# MathsStart

THE UNIVERSITY
*of* ADELAIDE

## Topic 3

# Quadratic Functions

$x = 3$

$$\begin{cases} y = x^2 - 6x + 8 \\ y = (x-2)(x-4) \\ y = (x-3)^2 - 1 \end{cases}$$

$(3, -1)$

## MATHS LEARNING CENTRE

# This Topic...

This topic introduces quadratic functions, their graphs and their important characteristics. Quadratic functions are widely used in mathematics and statistics. They are found in applied and theoretical mathematics, and are used to model non-linear relationships between variables in statistics. The module covers the algebra and graphing skills needed for analysing and using quadratic functions.

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## ── Prerequisites ────────────────────────────────

You will need to solve quadratic equations in this module. This is covered in the appendix.

## ── Contents ────────────────────────────────

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# 1
# Quadratic Functions and Parabolas

## 1.1 Quadratic functions

In a quadratic function, the highest power of $x$ is 2.

*Examples*

(a)     $y = x^2 + 2x + 3$ is a quadratic function of $x$.

(b)     $f(t) = t^2 - t + 10$ is quadratic in $t$.

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The general form of a quadratic function of $x$ is $ax^2 + bx + c$, for some numbers a, b and c.

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The graphs of quadratic functions are called *parabolas*. The important characteristics of quadratic functions are found from their graphs.

## 1.2 Examples of quadratic functions and parabolas

We often see parabolas in the world around us, in equipment and in visual design.

*Example*

The mirrors in torches and car headlights are shaped like parabolas; microwave receivers on the roofs of buildings and satellite TV receivers also have parabolic shapes. Parabolas have the special property that radiation generated at a point, called the *focus*, is reflected in parallel rays off the parabola. Torch bulbs are placed at this point.  Also, parallel rays of incoming radiation are concentrated at the *focus*. Receptor devices are placed at this point of high intensity.  Every parabola has this property, and no other type of curve does.
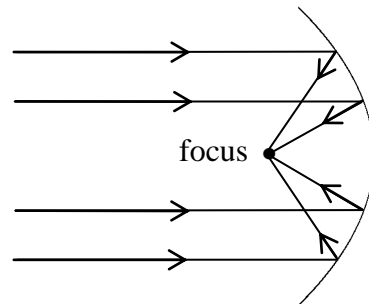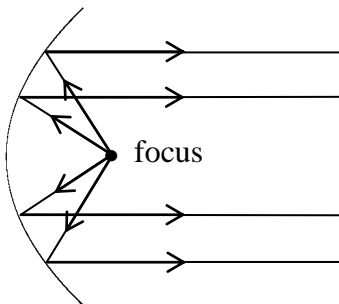
*Example*

The graph below shows how a population is changing (the dots are data points).  In this example, the relationship between population and time is curved, and is approximated by a quadratic function.  We think of the quadratic function as a model for the population decay. The model can be used to predict how fast the population will change in the future.
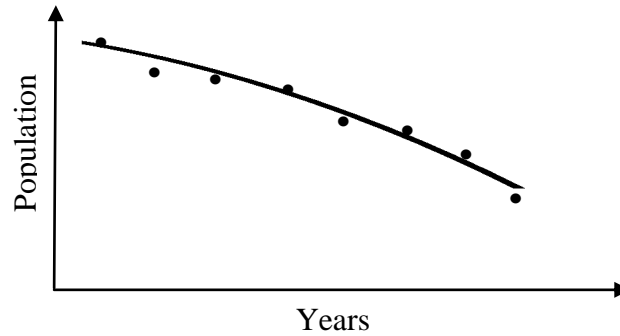


*Example*

A projectile has a parabolic path.  The equation of its path can be calculated from three data points, and can be used to estimate its maximum height reached and landing.
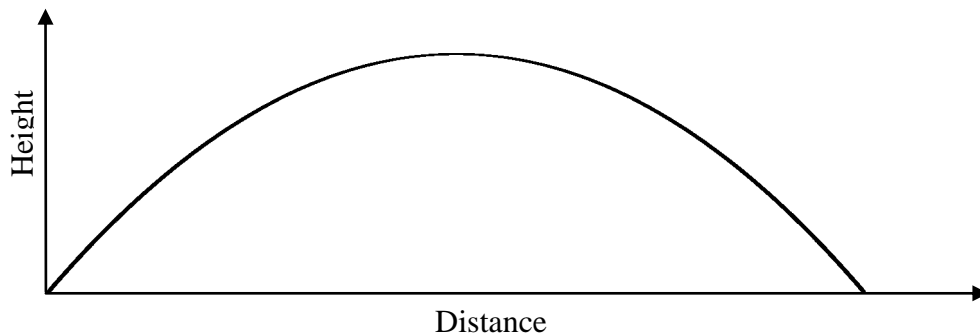


We use *x* and *y* as the independent and dependent variables when we study the general properties of quadratic functions, and place no restriction on the domain of the functions. The important characteristics of quadratic functions are found from their graphs.
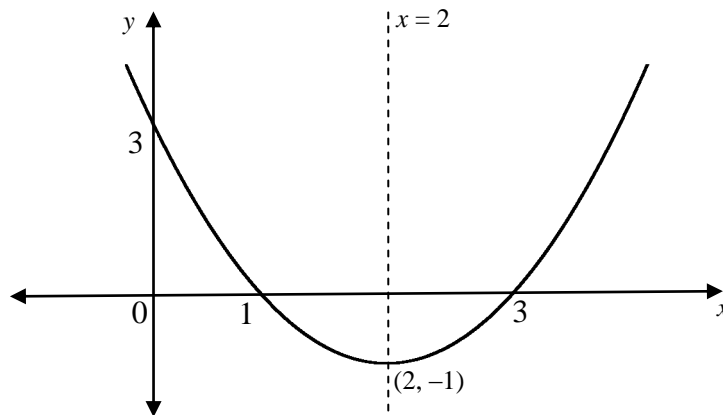
These are

- the *x*- and *y-intercepts*,
- the *vertex* (or *turning point*),
- the *line of symmetry*.

*Example*

The graph of the quadratic function $y = x^2 - 4x + 3$ is shown below. The $x$-intercepts of the parabola are (1, 0) and (3, 0), the $y$-intercept is (0, 3) and the vertex or turning point is (2, −1). You can see that the parabola is *symmetric about the line x = 2*, in the sense that this line divides the parabola into two parts, each of which is a mirror image of the other.



The parabola above was drawn with a mathematical graphing package. This module shows how to draw parabolas by hand. These are not intended to be accurate representations of parabolas, but are used to guide us when solving problems. They should display the most important features of parabolas: the intercepts, the vertex and the line of symmetry (unless there is a good reason not to). Although the focus is an important feature of a parabola, it is not used in most applications and is not usually shown on sketches.

## Problems 1

1. Check that $y = 1 - (x - 1)^2$ is a parabola by rewriting it in the general form $y = ax^2 + bx + c$.

2. Complete the following table, then use it to draw the parabolas $y = x^2$ and $y = 1 - (x - 1)^2$.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y = x^2$ | | | | | | | |
| $y = 1 - (x - 1)^2$ | | | | | | | |

3. What is the line of symmetry for each parabola?

4. What are the ranges of the quadratic functions $x^2$ and $1 - (x - 1)^2$?
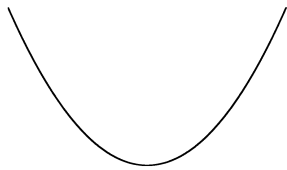
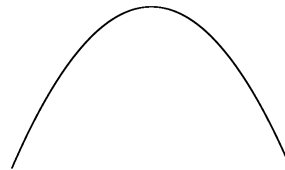We need to see the graph to work out the range of a function.

# 2

# Sketching Parabolas

## 2.1 The shape of a parabola

Parabolas have *two* orientations: concave up and concave down. (Note that the word "convex" has a different meaning in mathematics, so we do not use it in this context.)



Concave up                    Concave down

The orientation of a parabola can be found from its equation.

*Example*

The parabola $y = x^2$ is concave up. To see this, imagine how the value of $y$ will change when we substitute very large $x$-values into the equation, such as $x = 1\ 000$, $x = 1\ 000\ 000$, $x = 1\ 000\ 000\ 000$, etc. As $x$ is given larger and larger values, the value of $y$ becomes very large and *positive*. So the parabola must be concave up.

In comparison, the parabola $y = -x^2$ is concave down. As $x$ is given larger and larger values, the value of $y$ becomes very large and *negative*. So the parabola must be concave down.

*Example*

The parabola $y = 50x^2 - 230x + 107$ is concave up. To see this, imagine how $y$ changes when very large values of $x$ are substituted into the equation. When $x$ is very large, the value of $50x^2$ will be much larger than the value of $-230x$ and also much larger than 107, so the value of $y$ would be very large and *positive* (check this). This would be true for all very large values of $x$, so the parabola must be concave up.

> We don't include the case $a = 0$, as $y = bx + c$ is the equation of a straight line, not a parabola!

In general:

> The parabola $y = ax^2 + bx + c$ is concave up if $a > 0$ and concave down if $a < 0$.

## Problems 2A

Rewrite the following parabolas in the form $y = ax^2 + bx + c$, and state whether they are concave up or concave down.

(a)     $y = (x - 3)^2 + 4$         (b)     $y = 3 - (x + 2)^2$         (c)     $y = 10(x - 2)^2 - 15$

(d)     $y - x^2 + 2x + 3 = 0$     (e)     $s = (t - 3)^2 + 2t + 1$     (f)     $P = (w - 3)^2 + (w + 1)^2$

### 2.2 The intercepts of a parabola

Before we can sketch a parabola, we need to draw scales on the *x*- and *y*-axes. This can be done once the *x*- and *y*-intercepts are known.

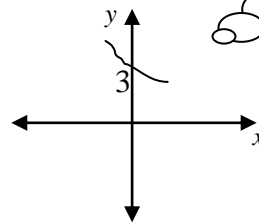The *y*-intercept is where the graph meets the *y*-axis.

*Example*

Find the *y*-intercept of $y = x^2 - 4x + 3$.

> The parabola will look something like this. The y-intercept shows what scale we need to use on the y-axis.

## Answer

Put $x = 0$, then

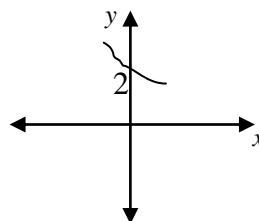$$y = x^2 - 4x + 3$$
$$= 0 - 0 + 3$$
$$= 3.$$

The *y*-intercept is (0, 3).
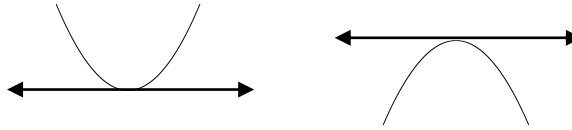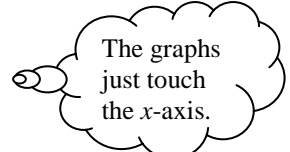


## Example

Find the *y*-intercept of $y = (2 - x)(1 - x)$.

*Answer*
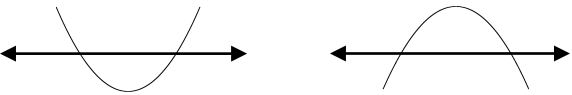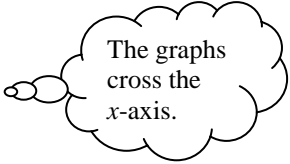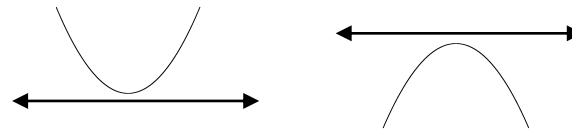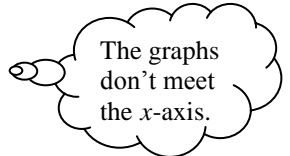
Put $x = 0$, then

$$y = (2 - x)(1 - x)$$
$$= (2 - 0)(1 - 0)$$
$$= 2.$$

The *y*-intercept is (0, 2).

The *x*-intercepts are where the graph meets the *x*-axis. There will be either

• two *x*-intercepts:

> The graphs cross the *x*-axis.

• exactly one intercept:

> The graphs just touch the *x*-axis.

• no intercepts:

> The graphs don't meet the *x*-axis.

*Example*

Find (i) the *x*-intercepts and (ii) the vertex of $y = (x-1)(x-3)$.

*Answer*  (See Appendix: Quadratic Equations)

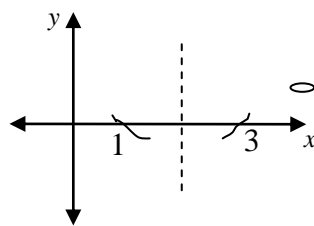(i) The *x*-intercepts.

Put $y = 0$, then

$$(x-1)(x-3) = 0.$$

So either   $x - 1 = 0$

> When a product of two numbers is zero, either one number or the other must be zero.

$$x = 1,$$

or   $x - 3 = 0$

$$x = 3.$$

The *x*-intercepts are (1, 0) and (3, 0).

> The *x*-intercepts show what scale should be used on the *x*- axis.

> First find the line of symmetry.

(ii) The line of symmetry and vertex.

The line of symmetry passes through the midpoint of (1, 0) and (3, 0), so it must have equation $x = 2$.

To find the vertex, put $x = 2$.

> Put $x = 2$ as the vertex is on the line of symmetry.

$$y = (x-1)(x-3)$$
$$= (2-1)(2-3)$$
$$= -1$$

The vertex is (2, −1)

*Example*

Find the $x$-intercepts and vertex of $y = 2 + x - x^2$.

*Answer*

(i) The $x$-intercepts.

Put $y = 0$, then

$$2 + x - x^2 = 0$$

Solve by factorising.

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0.$$
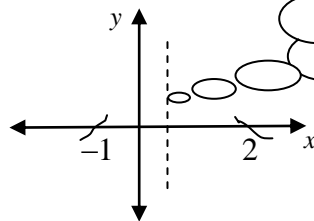
So either   $x + 1 = 0$

$$x = -1,$$

or   $x - 2 = 0$

$$x = 2.$$

The $x$-intercepts are $(-1, 0)$ and $(2, 0)$.

The midpoint of $-1$ and 2 is $\dfrac{(-1) + 2}{2}$ .



(ii) The line of symmetry and vertex.

The equation of the line of symmetry is $x = \dfrac{1}{2}$.

To find the vertex, put $x = \dfrac{1}{2}$.

The vertex is on the line of symmetry.

$$y = 2 + x - x^2$$

$$y = 2 + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{9}{4}$$



The vertex is $(\dfrac{1}{2}, \dfrac{9}{4})$.
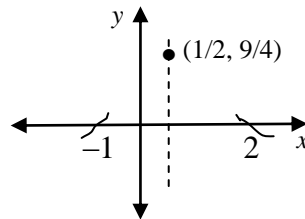
If you cannot solve a quadratic equation by factorisation, then try completing the square or the quadratic formula. These are described in the appendix. The line of symmetry can also be found from the formula below.

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The parabola $y = ax^2 + bx + c$ has line of symmetry $x = -\dfrac{b}{2a}$.

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*Example*

Find the *x*-intercepts and vertex of $y = 2x^2 - x - 2$.

*Answer*

(i) The *x*-intercepts.

Put $y = 0$, then

$$2x^2 - x - 2 = 0.$$

Using the quadratic formula:

$$a = 2, \; b = -1, \; c = -2$$

$$b^2 - 4ac = (-1)^2 - 4 \times 2 \times (-2) = 17 > 0 \;\Rightarrow\; \text{the equation has two solutions.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{17}}{2 \times 2}$$

$$= 1.2808 \text{ or } -0.7808 \quad \text{(4 d.p.)}$$

The intercepts are $(-0.7808, 0)$ and $(1.2808, 0)$.

(ii) Line of symmetry and vertex.

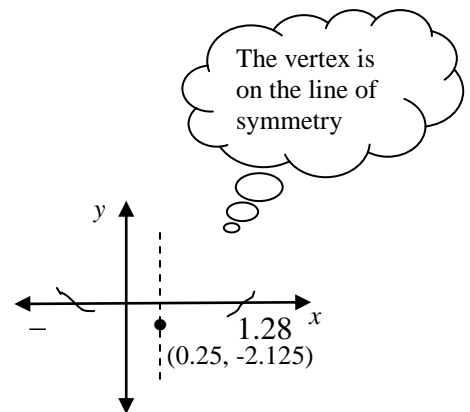The line of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{-1}{2 \times 2} = \dfrac{1}{4}$ .

To find the vertex, put $x = \dfrac{1}{4}$.

$$y = 2x^2 - x - 2 = 2 \times \frac{1}{4} \times \frac{1}{4} - \frac{1}{4} - 2 = -\frac{17}{8}$$

The vertex is $(\dfrac{1}{4}, -\dfrac{17}{8})$.



The vertex is on the line of symmetry

## 2.3 Sketching a parabola

A sketch of a parabola should show the intercepts, the line of symmetry and the vertex.

Parabolas without *x*-intercepts will be covered in section 3.

*Example*

Sketch the parabola $y = x^2 - 4x + 3$.

*Answer*

(a) Shape.

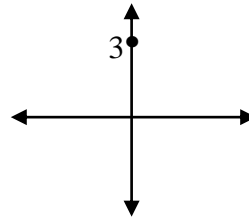The parabola is concave up as the coefficient of *x* is greater than 0.

(b) Intercepts.

Put $x = 0$, then

$$y = x^2 - 4x + 3$$
$$= 0 - 0 + 3$$
$$= 3.$$

The *y*-intercept is (0, 3).

Put $y = 0$, then

$$x^2 - 4x + 3 = 0$$
$$(x - 1)(x + 3) = 0.$$

So either $\quad x - 1 = 0 \Rightarrow x = 1,$

or $\quad x - 3 = 0 \Rightarrow x = 3$

The *x*-intercepts are (1, 0) and (3, 0).
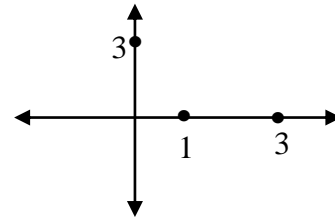
(c) Line of symmetry and vertex.

The line of symmetry is $x = 2$, as it passes through the midpoint of (1, 0) and (3, 0).

Check: $\quad x = -\dfrac{b}{2a} = -\dfrac{-4}{2 \times 1} = 2.$

Check using the formula for the line of symmetry.

Put $x = 2$, then

$$y = x^2 - 4x + 3$$
$$= 4 - 8 + 3$$
$$= -1.$$

The vertex is (2, −1).
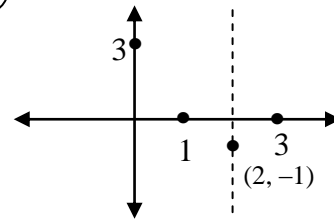
(d) Sketch.
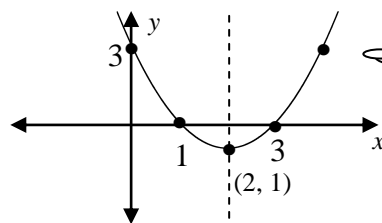
The point (4, 3) is on the parabola because the parabola is symmetric about the line $x = 2$.

*Example*

Sketch the parabola $y = 1 - (x - 1)^2$.

*Answer*

Expand brackets then simplify.

$$y = 1 - (x - 1)^2$$
$$= 1 - (x - 1)(x - 1)$$
$$= 1 - (x^2 - x - x + 1)$$
$$= -x^2 + 2x.$$

(a) Shape.

The parabola is concave down as the coefficient of $x$ is −1.

(b) Intercepts.

Put $x = 0$, then

$$y = -x^2 + 2x$$
$$= -0 + 0$$
$$= 0.$$

The $y$-intercept is (0, 0).

Put $y = 0$, then

$$-x^2 + 2x = 0$$
$$x^2 - 2x = 0$$
$$x(x - 2) = 0.$$
$$\text{So either} \quad x = 0,$$
$$\text{or} \quad x - 2 = 0$$
$$x = 2.$$
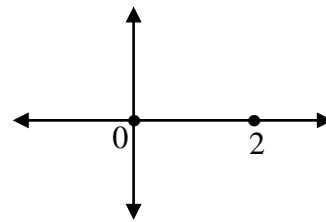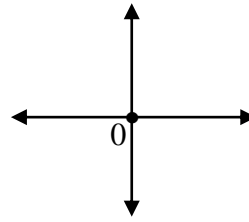
The $x$-intercepts are (0, 0) and (2, 0).

(c) Line of symmetry and vertex.

The line of symmetry is $x = 1$, as it passes through the midpoint of (0, 0) and (2, 0).

Check: $x = -\dfrac{b}{2a} = -\dfrac{2}{2 \times (-1)} = 1.$

Put $x = 1$, then

$$y = -x^2 + 2x$$
$$= -1 + 2 = 1.$$

The vertex is (1, 1).
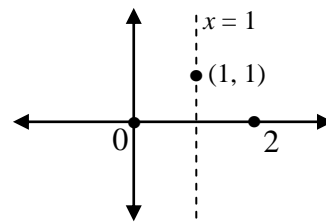
(d) Sketch.
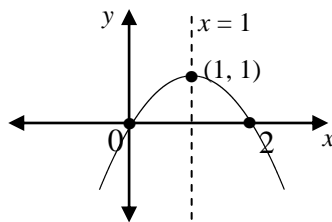
*Note.* Sometimes the method above does not produce enough points for a sketch. If this happens, then you should calculate more points by substitution and by using symmetry.

*Example*

The parabola $y = x^2$ has $y$-intercept $(0, 0)$, $x$-intercept $(0, 0)$ and line of symmetry $x = 0$.

Four more points were calculated for the sketch below: $(1, 1)$, $(-1, 1)$, $(2, 4)$, $(-2, 4)$.
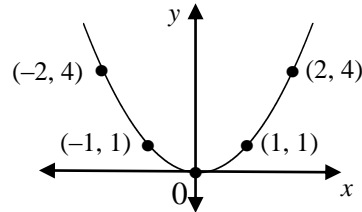


## Problems 2B

1. Sketch the following parabolas, showing their intercepts, line of symmetry and vertex. In each case state the domain and range of the quadratic function.

(a)   $y = x^2 - 4x$     (b)   $y = x^2 - 5x + 6$          (c)   $y = x^2 + 2x - 8$

(d)   $y = 6 - x - x^2$   (e)   $f(x) = x^2 - 4x + 4$   (f)   $y = x^2 + 2x - 8.3$

2. If rice plants are sown at a density of $x$ plants per square metre in a certain location, then the yield of rice is $0.01x(10 - 0.5x)$ kg per square metre. What is the maximum yield of rice per acre, and what density of plants gives this maximum?

# 3
# Transformations

The graph of a function is very useful in mathematics, as it displays the important features of the function. Graphical representations can be simplified by the use of transformations such as *translations, reflections*, and *dilations*.

## 3.1 Translations

A *translation* of a geometric figure is a transformation in which every point is moved the same distance in the same direction.

*Example*

In the diagram below, each point on the book is shifted 4 cm to the right.
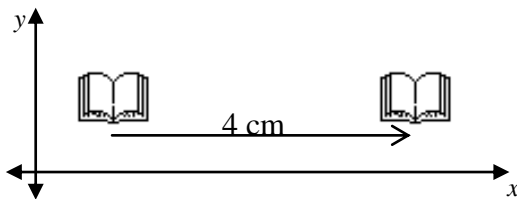


*Example*

In the diagram below, each point on the book is shifted 1 cm upwards.



Question: If the parabola $y = x^2$ is shifted $h$ units to the right, what will be its new equation?
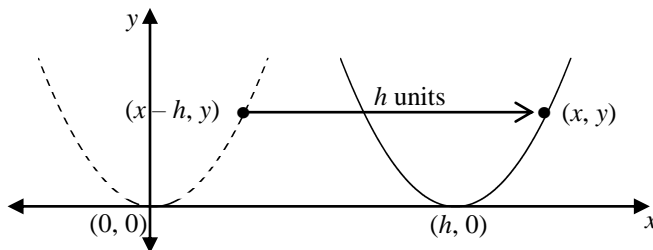
To answer this question, remember that the equation of a parabola is a formula which describes the relationship between the *x*- and *y*-coordinates of points on the parabola.

If  the point $(x, y)$ is on the *new* parabola, then the point $(x - h, y)$

> • must be on the *original*  parabola, and so
>
> • must satisfy $y = (x - h)^2$.

As the points $(x, y)$ on the new parabola satisfy the equation $y = (x - h)^2$, this is its equation.

*Check*. The point $(0, 0)$ is on the original parabola $y = x^2$, and the point $(h, 0)$ is on new parabola $y = (x - h)^2$.

When the parabola $y = x^2$ is shifted by *h* units *to the left*, we get a similar result.



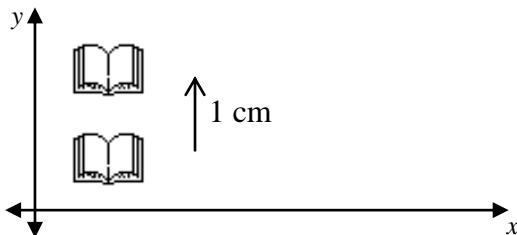If  the point $(x, y)$ is on the *new* parabola, then the point $(x + h, y)$

> • must be on the *original*  parabola, and so
>
> • must satisfy $y = (x + h)^2$.

*Check*. The point $(0, 0)$ is on the original parabola $y = x^2$, and the point $(-h, 0)$ is on new parabola $y = (x + h)^2$.

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If parabola $y = x^2$ is shifted by *h* units to the right, its new equation is $y = (x - h)^2$.

(We interpret $h < 0$ as being meaning shift to the left.)

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Question: If the parabola $y = x^2$ is shifted *k* units upwards, what will be its new equation?
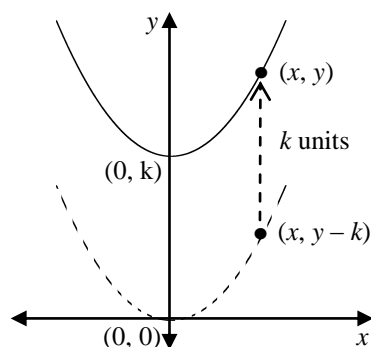


This question can be answered in a similar way to the previous question.

If  the point $(x, y)$ is on the *new* parabola, then the point $(x, \ y - k)$
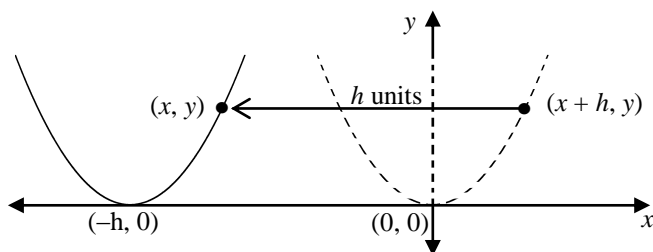
     • must be on the *original*  parabola, and so

     • must satisfy $y - k = x^2$ , ie. $y - \text{coordinate} = (x - \text{coordinate})^2$ , or $y = x^2 + k$ .

As the points $(x, y)$ on the new parabola satisfy the equation $y = x^2 + k$ , this is its equation.

*Check*. The point $(0, 0)$ is on the original parabola $y = x^2$ , and the point $(0, k)$ is on new parabola $y = x^2 + k$ .

> If the parabola $y = x^2$ is shifted $k$ units upwards, its new equation is $y = x^2 + k$ .
> (We interpret $k < 0$ as being meaning shift downwards.)

If we combine the horizontal and vertical transformations we get:

> If the parabola $y = x^2$ is shifted $h$ units to the right and $k$ units upwards, its new
> equation is $y = (x - h)^2 + k$ .

*Example*
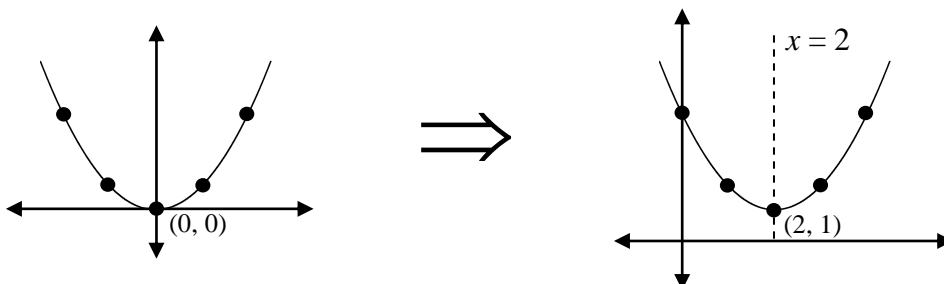
Sketch the parabola $y = x^2 - 4x + 5$ .

*Answer*

Write the equation in the form $y = (x - h)^2 + k$ by completing the square:

$$y = x^2 - 4x + 5$$
$$y - 5 = x^2 - 4x$$
$$y - 5 + 4 = x^2 - 4x + 4$$
$$y - 1 = x^2 - 4x + 4$$
$$y - 1 = (x - 2)^2$$
$$y = (x - 2)^2 + 1.$$

So the parabola can is obtained from $y = x^2$ by shifting it by 2 units to the right and 1 unit upwards.

This example shows how to sketch a parabola with no $x$-intercepts.  It also shows that the shape of the parabola $y = x^2 - 4x + 5$ is *exactly* the same as the shape of the parabola $y = x^2$!

## Problems 3A

1. Sketch the parabola $y = x^2$, then use translations of this to sketch

(a)    $y = x^2 - 6x + 8$              (b)        $y = x^2 - 6x + 9$            (c)        $y = x^2 - 6x + 10$

2. A parabola has the same shape as $y = x^2$ but with vertex at (2, 3).  What is its equation?

## 3.2  Reflections

A *reflection* of a geometric figure is a transformation that results in a mirror image of it.

*Example*

In the diagram below, the parabola $y = x^2$ is reflected across the $x$-axis. Each point $(x, y)$ on the original parabola is reflected to $(x, -y)$ on the reflected parabola.
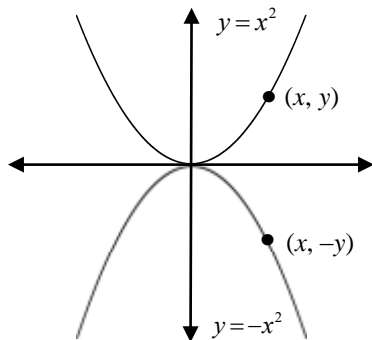


The equation of the reflection of  $y = x^2$ is $y = -x^2$.

*Example*

Sketch the parabola $y = -x^2 + 2x - 3$.

*Answer*

Observe that parabola $y = -x^2 + 2x - 3$ is the reflection of $y = x^2 - 2x + 3$ across the $x$-axis.

First sketch the parabola $y = x^2 - 2x + 3 = (x-1)^2 + 2$ by completing the square. This parabola is concave up with vertex (1, 2) and line of symmetry $x = 1$.

Next reflect your sketch across the $x$-axis.
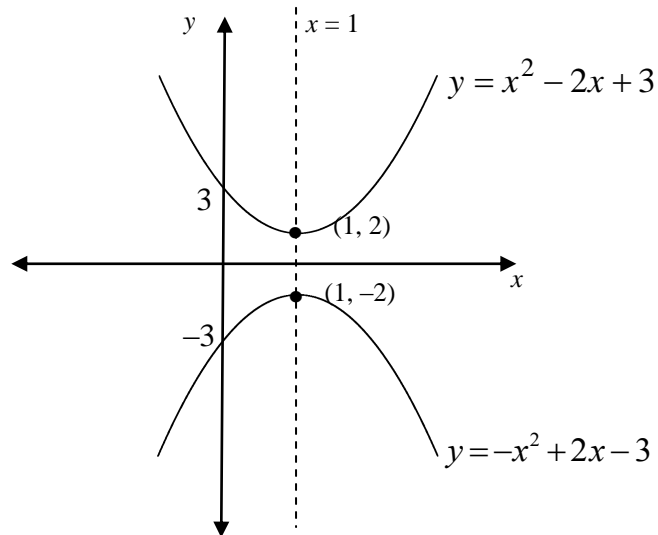
## Problems 3B

Sketch the parabola $y = x^2$, then use translations and reflections to sketch

(a)    $y = 6 + 2x - x^2$         (b)    $y = -6 + 2x - x^2$

## 3.3  Dilations

A *dilation* of a geometric figure is a transformation that results in a figure which is similar to the original, but which may be enlarged or reduced.

*Example*

In the diagram below, the second book is enlarged a factor of 2 in the $y$-direction, and the third book is by a factor of 3.
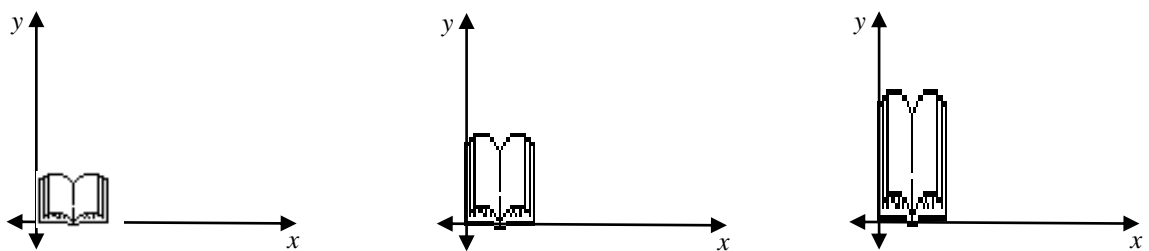


*Example*

In the diagram below, the parabola $y = x^2$ is enlarged by a factor of 2 in the $y$-direction.

As the $y$-coordinate of each point is doubled, the new parabola has equation $y = 2x^2$.
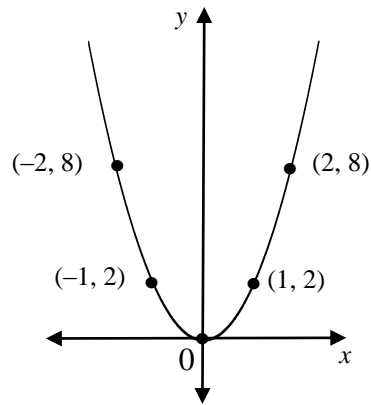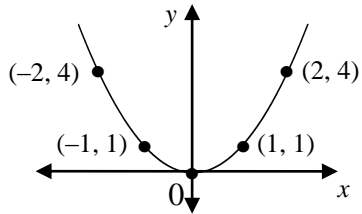
If the parabola $y = x^2$ is dilated by a factor of $a$ ( $> 0$) in the $y$-direction, its new equation is $y = ax^2$.

(The parabola is enlarged if $a > 1$ and is reduced if $a < 1$.)

If we combine translations with dilations, we have:

If the parabola $y = x^2$ is dilated by a factor of $a$, shifted $h$ units to the right and $k$ units upwards, its new equation is $y = a(x - h)^2 + k$.

(The dilation has to be done first for this to be correct, but the two translations can be done in either order.)

# A
# Appendix: Quadratic Equations

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$. This appendix gives three methods for solving quadratic equations: factorisation, completing the square, and the quadratic formula. The first two methods are emphasised in this module to give you more practice at algebra, but you are more likely to use the third method in real applications.

## 4.1 Solving equations by factorisation

When the product of two or more numbers is equal to zero, then one of the numbers must be zero. This idea can be used to solve quadratic equations.

*Example*

Solve the quadratic equation $(x+1)(x-2) = 0$.

*Answer*

$$(x+1)(x-2) = 0$$
$$\text{either} \quad x+1 = 0$$
$$x = -1$$
$$\text{or} \quad x-2 = 0$$
$$x = 2$$

Think of this as a product of two numbers equal to zero, where the two numbers are $x + 1$ and $x - 2$

Answers can be checked by substitution into the original equation.

The solutions are x $= -1$ and 2.

*Example*

Solve the quadratic equation $10x(x-1) = 0$.
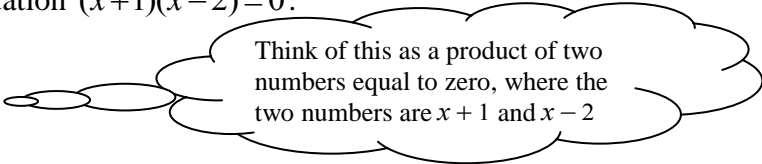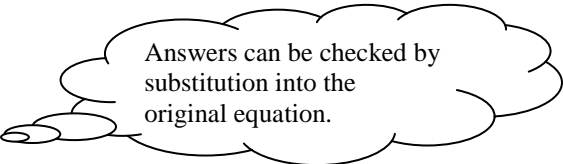
$$10x(x-2) = 0$$
$$\text{either} \quad x = 0$$
$$\text{or} \quad x-2 = 0$$
$$x = 2$$
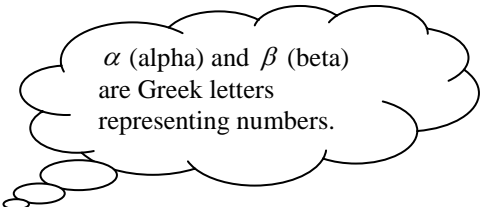
The solutions are x $= 0$ and 2.

$\alpha$ (alpha) and $\beta$ (beta) are Greek letters representing numbers.

To solve an equation like $x^2 + 5x + 4 = 0$, we first need to *factorise* $x^2 + 5x + 4$ into a product of two *factors* $(x+\alpha)(x+\beta)$, where $\alpha$ and $\beta$ are some numbers.

To understand how to do this, we expand the product $(x+\alpha)(x+\beta)$, giving

$$(x+\alpha)(x+\beta) = x^2 + \alpha x + \beta x + \alpha\beta = x^2 + (\alpha+\beta)x + \alpha\beta.$$

You can see that we can write $x^2 + 5x + 4$ in the form $(x + \alpha)(x + \beta)$ if we can find two numbers $\alpha$ and $\beta$ with $\alpha + \beta = 5$ and $\alpha\beta = 4$. *We now try to guess the numbers $\alpha$ and $\beta$.* The product of $\alpha$ and $\beta$ is 4, so the numbers could be 1 & 4 or 2 & 2. The sum of $\alpha$ and $\beta$ is 5, so $\alpha$ and $\beta$ must be 1 & 4. This shows that $x^2 + 5x + 4 = (x + 1)(x + 4)$. *If we couldn't guess these numbers, then we would need to use another method!*
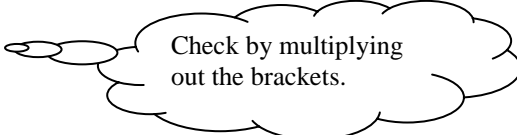
*Example*

Write $x^2 + x - 6$ as a product of two factors.

*Answer*

The numbers $\alpha$ and $\beta$ have product –6, and sum 1.  Pairs of numbers with product –6 are 1 & –6; 2 & –3; 3 & –2; and 6 & –1. Their sum is 1, so $\alpha$ and $\beta$ must be 3 & –2, and

$x^2 + x - 6 = (x + 3)(x - 2)$.       Check by multiplying out the brackets.
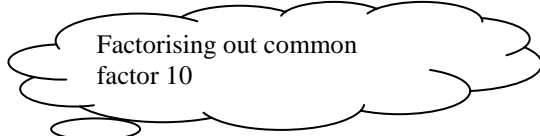
A common factor is a factor which is a factor of every term.  It is a good idea to factorise out the common factors first of all.

*Example*
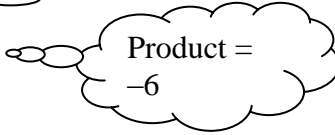
Factorise $10x^2 - 10x - 60$.

*Answer*

$$10x^2 - 10x - 60 = 10(x^2 - x - 6)$$
$$= 10(x - 3)(x + 2)$$

Factorising out common factor 10

Product = –6

*Example*

Solve $3x^2 + 3x - 36 = 0$ by factorising.

*Answer*

$$3x^2 + 3x - 36 = 3(x^2 + x - 12)$$
$$= 3(x + 4)(x - 3)$$

Factorising

$3(x + 4)(x - 3) = 0$

Solving

either   $x + 4 = 0 \Rightarrow x = -4$

or   $x - 3 = 0 \Rightarrow x = 3$

The solutions are $x = 3$ and –4.

*Example*

Solve $x^2 = 7x$.

$$x^2 = 7x$$
$$x^2 - 7x = 0$$
$$x(x - 7) = 0.$$

a common factor

So either $x = 0$,

or $x - 7 = 0 \Rightarrow x = 7$.

The answers are $x = 0$ and 7.

*Warning*

If we tried to solve $x^2 = 7x$ by dividing both sides of the equation by $x$, then we end up with only one solution: $x = 7$. The reason is that when we divide both sides by $x$ we need to assume that $x \neq 0$, so we end up with only the non-zero solution.

## Problems 4A

1.  Factorise the following quadratic expressions.

(a)    $x^2 + 4x + 3$          (b)    $x^2 + 6x + 5$          (c)    $a^2 + 8a + 7$

(e)    $x^2 + x - 2$          (f)    $b^2 - b - 2$          (g)    $n^2 - 4n - 5$

(h)    $s^2 - s - 12$          (i)    $x^2 + x - 12$          (j)    $x^2 + 8x + 12$

(k)    $t^2 - 11t - 12$    (l)    $x^2 - 9x + 14$    (m)    $y^2 - 13y - 14$

2.  Solve the following quadratic equations.

(a)    $x^2 + 5x + 6 = 0$          (b)    $x^2 - 5x + 6 = 0$          (c)    $x^2 + x - 6 = 0$

(d)    $a^2 + 7a + 12 = 0$          (e)    $b^2 + 8b + 12 = 0$          (f)    $p^2 - 12p + 36 = 0$

3.  Multiply out the denominators, then solve the equations below.

(a)    $\dfrac{x}{x^2 + 12} = \dfrac{1}{7}$    (b)    $1 + \dfrac{6}{x - 1} = \dfrac{1}{x + 1}$

## 4.2 Solving equations by completing the square

The method of completing the square can be used to solve any quadratic equation. You can also use it to find out when quadratic equations have no solutions.

The quadratic expressions

$$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2$$

and

$$(x-a)^2 = (x-a)(x-a) = x^2 - 2ax + a^2,$$

where $a$ is some number, are called perfect squares.

*Examples*

Each of the expressions below is a perfect square:

(a)     $(x+3)^2 = (x+3)(x+3) = x^2 + 2 \times 3x + 3^2 = x^2 + 6x + 9$

(b)     $(x+4)^2 = (x+4)(x+4) = x^2 + 2 \times 4x + 4^2 = x^2 + 8x + 16$

(c)     $(x-3)^2 = (x-3)(x-3) = x^2 - 2 \times 3x + 3^2 = x^2 - 6x + 9$

(d)     $(x-4)^2 = (x-4)(x-4) = x^2 - 2 \times 4x + 4^2 = x^2 - 8x + 16$

To *complete the square* means to add a number to a quadratic expression so that the result is a perfect square.

*Examples*

(a)     $x^2 + 6x$ can be made into a perfect square by adding 9, as

$$x^2 + 6x + 9 = x^2 + 2 \times 3x + 3^2 = (x+3)^2.$$

(b)     $x^2 - 8x$ can be made into a perfect square by adding 16, as

$$x^2 - 8x + 16 = x^2 - 2 \times 4x + 4^2 = (x-4)^2.$$

(c)     $x^2 + 6x + 5$ can be made into a perfect square by adding $9 - 5$, as

$$x^2 + 6x + 5 + (9-5) = x^2 + 6x + 9 = x^2 + 2 \times 3x + 3^2 = (x+3)^2.$$

(d)     $x^2 + 6x + 7$ can be made into a perfect square by adding $9 - 7$, as

$$x^2 + 6x + 7 + (9-7) = x^2 + 6x + 9 = x^2 + 2 \times 3x + 3^2 = (x+3)^2.$$

(e)     $x^2 - 8x + 20$ can be made into a perfect square by adding $16 - 20$, as

$$x^2 - 8x + 20 + (16-20) = x^2 - 8x + 16 = x^2 - 2 \times 4x + 4^2 = (x-4)^2.$$

The method of "completing the square" can be used to solve quadratic equations.

*Example*

Solve $x^2 + 6x + 5 = 0$.

*Answer*

$x^2 + 6x + 5 = 0$

$x^2 + 6x + 5 + 4 = 4$
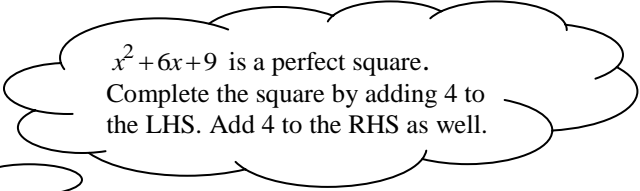
$x^2 + 6x + 9 = 4$

$(x + 3)^2 = 4$
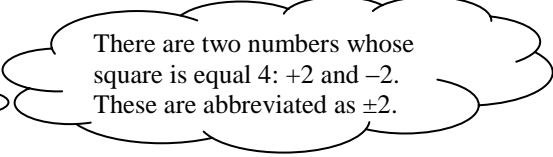
$x + 3 = \pm 2.$

So either   $x + 3 = +2 \Rightarrow x = -1,$

or   $x + 3 = -2 \Rightarrow x = -5.$

The solutions are $x = -1$ and $-5$.

> $x^2 + 6x + 9$ is a perfect square. Complete the square by adding 4 to the LHS. Add 4 to the RHS as well.

> There are two numbers whose square is equal 4: +2 and –2. These are abbreviated as ±2.

*Example*

Solve $x^2 + 6x + 7 = 0$.

> This equation cannot be solved using factorisation.

*Answer*

$x^2 + 6x + 7 = 0$

$x^2 + 6x + 7 + 2 = 2$

$x^2 + 6x + 9 = 2$

$(x + 3)^2 = 2$

$x + 3 = \pm\sqrt{2}.$
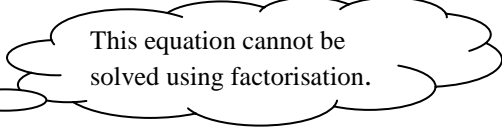
> Completing the square.

So either   $x + 3 = +\sqrt{2}$
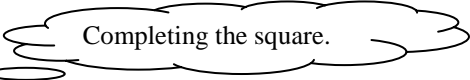
$x = \sqrt{2} - 3 \approx -1.58$   (2 d.p.),

or   $x + 3 = -\sqrt{2}$

$x = -\sqrt{2} - 3 \approx -4.42$   (2 d.p.)

The solutions are $x = \sqrt{2} - 3 \approx -1.58$ and $-\sqrt{2} - 3 \approx -4.42$   (2 d.p.).

*Example*

Solve $x^2 - 8x + 20 = 0$.
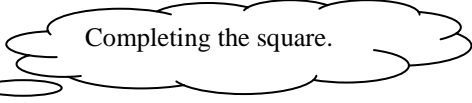
*Answer*

$x^2 - 8x + 20 = 0$

$x^2 - 8x + 20 - 4 = -4$

$x^2 - 8x + 16 = -4$

$(x - 8)^2 = -4$

> Completing the square.

There is no solution *as the square of a number cannot be negative*.

## Problems 4B

1. Solve the following quadratic equations by completing the square, if possible.

(a)    $x^2 - 4x + 3 = 0$          (b)    $x^2 - 4x + 4 = 0$          (c)

        $x^2 - 4x + 5 = 0$

(d)    $x^2 - 3x + 3 = 0$          (e)    $a^2 + 3a - 3 = 0$      (f)    $b^2 + b + 1 = 0$

2. Multiply out the denominators, then solve the equations below.

(a)    $\dfrac{x}{x^2 + 1} = \dfrac{1}{4}$          (b)    $\dfrac{3}{x - 1} = 1 - \dfrac{1}{x + 1}$

## 4.3 Solving equations by using the quadratic formula.

The quadratic formula is useful for solving quadratic equations when the equation has complicated terms.  If you only want an approximate answer, then the quadratic formula is best.  If you want an exact answer, then you may prefer the methods of factorisation or completing the square.  However, all three methods will lead to the same results.

---

**The Quadratic Formula**

(1)  If $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has no solutions.

(2)  If $b^2 - 4ac \geq 0$, then the equation $ax^2 + bx + c = 0$ has two solutions, given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ .$$

(3)  If $b^2 - 4ac = 0$ the two solutions are the same.

---

*Example*

Solve the equation $x^2 + x - 2.01 = 0$.

*Answer*

$$a = 1, b = 1, c = -2.01$$

$$b^2 - 4ac = 1^2 - 4 \times 1 \times (-2.01) = 9.04 > 0 \quad \Rightarrow \text{ there are two solutions}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{9.04}}{2}$$

$$= 1.0033 \text{ and } -2.0033 \ (4 \text{ d.p.})$$

*Note.* The quadratic formula can be proved by solving $ax^2 + bx + c = 0$ using the method of completing the square, eg. $ax^2 + bx + c = 0 \ \Rightarrow \ x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \ \Rightarrow \ $ etc.

## Problems 4C

1.  Without solving the equations below, find out which do not have solutions.

(a) $2x^2 - 3x + 1 = 0$        (b) $5x^2 - 3x + 10 = 0$        (c) $3x^2 + 5x + 2 = 0$

2.  Use the quadratic formula to solve the following quadratic equations, if possible.

(a) $4x^2 - 4x + 0.99 = 0$        (b) $4x^2 - 4x + 1 = 0$        (c) $4x^2 - 4x + 1.01 = 0$

# B

# Answers

## Section 1

1. $y = 1-(x-1)^2 = -x^2 + 2x \implies a = -1, b = 2, c = 0$

2.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y = x^2$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |
| $y = 1-(x-1)^2$ | $-15$ | $-8$ | $-3$ | $0$ | $1$ | $0$ | $-3$ |
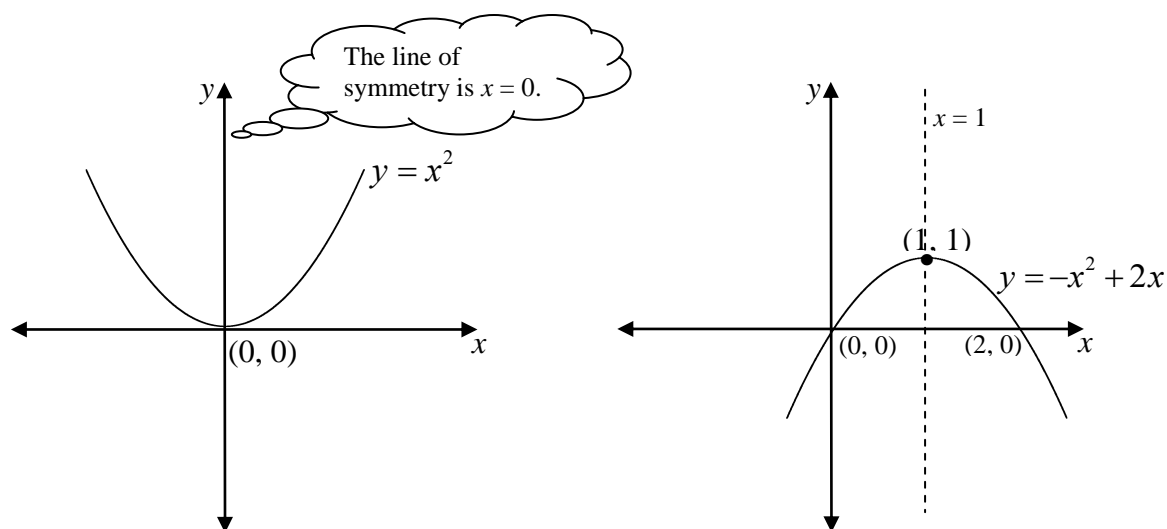


3. $x = 0$ and $x = 1$

4. $\{y : y \geq 0\}$ and $\{y : y \leq 1\}$

## Section 2.1

(a)     $y = (x-3)^2 + 4 = x^2 - 6x + 13$                     (concave up)

(b)     $y = 3 - (x+2)^2 = -x^2 - 4x - 1$          (concave down)

(c)     $y = 10(x-2)^2 - 15 = 10x^2 - 40x + 25$ (concave up)

(d)     $y = x^2 - 2x - 3$                     (concave up)

(e)     $s = t^2 - 4t + 10$          (concave up)

(f)     $P = 2w^2 - 4w + 10$     (concave up)

## Section 2.3

1.

| | *y*-intercept | *x*-intercepts | line sym. | vertex | domain | range |
|---|---|---|---|---|---|---|
| (a) | (0, 0) | (0, 0) & (4, 0) | $x = 2$ | (2, –4) | **R** | $\{y : y \geq -4\}$ |
| (b) | (0, 6) | (2, 0) & (3, 0) | $x = 2.5$ | (2.5, –0.25) | **R** | $\{y : y \geq -0.25\}$ |
| (c) | (0, –8) | (2, 0) & (–4, 0) | $x = -1$ | (–1, –9) | **R** | $\{y : y \geq -9\}$ |
| (d) | (0, 6) | (2, 0) & (–3, 0) | $x = -0.5$ | (–0.5, 5.25) | **R** | $\{y : y \leq 5.25\}$ |
| (e) | (0, 4) | (2, 0) | $x = 2$ | (2, 0) | **R** | $\{y : y \geq 0\}$ |
| (f) | (0, –8.3) | (2.05, 0) & (–4.05, 0) | $x = -1$ | (–1,–9.3) | **R** | $\{y : y \geq -9.3\}$ |

2. Sketch $y = 0.01x(10 - 0.5x)$

concave down;

*y*-intercept (0, 0), *x*-intercepts (0, 0) & (20, 0);

line sym. $x = 10$, vertex (10, 0.5).

Maximum yield occurs at the vertex

=> density = 10 plants per square metre & yield = 0.5 kg per square metre

## Section 3.1

1.

| | **(a)** | **(b)** | **(c)** |
|---|---|---|---|
| **Line sym**. | $x = 3$ | $x = 3$ | $x = 3$ |
| **vertex** | (3, –1) | (3, 0) | (3, 1) |

2. $y = (x - 2)^2 + 3$

## Section 3.2

1.

| | **(a)** | **(b)** |
|---|---|---|
| **concavity** | down | down |
| **Line sym.** | $x = 1$ | $x = 1$ |
| **vertex** | (1, 7) | (1, –5) |

## Section 4.1

| 1(a) | $(x+1)(x+3)$ | (b) | $(x+1)(x+5)$ | (c) | $(a+1)(a+7)$ |
|---|---|---|---|---|---|
| (e) | $(x+2)(x-1)$ | (f) | $(b-2)(b+1)$ | (g) | $(n-5)(n+1)$ |
| (h) | $(s-4)(s+3)$ | (i) | $(x+4)(x-3)$ | (j) | $(x+2)(x+6)$ |
| (k) | $(t-12)(t+1)$ | (l) | $(x-7)(x-2)$ | (m) | $(y-14)(y+1)$ |

3  *Quadratic Functions*

2(a)    –2 & –3        (b)    2 & 3        (c)    2 & –3

(d)    –3 & –4        (e)    –6 & –2        (f)    6

3(a)    $\dfrac{x}{x^2+12}=\dfrac{1}{7}$ $\Rightarrow$ $x^2-7x+12=0$ $\Rightarrow x=3\,\&\,4$

(b)    $1+\dfrac{6}{x-1}=\dfrac{1}{x+1}$ $\Rightarrow$ $x^2+5x+6=0$ $\Rightarrow$ $x=-3\,\&\,-2$

## Section 4.2

1(a)    $(x-2)^2=1$ $\Rightarrow$ $x=3$ and $1$        (b)    $(x-2)^2=0$ $\Rightarrow$ $x=2$

(c)    $(x-2)^2=-1$ $\Rightarrow$ no solution        (d)    $(x-\dfrac{3}{2})^2=-\dfrac{3}{4}$ $\Rightarrow$ no solution

(e)    $(a+\dfrac{3}{2})^2=\dfrac{21}{4}$ $\Rightarrow$ $a=-\dfrac{3}{2}+\sqrt{\dfrac{21}{4}}$ and $a=-\dfrac{3}{2}-\sqrt{\dfrac{21}{4}}$

(f)    $(b+\dfrac{1}{2})^2=-\dfrac{3}{4}$ $\Rightarrow$ no solution

2(a)    $\dfrac{x}{x^2+1}=\dfrac{1}{4}$ $\Rightarrow$ $x^2-4x+1=0$ $\Rightarrow x=2+\sqrt{3}$ and $2-\sqrt{3}$

(b)    $\dfrac{3}{x-1}=1-\dfrac{1}{x+1}$ $\Rightarrow$ $(x-2)^2=7$ $\Rightarrow$ $x=2+\sqrt{7}$ and $2-\sqrt{7}$

## Section 4.3

1(a)   $b^2-4ac=1>0\Rightarrow 2$ solutions

(b)   $b^2-4ac=-191<0\Rightarrow$ no solutions

(c)   $b^2-4ac=1>0\Rightarrow 2$ solutions

2(a)   $x=0.55\ \&\ 0.45$

(b)   $x=0.5$

(c)   no solutions