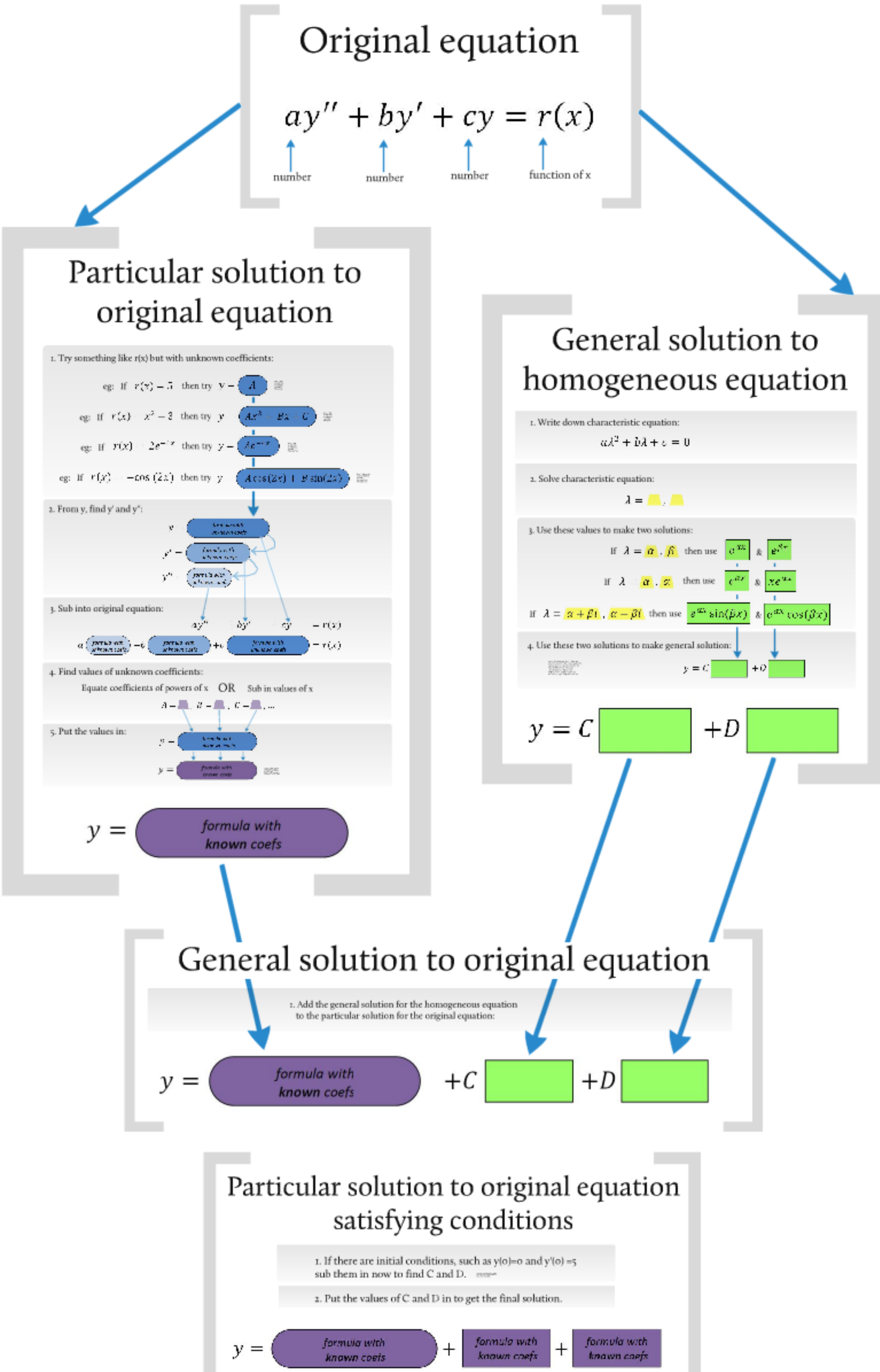
Solving linear second-order differential equations with constant coefficients



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Particular solution to original equation

1. Try something like r(x) but with unknown coefficients:

eg: If $r(x) = 5$ then try $y = A$

eg: If $r(x) = x^2 - 3$ then try $y = Ax^2 + Bx + C$

eg: If $r(x) = 2e^{-4x}$ then try $y = Ae^{-4x}$

eg: If $r(x) = -\cos(2x)$ then try $y = A\cos(2x) + B\sin(2x)$

2. From y, find y' and y":

$y =$ formula with unknown coefs

$y' =$ formula with unknown coefs

$y'' =$ formula with unknown coefs

3. Sub into original equation:

$$ay'' + by' + cy = r(x)$$

a (formula with unknown coefs) $+b$ (formula with unknown coefs) $+c$ (formula with unknown coefs) $= r(x)$

4. Find values of unknown coefficients:

Equate coefficients of powers of x OR Sub in values of x

$A = \_\_$, $B = \_\_$, $C = \_\_$, ...

5. Put the values in:

$y =$ formula with unknown coefs

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$y =$ **formula with known coefs**

General solution to homogeneous equation

1. Write down characteristic equation:

$$a\lambda^2 + b\lambda + c = 0$$

2. Solve characteristic equation:

$$\lambda = \square \, , \, \square$$

3. Use these values to make two solutions:

If $\lambda = \alpha \, , \, \beta$ then use $e^{\alpha x}$ & $e^{\beta x}$

If $\lambda = \alpha \, , \, \alpha$ then use $e^{\alpha x}$ & $x e^{\alpha x}$

If $\lambda = \alpha + \beta i \, , \, \alpha - \beta i$ then use $e^{\alpha x} \sin(\beta x)$ & $e^{\alpha x} \cos(\beta x)$

4. Use these two solutions to make general solution:

Note: The theory says that if I have two solutions that aren't multiples of each other, then all the rest of the solutions must be linear combinations of these two. That's why we just make a linear combination of them (with arbitrary coefficients) to get the general solution.

$$y = C \, \square + D \, \square$$

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General solution to original equation

1. Add the general solution for the homogeneous equation to the particular solution for the original equation:

$y =$ *formula with known coefs* $+C$ [ ] $+D$ [ ]

Particular solution to original equation satisfying conditions

1. If there are initial conditions, such as y(0)=0 and y'(0) =5 sub them in now to find C and D.

2. Put the values of C and D in to get the final solution.

$y =$ *formula with known coefs* $+$ *formula with known coefs* $+$ *formula with known coefs*

Example

Solve $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 13y = e^{2x}$

subject to the conditions $y(0) = 1$ and $y'(0) = 0$.

Particular solution to original equation

$$\begin{aligned} \text{Try} \quad y &= Ae^{2x} \\ y' &= 2Ae^{2x} \\ y'' &= 4Ae^{2x} \end{aligned}$$

$$\text{Now} \quad y'' + 6y' + 13y = e^{2x}$$

$$\begin{aligned} \text{So} \quad 4Ae^{2x} + 6 \times 2Ae^{2x} + 13 \times Ae^{2x} &= e^{2x} \\ 29Ae^{2x} &= e^{2x} \\ 29A &= 1 \\ A &= \tfrac{1}{29} \end{aligned}$$

So a particular solution is:

$$y = \tfrac{1}{29}e^{2x}$$

General solution to homogeneous equation

Characteristic equation: $\lambda^2 + 6\lambda + 13 = 0$.

$$\begin{aligned} \lambda &= \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 13}}{2 \times 1} \\ &= \frac{-6 \pm \sqrt{-16}}{2} \\ &= \frac{-6 \pm 4i}{2} \\ &= -3 \pm 2i \end{aligned}$$

So two solutions are $e^{-3x}\cos(2x)$ and $e^{-3x}\sin(2x)$.
Thus general solution to homogeneous equation is:

$$y = Ce^{-3x}\cos(2x) + De^{-3x}\sin(2x)$$

General solution to original equation:

$$y = \frac{1}{29}e^{2x} + Ce^{-3x}\cos(2x) + De^{-3x}\sin(2x)$$

Particular solution to original equation satisfying conditions

$$\begin{aligned} y &= \tfrac{1}{29}e^{2x} + Ce^{-3x}\cos(2x) + De^{-3x}\sin(2x) \\ y' &= \tfrac{2}{29}e^{2x} - 3Ce^{-3x}\cos(2x) - 2Ce^{-3x}\sin(2x) \\ &\quad - 3De^{-3x}\sin(2x) + 2De^{-3x}\cos(2x) \end{aligned}$$

Substituting $y(0) = 1$ into the first equation gives:

$$\begin{aligned} 1 &= \tfrac{1}{29}e^{0} + Ce^{0}\cos(0) + De^{0}\sin(0) \\ 1 &= \tfrac{1}{29} + C \\ C &= 1 - \tfrac{1}{29} \\ C &= \tfrac{28}{29} \end{aligned}$$

Substituting $y'(0) = 0$ into the second equation gives:

$$\begin{aligned} 0 &= \tfrac{2}{29}e^{0} - 3Ce^{0}\cos(0) - 2Ce^{0}\sin(0) \\ &\quad - 3De^{0}\sin(0) + 2De^{0}\cos(0) \\ 0 &= \tfrac{2}{29} - 3C + 2D \\ 0 &= \tfrac{2}{29} - 3 \times \tfrac{28}{29} + 2D \\ 0 &= \tfrac{2}{29} - \tfrac{84}{29} + 2D \\ 0 &= -\tfrac{82}{29} + 2D \\ 2D &= \tfrac{82}{29} \\ D &= \tfrac{41}{29} \end{aligned}$$

So the particular solution is:

$$y = \frac{1}{29}e^{2x} + \frac{28}{29}e^{-3x}\cos(2x) + \frac{41}{29}e^{-3x}\sin(2x)$$

Example

$$\text{Solve } 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 2y = 3x$$

subject to the conditions $y(0) = -2$ and $y'(0) = 2$.

Particular solution to original equation

$$\begin{aligned} \text{Try} \quad y &= Ax + B \\ y' &= A \\ y'' &= 0 \end{aligned}$$

$$\text{Now} \quad 8y'' - 8y' + 2y = 3x$$

$$\begin{aligned} \text{So} \quad 8 \times 0 - 8A + 2(Ax + B) &= 3x \\ -8A + 2Ax + 2B &= 3x \\ 2Ax + (2B - 8A) &= 3x \end{aligned}$$

$$\text{So } 2A = 3 \qquad 2B - 8A = 0$$

$$A = -\tfrac{3}{2} \qquad 2B - 8 \times (-\tfrac{3}{2}) = 0$$

$$\begin{aligned} 2B + 12 &= 0 \\ 2B &= -12 \\ B &= -6 \end{aligned}$$

So a particular solution is:

$$y = -\tfrac{3}{2}x - 6$$

General solution to homogeneous equation

Characteristic equation: $8\lambda^2 - 8\lambda + 2 = 0$.

$$\begin{aligned} 4\lambda^2 - 4\lambda + 1 &= 0 \\ (2\lambda - 1)^2 &= 0 \\ 2\lambda - 1 &= 0 \\ 2\lambda &= 1 \\ \lambda &= \tfrac{1}{2} \end{aligned}$$

So two solutions are $e^{\frac{1}{2}x}$ and $xe^{\frac{1}{2}x}$.

Thus general solution to homogeneous equation is:

$$y = Ce^{\frac{1}{2}x} + Dxe^{\frac{1}{2}x}$$

General solution to original equation:

$$y = -\tfrac{3}{2}x - 6 + Ce^{\frac{1}{2}x} + Dxe^{\frac{1}{2}x}$$

Particular solution to original equation satisfying conditions

$$y = -\tfrac{3}{2}x - 6 + Ce^{\frac{1}{2}x} + Dxe^{\frac{1}{2}x}$$
$$y'' = -\tfrac{3}{2} + \tfrac{1}{2}Ce^{\frac{1}{2}x} + De^{\frac{1}{2}x} + \tfrac{1}{2}Dxe^{\frac{1}{2}x}$$

Substituting $y(0) = -2$ into the first equation gives:

$$\begin{aligned} -2 &= 0 - 6 + Ce^0 + D \times 0 \\ -2 &= -6 + C \\ C &= 4 \end{aligned}$$

Substituting $y'(0) = 2$ into the second equation gives:

$$\begin{aligned} 2 &= -\tfrac{3}{2} + \tfrac{1}{2}Ce^0 + De^0 + \tfrac{1}{2}D \times 0 \\ 2 &= -\tfrac{3}{2} + \tfrac{1}{2}C + D \\ 2 &= -\tfrac{3}{2} + \tfrac{1}{2} \times 4 + D \\ 2 &= -\tfrac{3}{2} + 2 + D \\ 0 &= -\tfrac{3}{2} + D \\ D &= \tfrac{3}{2} \end{aligned}$$

So the particular solution is:

$$y = -\tfrac{3}{2}x - 6 + 4e^{\frac{1}{2}x} + \tfrac{3}{2}xe^{\frac{1}{2}x}$$