Maths Learning Service: Revision ***Mathematics IA***

Complex Numbers

The *imaginary number* $i = \sqrt{-1}$ is an extension to the real number system which allows us to solve equations such as

$$x^2 = -1.$$

A *complex number* is any number of the form $z = a + bi$, where a and b are real numbers.

Note: All numbers involving i can be written in this form.

Examples: (a) $i^2 + i^3$

$$\begin{aligned} &= -1 + i^2 i \\ &= -1 - i \end{aligned}$$

(b) $\dfrac{2i-3}{i+1}$

$$\begin{aligned} &= \frac{2i-3}{i+1} \times \frac{i-1}{i-1} \\ &= \frac{2i^2 - 5i + 3}{i^2 - 1} \\ &= \frac{-2 - 5i + 3}{-1 - 1} \\ &= \frac{1 - 5i}{-2} = -\frac{1}{2} + \frac{5}{2}i \end{aligned}$$

Notes: (1) In $z = a + bi$, a is the *real part of* z.
b is the *imaginary part of* z.
(2) If $b = 0$, z is a real number.
If $a = 0$, z is a *purely imaginary number*.

$\overline{z} = a - bi$ is the *complex conjugate* of $z = a + bi$. The solutions to quadratic equations are complex conjugates.

Example: Solve $x^2 - 2x + 10 = 0$.

Solution: Using the quadratic formula with $a = 1, b = -2$ and $c = 10$,

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 10}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{-36}}{2} \\ &= \frac{2 \pm \sqrt{36 \times -1}}{2} \\ &= \frac{2 \pm 6i}{2} \\ &= 1 - 3i \text{ or } 1 + 3i. \end{aligned}$$

Exercises

(1) Rewrite the following in the form $a + bi$:

(a) $(2i + 6) + (5i - 1)$ (b) $(3i + 2)(i - 1)$

(c) $\dfrac{i}{1 + 2i}$ (d) $\dfrac{1}{2 - i} + \dfrac{2}{2 + i}$

(2) If $z_1 = 1 - 2i$ and $z_2 = 2 + i$, find:

(a) $\overline{z_1} + \overline{z_2}$ (b) $\overline{z_1 + z_2}$

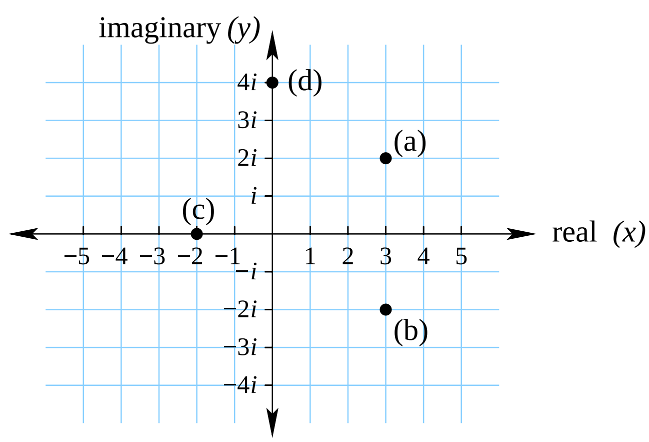
(3) Solve for x:

(a) $x^2 - 10x + 29 = 0$ (b) $\dfrac{1}{x} = 1 - 2x$

The Complex (Argand) Plane

The complex number $z = a + bi$ can be represented on a number *plane* (rather than a number *line*) with co-ordinates (a, b). The $x-$axis represents the real component of z and the $y-$axis represents the imaginary component.

Examples: (a) $3 + 2i$ (b) $3 - 2i$ (c) -2 (d) $4i$



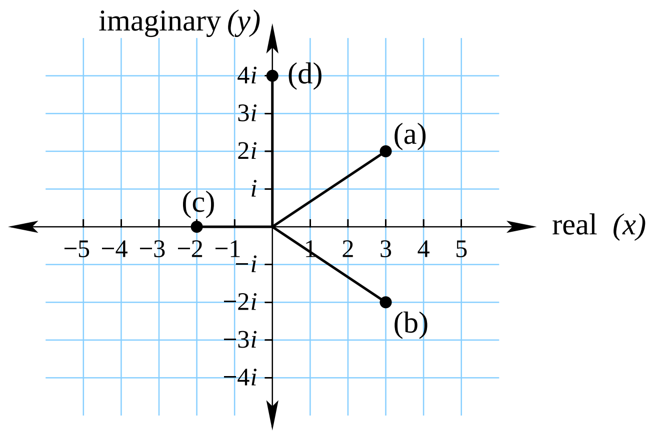
The distance of a complex number from the origin of the Argand Plane is called the *modulus* of the complex number z (or $|z|$). By Pythagoras' Theorem:

$$|z| = \sqrt{a^2 + b^2}.$$

Examples: (a) $|3 + 2i|$ (b) $|3 - 2i|$ (c) $|-2|$ (d) $|4i|$

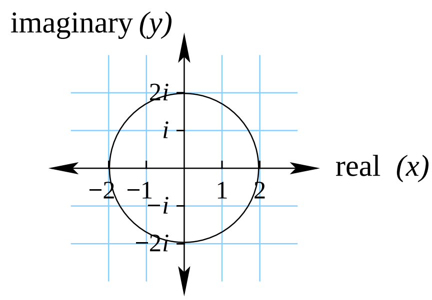
$= \sqrt{3^2 + 2^2}$ $= \sqrt{3^2 + (-2)^2}$ $= \sqrt{(-2)^2 + 0}$ $= \sqrt{0 + 4^2}$

$= \sqrt{13}$ $= \sqrt{13}$ $= 2$ $= 4$



Example: Find all complex numbers $z = x + iy$ which satisfy $|z| = 2$.

Solution: This statement represents all complex numbers whose modulus (or in vector terms length) is 2. Hence the solution is a circle of radius 2 centered on $(0, 0)$ in the Argand Plane.



An algebraic solution is as follows:

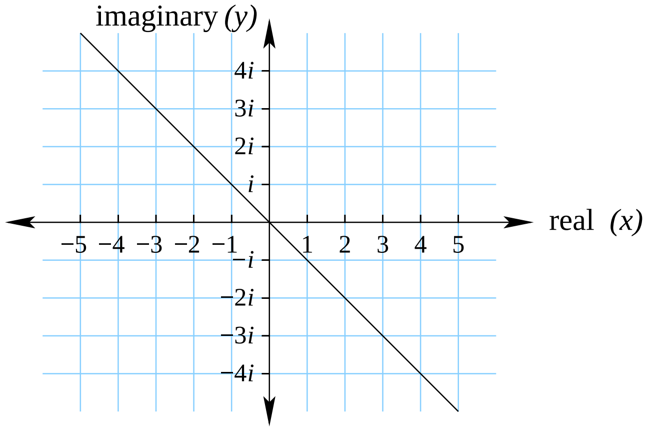
$$\begin{aligned} |z| &= 2 \\ \Rightarrow \quad |x+iy| &= 2 \\ \Rightarrow \quad \sqrt{x^2+y^2} &= 2 \\ \Rightarrow \quad x^2+y^2 &= 4, \end{aligned}$$

which is the cartesian equation of a circle centered on $(0, 0)$.

Example: Find all complex numbers $z = x + iy$ which satisfy $|z + 1| = |z - i|$.

Solution: This can be solved geometrically as well but here is an algebraic solution:

$$\begin{aligned} |z+1| &= |z-i| \\ \Rightarrow \quad |x+iy+1| &= |x+iy-i| \\ \Rightarrow \quad |(x+1)+iy| &= |x+i(y-1)| \\ \Rightarrow \quad \sqrt{(x+1)^2+y^2} &= \sqrt{x^2+(y-1)^2} \\ \Rightarrow \quad (x+1)^2+y^2 &= x^2+(y-1)^2 \\ \Rightarrow \quad x^2+2x+1+y^2 &= x^2+y^2-2y+1 \\ \Rightarrow \quad 2x &= -2y \\ \Rightarrow \quad y &= -x. \end{aligned}$$



(The reader is left to investigate why this solution works geometrically.)

Exercises

(4) Find:

(a) $|1+i|$ (b) $|1-i|$ (c) $|-6i|$

(5) Find all complex numbers $z = x + iy$ which satisfy:

(a) $|z+1| = 1$ (b) $|z| = |z+1|$

Answers to Exercises

(1) (a) $5+7i$ (b) $-5-i$

(c) $\frac{2}{5}+\frac{1}{5}i$ (d) $\frac{6}{5}-\frac{1}{5}i$

(2) (a) $3+i$ (b) $3+i$

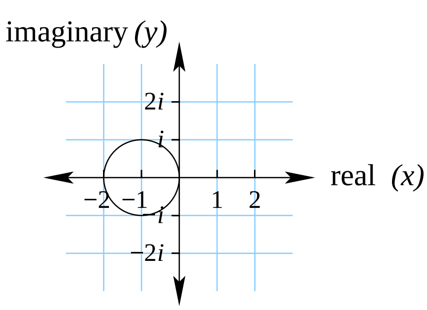
(This example demonstrates the property that $\overline{z_1}+\overline{z_2} = \overline{z_1+z_2}$.)

(3) (a) $5-2i$ or $5+2i$ (b) $\frac{1}{4}-\frac{\sqrt{7}}{4}i$ or $\frac{1}{4}+\frac{\sqrt{7}}{4}i$

(4) (a) $\sqrt{2}$ (b) $\sqrt{2}$ (c) 6

(5) (a) $(x+1)^2+y^2 = 1$

(A circle of radius 1 centred on $(-1,0)$)



(b) $x = -\frac{1}{2}$

(A vertical line)