Maths Learning Service: Revision ***Mathematics IA***

Matrices

Mathematics IMA

A matrix is an array of numbers, written within a set of [ ] brackets, and arranged into a pattern of rows and columns. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 21 & 7 & -4 & 9 \end{bmatrix}$$

The **order** (or **size**, or **dimension**) of a matrix is written as "$m \times n$" where $m =$ the number of rows, and $n =$ the number of columns. For example, the matrices above have dimensions

$$2 \times 3, \quad 3 \times 3 \quad \text{and} \quad 1 \times 4.$$

Basic Matrix Operations

Addition (or subtraction) of matrices is performed by adding (or subtracting) elements in corresponding positions. Addition is only valid if the two matrices have the same order.

Examples:

(i) $$\begin{bmatrix} 2 & -4 & 0 \\ -1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 7 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2+3 & -4+4 & 0+(-1) \\ -1+7 & 3+0 & 5+(-2) \end{bmatrix} = \begin{bmatrix} 5 & 0 & -1 \\ 6 & 3 & 3 \end{bmatrix}$$

(ii) $$\begin{bmatrix} 3 & 4 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 7 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} 3-1 & 4-7 \\ -2-(-8) & 0-1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 6 & -1 \end{bmatrix}$$

(iii) $\begin{bmatrix} 2 & -4 & 0 \\ -1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -2 & 0 \end{bmatrix}$ cannot be done as the orders are different.

When a matrix is multiplied by a real number (called a *scalar*), each element is multiplied by the scalar. The result is another matrix of the same order.

Examples:

(i) $$4\begin{bmatrix} 2 & 1 \\ -3 & 9 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 4 \times 2 & 4 \times 1 \\ 4 \times -3 & 4 \times 9 \\ 4 \times 0 & 4 \times -5 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -12 & 36 \\ 0 & -20 \end{bmatrix}$$

(ii) $$\frac{1}{2}\begin{bmatrix} 7 & 8 & -10 & 6 & 0.4 \end{bmatrix} = \begin{bmatrix} 3.5 & 4 & -5 & 3 & 0.2 \end{bmatrix}$$

(iii) $$2\begin{bmatrix} 5 & -3 \\ 0 & -6 \end{bmatrix} - 3\begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 10 & -6 \\ 0 & -12 \end{bmatrix} - \begin{bmatrix} 9 & 12 \\ -3 & 21 \end{bmatrix} = \begin{bmatrix} 1 & -18 \\ 3 & -33 \end{bmatrix}$$

When giving matrices a name, use capital letters such as A, B, etc to distinguish them from algebraic scalars such as a, b, etc.

Exercises

(1) Given that

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 4 & 5 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 1 & -3 \\ 2 & 0 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ -4 & 9 \end{bmatrix} \quad D = \begin{bmatrix} 11 & 5 \\ 0 & -2 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 3 \end{bmatrix}$$

find the following (if possible):

(a) $A + B$ (b) $B + A$ (c) $C + D$ (d) $C - D$ (e) $D - C$

(f) $A + E$ (g) $B - D$ (h) $3A$ (i) $2C + D$ (j) $5B - 4E$

Matrix Multiplication

The rule for multiplying matrices can be represented as follows:

$$AB = \begin{bmatrix} \text{row 1 of } A \times \text{col 1 of } B & \text{row 1 of } A \times \text{col 2 of } B & \text{row 1 of } A \times \text{col 3 of } B & \cdots \\ \text{row 2 of } A \times \text{col 1 of } B & \text{row 2 of } A \times \text{col 2 of } B & \text{row 2 of } A \times \text{col 3 of } B & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

where "row i of $A \times$ col j of B" is a single number and stands for "each entry in row i of A is multiplied by the corresponding entry in column j of B and the results are added together".

Examples:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -5 \\ -1 & -2 \\ 0 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(i) $$CA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + 2 \times 1 & 1 \times (-1) + 2 \times 4 & 1 \times 3 + 2 \times 5 \\ 3 \times 2 + 4 \times 1 & 3 \times (-1) + 4 \times 4 & 3 \times 3 + 4 \times 5 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 13 \\ 10 & 13 & 29 \end{bmatrix}$$

(ii) $$AB = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + (-1) \times (-1) + 3 \times 0 & 2 \times (-5) + (-1) \times (-2) + 3 \times 3 \\ 1 \times 4 + \quad 4 \times (-1) + 5 \times 0 & 1 \times (-5) + \quad 4 \times (-2) + 5 \times 3 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 0 & 2 \end{bmatrix}$$

(iii) $$CB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -1 & -2 \\ 0 & 3 \end{bmatrix}$$

This is not possible because there are fewer entries in the rows of C (two) than in the columns of B (three).

$$\boxed{\begin{array}{c}\textit{Matrix multiplication is only defined when the number of columns}\\ \textit{in the first matrix equals the number of rows in the second.}\end{array}}$$

(iv) $CD = \begin{bmatrix} 13 & 19 \\ 27 & 43 \end{bmatrix}$ but $DC = \begin{bmatrix} 16 & 22 \\ 27 & 40 \end{bmatrix}$ so $CD \neq DC$.

$$\boxed{\textit{In general } AB \neq BA \textit{ for matrices.}}$$

(v) $$CI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times 1+2\times 0 & 1\times 0+2\times 1 \\ 3\times 1+4\times 0 & 3\times 0+4\times 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = C \text{ (unchanged)}$$

(vi) $$IC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = C \text{ (unchanged)}$$

$$\boxed{\begin{array}{c}\textit{The matrix } I \textit{ is an identity matrix and is the}\\ \textit{matrix equivalent of the number 1 in scalar multiplication.}\end{array}}$$

Notes: 1. The identity is an exception to the general rule for matrix multiplication since $CI = IC = C$.

2. Identity matrices only exist for square matrices. The matrix I used in Examples (v) and (vi) is called "the identity matrix for a 2×2 matrix". The identity matrix for a 3×3 matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Exercises

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 4 \end{bmatrix} \quad E = \begin{bmatrix} -3 & 2 \\ 1 & 7 \end{bmatrix}$$

(2) Using the above matrices, calculate the following (if possible):

(a) AB (b) BA (c) DI (d) ID (e) CD

(f) DC (g) BC (h) CB (i) E^2 (j) B^2

Inverse of a Square Matrix

In scalar algebra, we know that

$$a \times \frac{1}{a} = aa^{-1} = a^{-1}a = 1 \quad (a \neq 0).$$

We call a^{-1} the *multiplicative inverse* of a.

For square matrices, we define the inverse "A^{-1}" as having the property that

$$A \times A^{-1} = A^{-1} \times A = I.$$

The inverse of a 2×2 matrix is found by the formula below.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $\det(A) = |A| = \textit{determinant}$ of $A = ad - bc$.

Notes:

1. A^{-1} can not be found by rearrangement ($A^{-1} = I \div A$), because division is not defined for matrices.
2. A matrix has no inverse if its determinant ($ad - bc$) equals 0 (hence the name "determinant").
3. A^{-1}, if it exists, is unique.

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A^{-1} = \dfrac{1}{1 \times 4 - 2 \times 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\dfrac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Check this by calculating AA^{-1} and $A^{-1}A$ to see if both equal I.

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \checkmark$$

$A^{-1}A = \ldots$ (try this as an exercise).

Exercises

(3) Show that the following pairs of matrices are inverses.

(a) $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ and $\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

(4) Using the formula, find the inverses of the following matrices (if they exist).

(a) $\begin{bmatrix} 7 & 18 \\ 2 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 1 \end{bmatrix}$

An application of the inverse: Solving Simultaneous Equations

A pair of simultaneous linear equations such as

$$\begin{aligned} -x + 2y &= 0 \\ x + y &= 3 \end{aligned}$$

can be written in matrix notation as

$$\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

or $\quad A \quad X = B.$

If a unique solution exists, we can use the inverse matrix to solve the system, as follows:

$$\begin{aligned} AX &= B \\ \Rightarrow \quad A^{-1}AX &= A^{-1}B \end{aligned}$$

Note that A^{-1} has been *pre*-multiplied on both sides. Since order of multiplication is important, we can't use BA^{-1} (i.e. *post*-multiplication) on the RHS since we pre-multiplied on the LHS.

$$\begin{aligned} \Rightarrow \quad IX &= A^{-1}B \qquad \text{since } A^{-1}A = I \\ \Rightarrow \quad X &= A^{-1}B \qquad \text{since } IX = X \end{aligned}$$

In this example, we have $A = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$ and hence $A^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$ so

$$\begin{aligned} X &= A^{-1}B \\ &= -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= -\frac{1}{3}\begin{bmatrix} -6 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

Hence $x = 2$ and $y = 1$ is the answer (check by substituting back into the original equations).

Exercises

(5) Rewrite the following pairs of equations in the form of a matrix equation, $AX = B$, and solve (if a unique solution exists) using the inverse matrix of A.

(a) $\begin{aligned} x - y &= 5 \\ x + y &= 1 \end{aligned}$ (b) $\begin{aligned} 5x + y &= 7 \\ 3x - 4y &= 18 \end{aligned}$ (c) $\begin{aligned} x + 2y &= 8 \\ 3x + 6y &= 15 \end{aligned}$

(d) $\begin{aligned} 2x + 3y &= 11 \\ 6x + 9y &= 33 \end{aligned}$

Determinant of a 3×3 matrix

The inverse of a 3×3 matrix can be found using *row operations* (see revision sheet on Solving Linear Equations) but the determinant is as follows:

$$\text{If } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ then}$$

$$\det(A) = |A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

This may seem like a complicated definition but the determinant can be thought of as a "first row expansion", where each entry in the first row is multiplied by the 2×2 determinant created by removing the row and column containing that entry. Notice also that the signs connecting the three terms alternate $(+, -, +)$.

Examples:

(i) $$\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} = 3(0-4) - 2(12-2) + (8-0) = -24$$

(ii) $$\begin{vmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 2 & 2 & -2 \end{vmatrix} = -\begin{vmatrix} 0 & -2 \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 2 & -2 \end{vmatrix} + 0\begin{vmatrix} 3 & 0 \\ 2 & 2 \end{vmatrix} = -(0+4) - 2(-6+4) + 0 = 0$$

Note: The matrix in Example (ii) has no inverse.

Exercises

(6) Find the following determinants:

(a) $\begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 4 & 2 & 2 \end{vmatrix}$ (c) $\begin{vmatrix} -3 & 1 & -2 \\ 2 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$ (d) $\begin{vmatrix} 3 & 3 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix}$

(7) Show that the *upper triangular* matrix $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$ has determinant aei.

Answers to Exercises

(1) (a) $\begin{bmatrix} 6 & 3 & -3 \\ 6 & 5 & 9 \end{bmatrix}$ (b) same as (a) (c) $\begin{bmatrix} 12 & 7 \\ -4 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} -10 & -3 \\ -4 & 11 \end{bmatrix}$

(e) $\begin{bmatrix} 10 & 3 \\ 4 & -11 \end{bmatrix}$ (f) not possible (g) not possible (h) $\begin{bmatrix} -3 & 6 & 0 \\ 12 & 15 & 9 \end{bmatrix}$

(i) $\begin{bmatrix} 13 & 9 \\ -8 & 16 \end{bmatrix}$ (j) not possible

(2) (a) $\begin{bmatrix} 2 & 3 & 1 \\ -8 & -5 & -5 \end{bmatrix}$ (b) not possible (c) D (d) D

(e) not possible (f) $\begin{bmatrix} 3 \\ 6 \\ 15 \end{bmatrix}$ (g) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ (h) not possible

(i) $E^2 = EE = \begin{bmatrix} 11 & 8 \\ 4 & 51 \end{bmatrix}$ (j) not possible

(4) (a) $\dfrac{1}{20}\begin{bmatrix} 8 & -18 \\ -2 & 7 \end{bmatrix}$ or $\begin{bmatrix} \frac{2}{5} & -\frac{9}{10} \\ -\frac{1}{10} & \frac{7}{20} \end{bmatrix}$ (b) $\dfrac{1}{4}\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$

(c) determinant $= 0$ so no inverse exists (d) $-\dfrac{1}{14}\begin{bmatrix} 4 & -6 \\ -3 & 1 \end{bmatrix}$

(e) $\dfrac{1}{13}\begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix}$ (f) no inverse as the matrix is not square

(5) (a) $x = 3, y = -2$ (b) $x = 2, y = -3$

(c) no solution ($\det(A) = 0$). (The lines are parallel.)

(d) no unique solution ($\det(A) = 0$). (The two equations represent the same line since $3(2x + 3y = 11)$ gives $6x + 9y = 33$.)

(6) (a) 2 (b) 0 (c) 9 (d) 0