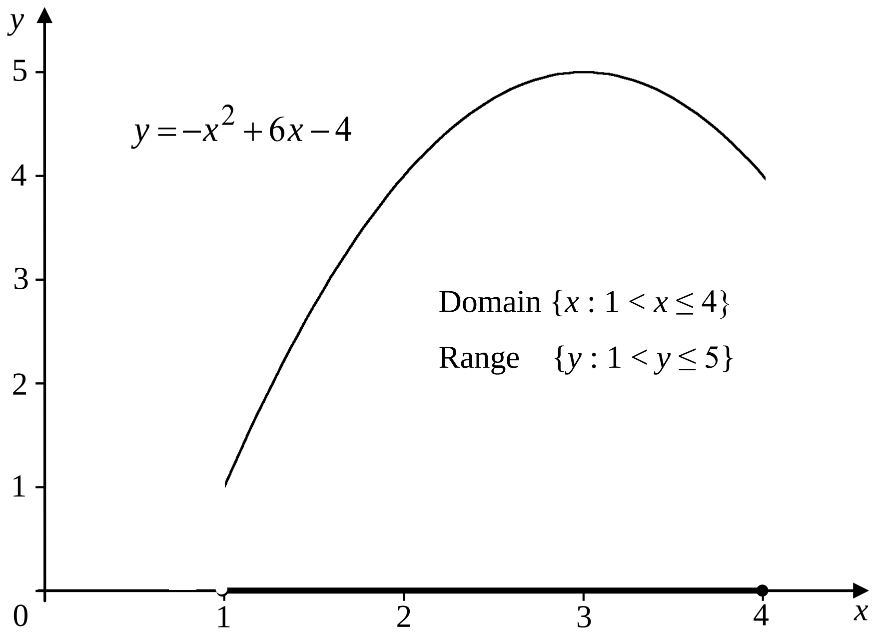
MathsStart

(NOTE Feb 2013: This is the old version of MathsStart.
New books will be created during 2013 and 2014)

THE UNIVERSITY of ADELAIDE

Topic 1

Numbers & Functions



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This Topic...

This module introduces the real number system and ways of describing numbers. It also introduces functions, which are the most common type of mathematical formulas.

Mathematics is built on numbers and proceeds by inventing, analysing and then using mathematical expressions and formulas.

(Author of this Topic: Paul Andrew)

— Prerequisites

A scientific calculator is needed for MathsStart.

— Contents

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1
The Real Number System

The real number system evolved over a long period of time.[1]

1.1 Natural numbers (N)

Numbers like 1, 2, 3, . . . were first used for counting things such as food, animals, money, etc. These are called the *natural numbers* (sometimes the *counting numbers*) and were invented independently in many countries.

We use curly brackets {} to indicate a *set* or *collection* of numbers, so the *set of all natural numbers* is written as:

$$\{1, 2, 3, \ldots\}.$$

Using 3 dots is a way of showing that the list of numbers goes on forever.

This set is traditionally represented by the capital letter **N**, so we can write:

$$\mathbf{N} = \{1, 2, 3, \ldots\}.$$

The set of natural numbers **N** has some special properties:

- if two natural numbers are added, then the result is also a natural number.
- if two natural numbers are multiplied, then the result is also a natural number.

We describe this by saying that the set **N** is *closed under addition and under multiplication*.

However the set **N** is *not* closed under subtraction. *Why not?*

Closed means that these operations don't result in numbers outside the set.

1.2 Whole numbers

After a while, *zero* was included as a number. If a shopkeeper had 20 apples, and sold 19 then there would be 1 apple left. If all 20 were sold, then there would be 0 left.

The *whole numbers* are the natural numbers together with zero:

$$\{0, 1, 2, 3, \ldots\}.$$

No special symbol is used to represent the set of whole numbers.

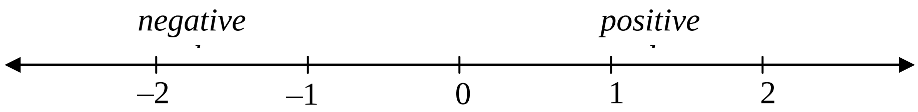
Are the whole numbers closed under $+$ *and* $\times$?

[1] Oxford Dictionary: A system is an organised or connected group of objects.

1.3 Integers

Negative numbers are useful for indicating deficits. If income is \$500 dollars and expenditure is \$600, then the balance is \$500 – \$600 = –\$100.

Negative numbers gained general acceptance in the late 16th century. Before then there was some doubt as to whether negative numbers really existed. '*Show me –2 apples*!' This suspicion was resolved by a new interpretation of numbers as points on a *number line*. Numbers on the right of the *origin* 0 were taken as positive and numbers on the left were taken as negative.



The *integers* are the natural numbers, their negatives and zero:

A *definition* is a precise statement of what something means.

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$$

> **Definition**
> An *integer* is a natural number, the negative of a natural number, or zero.

The *set of all integers* is an important set of numbers and is represented by the letter **Z**, i.e.

$$\mathbf{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The set of integers **Z** *contains* the set of natural numbers **N**, and is

- *closed under addition*
- *closed under subtraction*
- *closed under multiplication*.

However **Z** is *not* closed under division.

Why not?

1.3 Rational numbers (Q)

Simple fractions like $\frac{1}{2}$, $\frac{3}{4}$, . . . were probably first used in commerce. '*Can I have half a melon, please*?' They were invented independently in many countries and are examples of *rational numbers*. The set of rational numbers is the set of all possible fractions and their negatives.

> **Definition**
> A *rational number* is a number that can be written in the form $\frac{m}{n}$, where *m* and *n* are integers with $n \neq 0$.[2]

Why is the condition n ≠ 0 used in this definition?

Examples

(a) $\frac{3}{4}$ is a rational number as 3 and 4 are integers.

(b) $-\frac{3}{4}$ is a rational number as it can be written as $\frac{-3}{4}$, where -3 and 4 are integers.

(c) $1\frac{2}{5}$ is a rational number as it can be written as $\frac{7}{5}$.

(d) 1.234 is a rational number as it can be written as $\frac{1234}{1000}$.

The *set of rational numbers* is represented by the letter **Q.** It is too complicated to be written as a list of numbers in any reasonable way. Instead, we use set notation like this:

$$\mathbf{Q} = \{ \frac{m}{n} \mid m, n \in \mathbf{Z} \}.$$

The set of rational numbers **Q** contains the set of integers **Z**, and is

- *closed under addition*
- *closed under subtraction*
- *closed under multiplication*
- *closed under division*.

Why is each integer also a rational number?

It was believed 2,500 years ago that *every* number was a rational number (reflecting the logic and perfection of the universe) until a member of Pythagoras's secret society of mathematicians and mystics showed that $\sqrt{2}$ was not a rational number. This caused a huge impact at the time.

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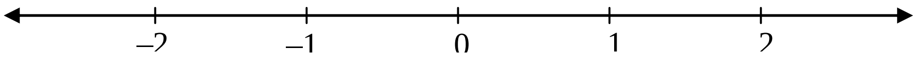
[2] The word *rational* comes from the word *ratio*. In the fraction $\frac{m}{n}$, *m* is the *numerator* and the *n* is the denominator.

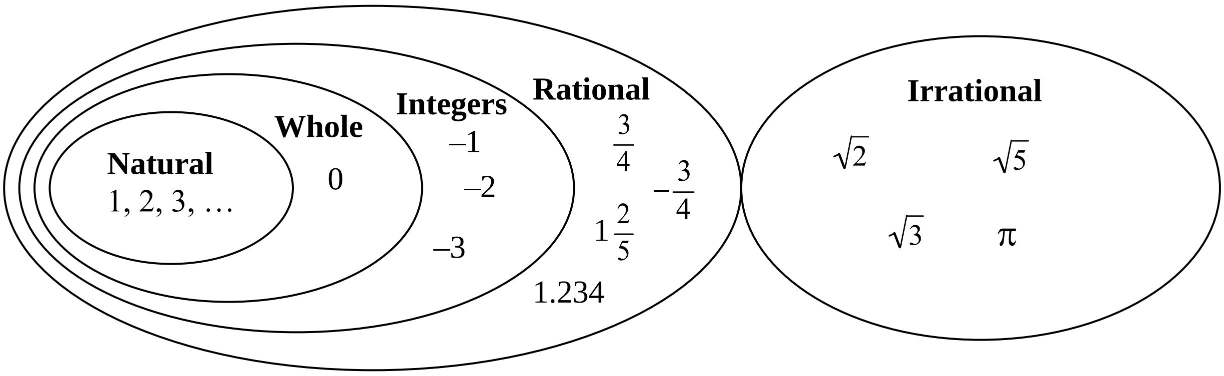
Numbers which are not rational are called *irrational numbers*. Other examples of irrational numbers are $\sqrt{3}$, $\sqrt{5}$ and π.

1.4 Real numbers

The *real numbers* are the 'measuring numbers'. They are all the numbers that can be found on a number line (commonly called *the real line*).



The set of real numbers is represented by the letter **R**. It is impossible to write it out as a list in any way. It contains the natural numbers **N**, the whole numbers, the integers **Z** , the rational numbers **Q**, and the irrational numbers.

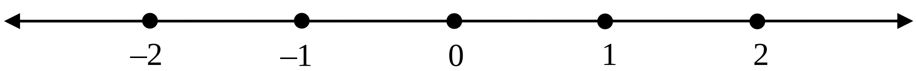


The set of real numbers **R** is

- *closed under addition*
- *closed under subtraction*
- *closed under multiplication*
- *closed under division*.

There are other useful numbers that are not real numbers – you'll meet these in a later course.

When the real line is examined, it is easy to see the pattern of the natural numbers and integers.



Do the rational numbers also have a visible pattern?

To answer this question, look at some of the rational numbers between 0 and 1.

You can see that:

- {0.1, 0.2, . . . , 0.9} are rational numbers and are evenly spaced between 0 and 1,
- {0.01, 0.02, . . . , 0.99} are rational and also evenly spaced between 0 and 1,
- {0.001, 0.002, . . . , 0.999} are rational and similarly evenly spaced between 0 and 1.

Continuing this way, it can be seen that the *rational numbers are dense between 0 and 1*, in the sense that here is no section of the real line between 0 and 1 that does not contain some rational numbers. If each rational number was coloured red, then the visible result would be a red line from 0 to 1! The same is true for any other section of the real line.

The set of all irrational numbers is also dense in every section of the real line.

Problems 1

1. Are $1\frac{1}{2}$ and – 0.2 rational numbers?

2. Confirm that **Q** is closed under addition, subtraction, multiplication and division by showing that $\frac{1}{2} + \frac{3}{4}$, $\frac{1}{2} - \frac{3}{4}$, $\frac{1}{2} \times \frac{3}{4}$, and $\frac{1}{2} \div \frac{3}{4}$ are all rational numbers.

3. Explain why every integer is a rational number.

4. What makes an irrational number different from a rational number?

5. Evaluating $\sqrt{2}$ using my calculator gives 1.4142136.

a. Explain why this is not the same as $\sqrt{2}$.

b. Find a rational number that is

i. closer than 0.001 to $\sqrt{2}$ and less than $\sqrt{2}$.

ii. closer than 0.001 to $\sqrt{2}$ and greater than $\sqrt{2}$.

6. Explain why $\sqrt{2}$ + 1 is an irrational number.

2
Functions

One of the things we use mathematics for is to investigate the relationships between variables.

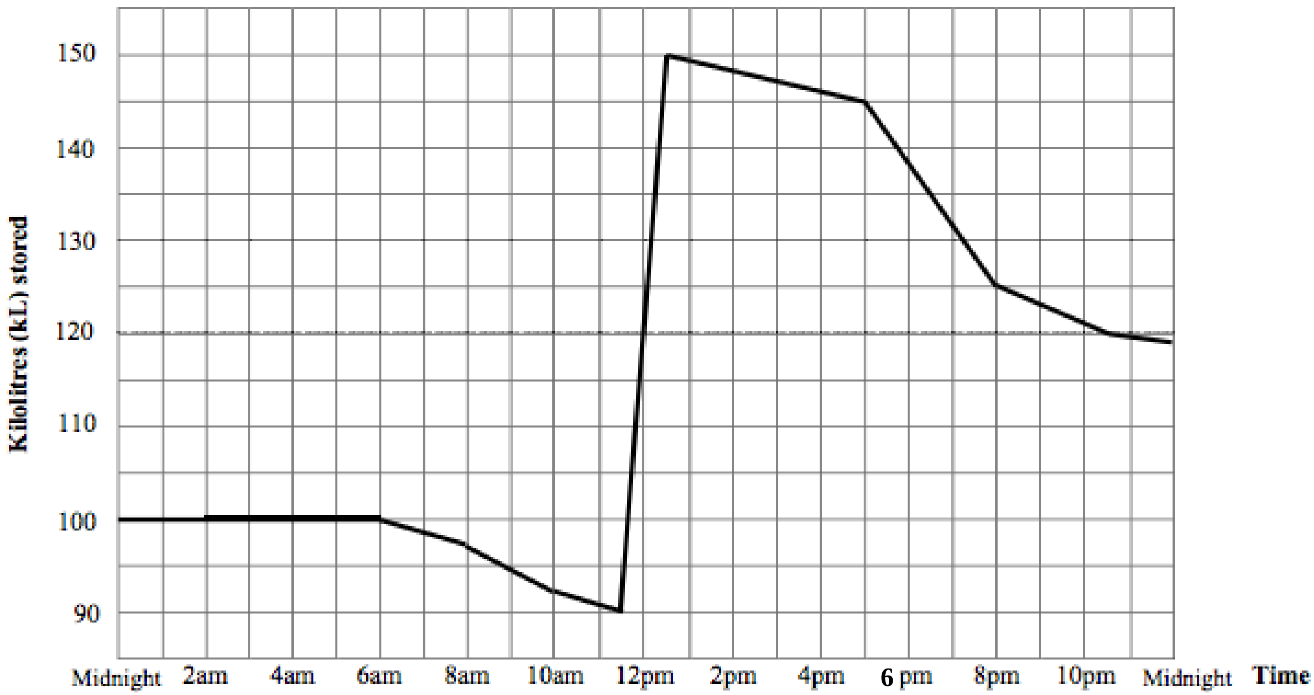
2.1 Functions

The word *function* has a number of meanings in everyday conversation. We might talk about a *social function* (a social gathering) or the *function* (the use) of a new machine. In mathematics the word *function* is used to describe the relationship between two variables.

> **Definition**
> One variable is said to be a *function* of another variable when its value depends on the value of the other variable. It is called the *dependent variable*, and the variable it depends upon is called the *independent variable*.

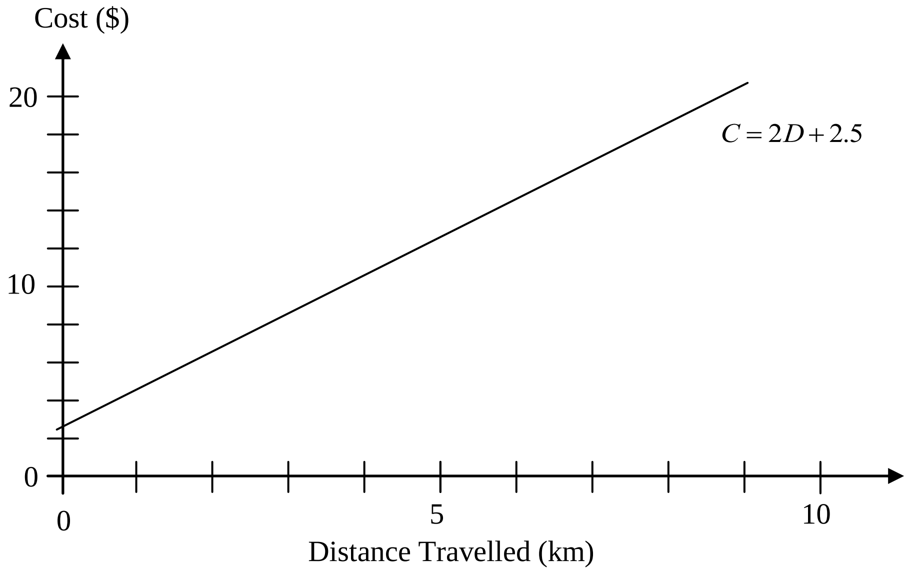
Examples

(a) The graph below shows the amount of petrol in storage at a petrol station between opening time at 4 am and closing time at midnight.



In this example, *Kilolitres* is a function of *Time* as the amount in storage depends upon the time. *Kilolitres* is the dependent variable and *Time* is the independent variable.

(b) This graph shows how the cost of a taxi ride is related to distance travelled.



The equation of this straight line is

$$C = 2D + 2.5,$$

where C is the cost and D is the distance travelled in kilometres.

We say

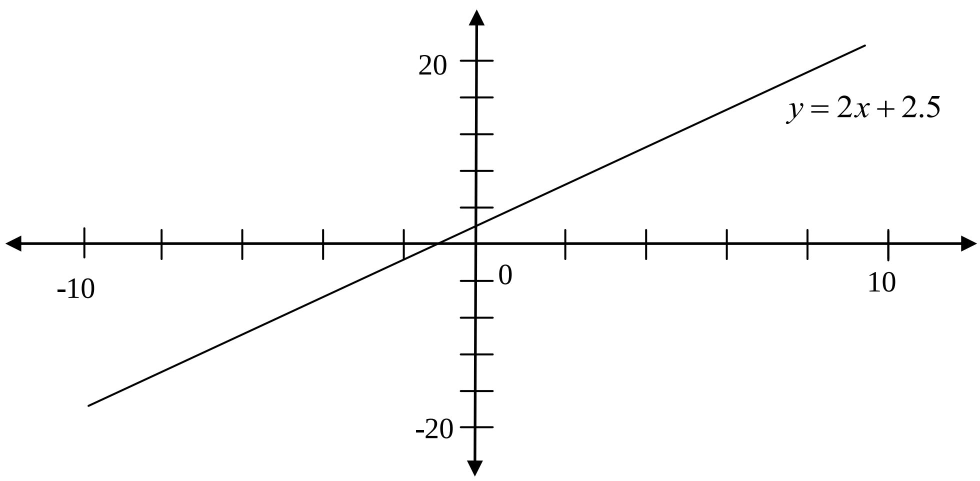
- C is a function of D, as the value of variable C depends upon the value of D,

and also that

- $2D + 2.5$ is a function of D, as the value of $2D + 2.5$ depends on the value of D.

(c) In mathematics we study the characteristics of functions in anticipation of using them in future applications – *just like we study the rules and moves of a game before playing it*. In situations when we have no specific applications in mind, it is common to use letters like x and y to stand for the independent and dependent variables.

For example, the graph of the $y = 2x + 2.5$ is



2.2 Function Notation

Function symbols like '*Kilolitres*(10)' are commonly used to indicate the value of the dependent variable for a specific value of the independent variable (eg. *Time* = 10). This is called *function notation* and is a very useful when writing out mathematics.

Examples

(a) If you return to petrol storage graph, then you can see that

- *Kilolitres*(4) = 100
- *Kilolitres*(12) = 120
- *Kilolitres*(20) = 125

Writing *Kilolitres*(4) = 100 is a lot easier than writing

'*at 4 hours past midnight, there was 100 kL of petrol in storage*'.

(b) If the cost ($) of a taxi ride is $C = 2D + 2.5$, where D is the distance travelled in kilometres, then

- $C(0) = 2.5$ (at flagfall)
- $C(10) = 22.5$ (cost of a 10 km journey)

(c) If $y = 2x + 2.5$, then

- $y(0) = 2.5$
- $y(10) = 22.5$
- $y(-10) = -17.5$

Problems 2

1. Use the petrol storage graph to

- **a.** evaluate *Kilolitres*(8)
- **b.** solve *Kilolitres*(t) = 115
- **c.** solve *Kilolitres*(t) = 145

2. If the cost of a taxi ride is $C = 2D + 2.5$,

- **a.** evaluate $C(5)$
- **b.** solve $C(D) = 8.5$

3. If $y = 2x + 2.5$,

- **a.** evaluate $y(-2)$
- **b.** solve $y(x) = 16.5$
- **c.** evaluate $y(w)$
- **d.** evaluate $y(w + 1)$

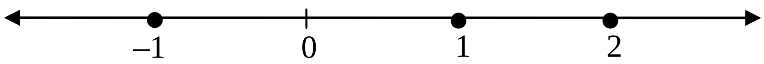
3

Intervals on Number Lines

We often need to describe the values that variables take, for example in the petrol storage example, *Petrol* had values between between 90 kL and 150 kL when Time took values from 4 hours to 24 hours. We use inequality signs to describe intervals on number lines.

3.1 Inequality signs

When two real numbers are compared, the number that is further to the right on the real line is said to be larger, and the number that is to the left is said to be smaller.



Example

(a) 2 is greater than both 1 and –1

(b) 0 is less than 2 but greater than –1

Four *inequality signs* are used to compare numbers:

Sign	**Interpretation**	**Examples**
$<$	'is less than'	$0 < 2$, $-5 < -4$, $3 < 4 < 5$
$\leq$	'is less than or is equal to'	$0 \leq 2$, $3 \leq 3$, $-5 \leq -4 \leq -3$
$>$	'is greater than'	$2 > 0$, $-4 > -5$, $5 > 4 > 3$
$\geq$	'is greater than or is equal to'	$2 \geq 0$, $3 \geq 3$, $-3 \geq -4 \geq -5$

When more than two numbers are compared, then they are should be compared *in ascending order*, in the same order as on a number line.

Example

(a) $0 < 1 < 2$ is correct

(b) $2 > 1 > 0$ is written wrongly

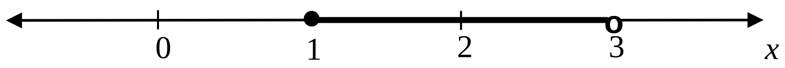
(c) $0 < 2 > 1$ looks very confusing.

3.2 Intervals

The most common subsets of the real line are intervals. An *interval* is the part of the real line between two *endpoints*.

Example

The interval drawn on the x-axis below represents the set of x-values between endpoints 1 and 3. The filled in circle at 1 indicates that 1 is included in the interval, and the empty circle at 3 indicates that 3 is not included in the interval.



This interval can also be represented in symbols by $\{x : 1 \le x < 3\}$.

- curly brackets {} indicate a set or collection of numbers,
- x is the variable taking the values on the number line
- the colon ‘ : ’stands for ‘such that’ or ‘for which’
- the inequality shows the actual values that x takes.

The notation $\{x : 1 \le x < 3\}$ is read aloud as

those values of x such that x is greater than or equal to 1 and less than 3.

This interval is said to be

- *closed at 1* as it contains endpoint 1, and
- *open at 3* as it doesn’t contain endpoint 3.

Other examples

(a) The whole number line doesn’t contain any endpoints. It can be written in set notation as

$$\{x : -\infty < x < \infty\}.$$

(b) The set of all positive numbers can be written as either

$$\{x : 0 < x < \infty\} \quad \text{or} \quad \{x : x > 0\}.$$

Problems 3

1. Write the following intervals in set notation.

a.

b.

c.

2. Draw the following intervals.

a. $\{x : 0 < x < 1\}$

b. $\{y : -2 \leq y \leq 1\}$

c. $\{L : 1 < L \leq 2\}$

d. $\{t : 1 \leq t < 2\}$

3.3 Domains and Ranges of Functions

Definition

The *domain* of a function is the set of values the independent variable takes.

The *range* of a function is the set of values taken by the function.

Example

In the petrol storage example, *Kiolitres*(*T*) is a function of *Time* (*T*) with

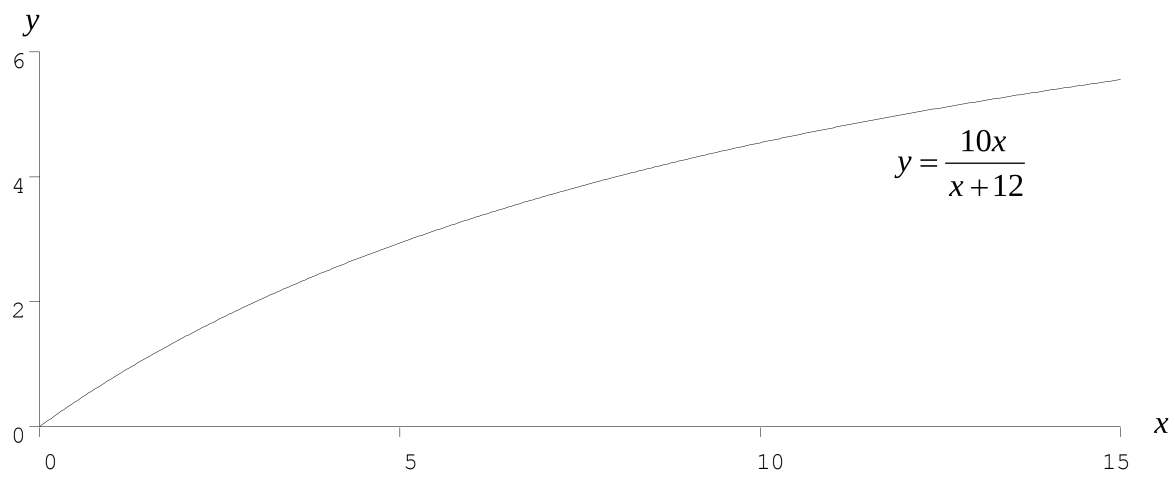
- domain $\{T : 4 \leq T \leq 24\}$
- range $\{K : 90 \leq K \leq 150\}$

Problems 3 (cont.)

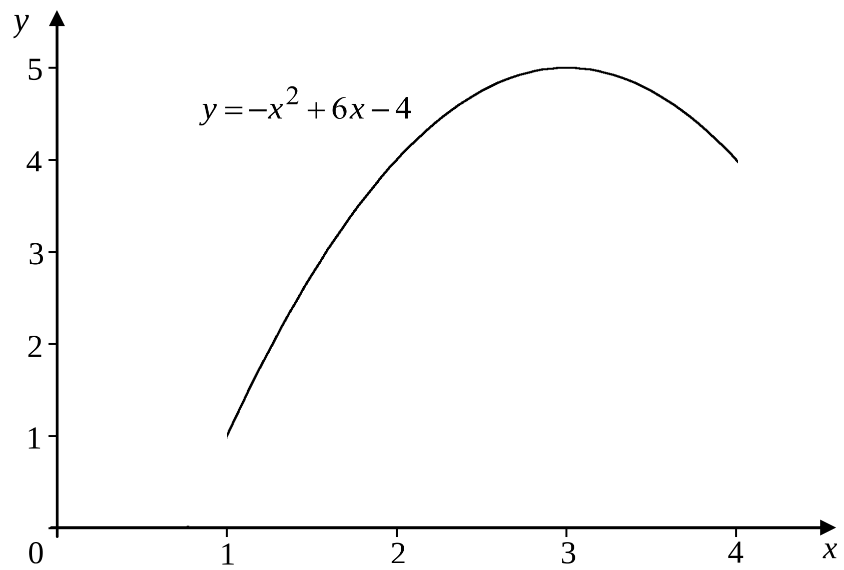
3. Children with aged between 4 and 12 years are given the dose

$$d = \frac{10a}{a+12} \text{ ml},$$

of a certain medication, where a is age in years. Write down the domain of this function, and use the graph below to find the range.



4. The graph of the function $y = -x^2 + 6x - 4$ is shown below using a particular domain. What is the domain and range of this function according to this graph?



A

Appendix: List of Words & Symbols

Section	**Words**	**Symbols**
1.1	natural numbers	**N**
	set	{}
	closed under addition and multiplication	
1.2	whole numbers	
1.3	number line	
	origin	**0**
	integers	**Z**
	closed under subtraction	
1.4	rational number	**Q**
	closed under division	
	irrational number	
1.4	real number	**R**
	real line	
	dense	
2.1	function	
	dependent and independent variables	
2.2	function notation	variable(x)
3.1	inequality sign	$<, \leq, >, \geq$
3.2	endpoints	
	interval	{ : }
3.3	domain	
	range	

.

B

Appendix: Answers

Answers 1

1. As each can be written in the form $\frac{m}{n}$ for integers *m* and *n*, they are rational numbers. (eg. $1\frac{1}{2} = \frac{3}{2}$ and $-0.2 = \frac{-2}{10}$)

2. $\frac{1}{2} + \frac{3}{4} = \frac{10}{8}$; $\frac{1}{2} - \frac{3}{4} = \frac{-2}{8}$; $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$; $\frac{1}{2} \div \frac{3}{4} = \frac{2}{3}$

3. Every integer can be written as a fraction with the integer in the numerator, and 1 in the denominator, eg $\frac{m}{1}$.

4. An irrational number can't be written as a fraction with integers in the numerator and denominator.

5. **a.** $\sqrt{2}$ is irrational number and 1.4142136 is a rational number

b. 1.414 and 1.415

6. If $\sqrt{2} + 1$ is a rational number, then it is equal to a fraction $\frac{m}{n}$ for integers *m* and *n*. This means that $\sqrt{2} = \frac{m}{n} - 1$. But this is <u>false</u> as $\sqrt{2}$ is irrational, so $\sqrt{2} + 1$ is not a rational number.

Answers 2

1. **a.** $Kilolitres(8) = 97.5$ **b.** $t = 12$ **c.** $t =$ 12.5 or 17

2. **a.** $C(5) = 12.5$ **b.** $D = 3$

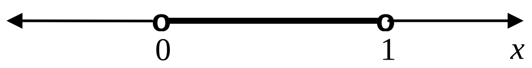
3. **a.** $y(-2) = -1.5$ **b.** $x = 7$ **c.** $y(w) = 2w + 2.5$

d. $y(w+1) = 2(w+1) + 2.5 = 2w + 4.5$

Answers 3

1. **a.** $\{x : 0 < x < 10\}$ **b.** $\{y : 0 < y \le 10\}$ **c.** $\{L : L \ge 5\}$

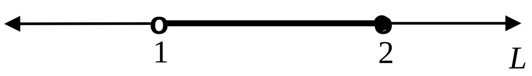
2.



$\{x : 0 < x < 1\}$

$\{x : -2 \le x \le 1\}$

3.



$\{L : 1 < L \le 2\}$

$\{t : 1 \le t < 2\}$

a. domain $\{a : 4 \le a \le 12\}$ **b.** range $\{d : 2.5 \le d \le 5\}$

4. **a.** domain $\{x : 1 \le x \le 4\}$ **b.** range $\{y : 1 \le y \le 5\}$