 + bx + 3$. Find a and b and all zeros of the polynomial.

Theorem on rational roots

If $\frac{p}{q}$ is a rational root of $P(x) = 0$, where the coefficients $(a_0, \ldots, a_n)$ are integers, then p is a factor of a_0 and q is a factor of a_n.

This theorem allows us to "hunt" for zeroes using the factors of a_0 and a_n. Substitute the ratios of these factors into $P(x)$ until one produces 0. It's a good idea to start with the easiest options first, in particular choosing $q = 1$, so that you work with whole numbers first!

Example: Factorize $x^3 - 7x - 6$ into linear factors. (In this case $a_3 = 1$ so the rational zero or zeroes of this polynomial have to be whole numbers because q can only be 1.) The factors of $a_0 = -6$ are $\pm 1, \pm 2, \pm 3$ and ± 6:

$$
\begin{aligned}
P(1) &= 1^3 - 7 \times 1 - 6 = 1 - 7 - 6 = -12, && \text{so } (x-1) \text{ is } \textbf{not} \text{ a factor.} \\
P(-1) &= (-1)^3 - 7 \times (-1) - 6 = -1 + 7 - 6 = 0, && \text{so } (x+1) \text{ is a factor.}
\end{aligned}
$$

Dividing $x^3 - 7x - 6$ by $(x + 1)$ leaves a remainder of $x^2 - x - 6$, which factorizes further to $(x + 2)(x - 3)$. Hence

$$x^3 - 7x - 6 = (x + 1)(x + 2)(x - 3).$$

Exercise

6. Factorize into linear factors.

   (a) $x^3 - 2x^2 - 5x + 6$ (b) $4x^3 - 8x^2 + x + 3$ (c) [harder] $2x^3 - 9x^2 + 6x - 1$

7. Find all the zeros of the following.

   (a) $3x^3 + 10x^2 + 4x - 8$ (b) $x^3 - 3x^2 - 3x + 1$ (c) $4x^4 - 4x^3 - 25x^2 + x + 6$

ANSWERS

1. (a) $x = 0,\ \pm\sqrt{5}$ (b) $x = -\frac{1}{2},\ \pm\sqrt{3}$ (c) $x = 0,\ -1 \pm \sqrt{5}$

2. $a = 2\sqrt{3}, b = -2\sqrt{3}$ (or vice versa). $x = -\sqrt{3} \pm \sqrt{2}$ or $x = \sqrt{3} \pm \sqrt{2}$

3. (a) $x - \dfrac{3}{x+2}$ (b) $x - 4 - \dfrac{3}{x-1}$

   (c) $2x^2 + 10x + 16 + \dfrac{35}{x-2}$ (d) $x^2 + x - 2, \quad x \neq -\frac{1}{2}$

   (e) $x^2 + 4x + 4 + \dfrac{11}{3x-1}$ (f) $x^3 - 2x^2 + \frac{5}{2}x - \frac{1}{4} + \dfrac{\frac{19}{4}}{2x+3}$

4. $z^2 - 4z + 8$

5. $a = -2,\ b = -5.\ x = -3,\ 1,\ \dfrac{-1 \pm \sqrt{5}}{2}$

6. (a) $(x-1)(x+2)(x-3)$ (b) $(x-1)(2x-3)(2x+1)$

   (c) $(2x-1)(x-2-\sqrt{3})(x-2+\sqrt{3})$

7. (a) $\frac{2}{3},\ -2$ (repeated) (b) $-1,\ 2 \pm \sqrt{3}$

   (c) $\pm\frac{1}{2},\ 3,\ -2$