Maths Learning Service: Revision

Logarithms

Mathematics IMA

You are already familiar with some uses of powers or indices. For example:

$$\begin{aligned} 10^4 &= 10 \times 10 \times 10 \times 10 = 10,000 \\ 2^3 &= 2 \times 2 \times 2 = 8 \\ 3^{-2} &= \frac{1}{3^2} = \frac{1}{9} \end{aligned}$$

Logarithms pose a related question. The statement

$$\log_{10} 100$$

asks "what power of 10 gives us 100?" The answer is clearly 2, so we would write

$$\log_{10} 100 = 2.$$

Similarly

$$\log_{10} 10,000 = 4 \quad \text{and} \quad \log_2 8 = 3$$

In general:

$$\boxed{a^x = b \quad \Leftrightarrow \quad \log_a b = x}$$

The number appearing as the subscript of the log is called the *base* so "$\log_{10}$" is read as "logarithm to base 10". The two most common bases you will encounter are 10 and the *exponential base* $e = 2.71828\ldots$. (The letter e is used in place of this inconvenient infinite decimal value.) Your calculator will work out both of these types of logs for you. On most calculators $\log_{10}$ appears as $\boxed{\text{log}}$ and $\log_e$ appears as $\boxed{\text{ln}}$. (The related operations of 10^x and e^x are usually "second functions" on the same key).

Exercises

(1) Find *without using a calculator*:

(a) $\log_{10} 1000$ (b) $\log_4 16$ (c) $\log_2 64$

(d) $\log_3 27$ (e) $\log_9 81$ (f) $\log_e e^2$

(g) Check (a) and (f) on the calculator.

(2) Solve the following equations:

(a) $\log_{10} x = 5$ (b) $\log_2 y = 5$ (c) $\log_3 z = 4$

(3) Find *without using a calculator*:

(a) $\log_{10} 10$ (b) $\log_4 1$ (c) $\log_{10} 0.1$

(d) $\log_2 0.25$ (e) $\log_{10} 1$ (f) $\log_e \frac{1}{e^2}$

(g) Check (a), (c), (e) and (f) on the calculator.

Laws of Logarithms

Given the link between indices and logarithms, we should be able to derive laws for logarithms based on the index laws.

Consider the following argument:

The definition of a logarithm allows us to write the number A as $b^{\log_b A}$ for some base b. Similarly, we could write

$$\begin{aligned} B &= b^{\log_b B} \\ \text{and} \quad A \times B &= b^{\log_b (A \times B)} \end{aligned} \qquad (1)$$

On the other hand, using the index laws, we get

$$A \times B = b^{\log_b A} \times b^{\log_b B} = b^{(\log_b A + \log_b B)}.$$

Comparing this expression for $A \times B$ with (1) we have

$$A \times B = b^{\log_b A + \log_b B} = b^{\log_b (A \times B)}.$$

Since the bases are the same,

$$\log_b A + \log_b B = \log_b (A \times B)$$

By similar arguments the Laws of Logarithms are as follows:

$$\boxed{\begin{aligned} \log_b A + \log_b B &= \log_b (A \times B) \\ \log_b A - \log_b B &= \log_b \left(\frac{A}{B}\right) \\ \log_b (A^n) &= n \log_b A \end{aligned}}$$

Here are a few examples where these laws can be used to solve equations.

(a) Find x such that $2\log_b 4 - 3\log_b 2 + \log_b 2 = \log_b x$.

$$\begin{aligned} \log_b (4^2) - \log_b (2^3) + \log_b 2 &= \log_b x \\ \log_b 16 - \log_b 8 + \log_b 2 &= \log_b x \\ \log_b \left(\tfrac{16}{8}\right) + \log_b 2 &= \log_b x \\ \log_b \left(\tfrac{16 \times 2}{8}\right) &= \log_b x \\ \log_b 4 &= \log_b x \\ \text{so} \quad x &= 4. \end{aligned}$$

(b) Find t such that $1000 = 100\left(2^{\frac{t}{5}}\right)$.

$$\begin{aligned}
10 &= 2^{\frac{t}{5}} \\
\log_{10} 10 &= \log_{10}\left(2^{\frac{t}{5}}\right) \qquad \text{(or any other base, such as } e\text{)} \\
1 &= \frac{t}{5}\log_{10} 2 \\
t &= \frac{5}{\log_{10} 2} \\
&= \frac{5}{0.30103} \\
&= 16.609\ldots
\end{aligned}$$

(c) In the previous example we chose $\log_{10}$ since this made $\log_{10} 10$ very easy and $\log_{10} 2$ could be found on a calculator. If we had used $\log_2$ we would have had to find $\log_2 10$, for which there is no calculator button.

It is possible to find logs to any base by noting the following argument:

$$\begin{aligned}
\text{Let } y = \log_a b \quad \Leftrightarrow \quad a^y &= b \\
\ln(a^y) &= \ln b \\
y \ln a &= \ln b \\
y &= \frac{\ln b}{\ln a}.
\end{aligned}$$

(Using $\log_{10}$ works just as well of course.) For example

$$\begin{aligned}
\log_2 8 &= \frac{\ln 8}{\ln 2} & &= \frac{\log_{10} 8}{\log_{10} 2} \\
&= \frac{2.07944\ldots}{0.69314\ldots} & &= \frac{0.9031\ldots}{0.3010\ldots} \\
&= 3 & &= 3.
\end{aligned}$$

Exercises

(4) Express as a single logarithm:

(a) $\log_b 8 - \log_b 2$ (b) $2\log_b 3 + \log_b 2$ (c) $1 - \log_{10} 4$
(d) $\log_b a + \log_b\left(\frac{1}{a}\right)$

(5) Write in terms of $\log_b 2$ and $\log_b 3$:

(a) $\log_b 6$ (b) $\log_b 8$ (c) $\log_b 24$

(6) Find, using a calculator (to 4 decimal places):

(a) $\log_2 6$ (b) $\log_3 8$ (c) $\log_3 1000$ (d) $\log_3 100,000$
(e) $\log_3 0.001$ (f) $\log_3 0.00001$ (g) $\log_3 1$

(7) Solve for x:

(a) $9 = 10\left(2^{-\frac{x}{1620}}\right)$ (b) $3^{5x+2} = 10$

Answers to Exercises

(1) (a) 3 (b) 2 (c) 6 (d) 3 (e) 2 (f) 2

(2) (a) $x = 10^5 = 100,000$ (b) $y = 2^5 = 32$ (c) $z = 3^4 = 81$

(3) (a) 1 (b) 0 (c) -1 (d) -2 (e) 0 (f) -2

(4) (a) $\log_b 4$ (b) $\log_b 18$ (c) $\log_{10}\left(\frac{5}{2}\right)$ (d) $\log_b 1 = 0$

(5) (a) $\log_b 2 + \log_b 3$ (b) $3\log_b 2$ (c) $3\log_b 2 + \log_b 3$

(6) (a) 2.5850 (b) 1.8928 (c) 6.2877 (d) 10.4795 (e) -6.2877

(f) -10.4795 (g) 0

(7) (a) 246.245 (b) 0.01918