MathsStart

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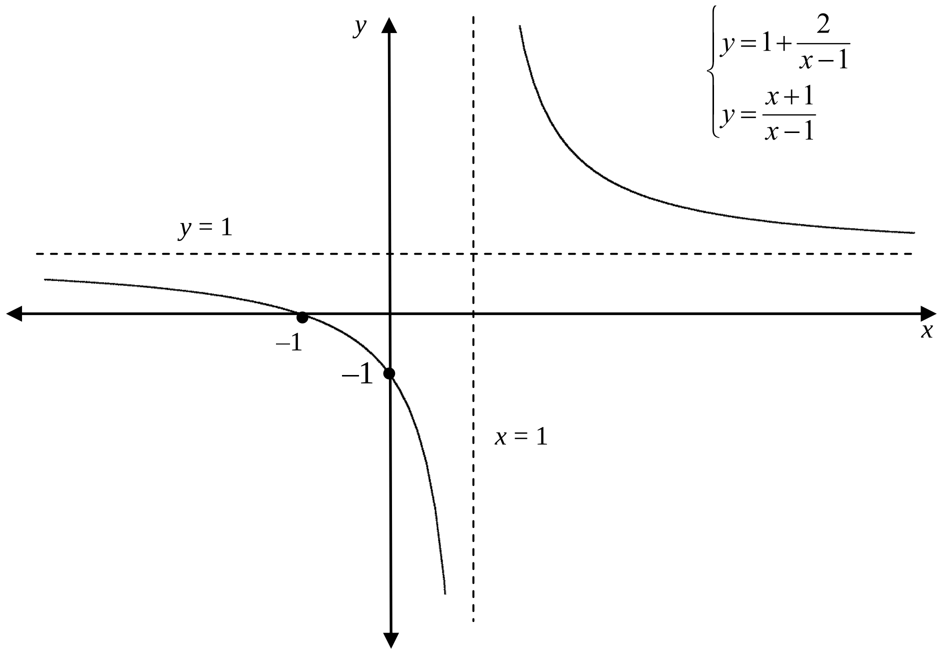
THE UNIVERSITY of ADELAIDE

Topic 4

Rational Functions



MATHS LEARNING CENTRE
Level 3, Hub Central, North Terrace Campus
The University of Adelaide, SA, 5005
TEL 8313 5862 | FAX 8313 7034 | EMAIL mathslearning@adelaide.edu.au
www.adelaide.edu.au/mathslearning/

This Topic...

This topic introduces rational functions, their graphs and their important characteristics. Rational functions arise in many practical and theoretical situations, and are frequently used in mathematics and statistics. The module also introduces the idea of a limit, and shows how this can be used for graph sketching.

Author of Topic 4: Paul Andrew

— Prerequisites

A Guide to Scientific Calculators.

— Contents

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1

Rational Functions and Hyperbolas

1.1 Rational expressions

Expressions which involve fractions are called rational expressions. These can be described in the same way as rational numbers.

	Proper	Improper	Mixed
Rational Numbers	$\frac{3}{4}$	$\frac{11}{4}$	$2\frac{3}{4}$
Rational Expressions	$\frac{3}{x}$	$\frac{2x+3}{x}$	$2+\frac{3}{x}$

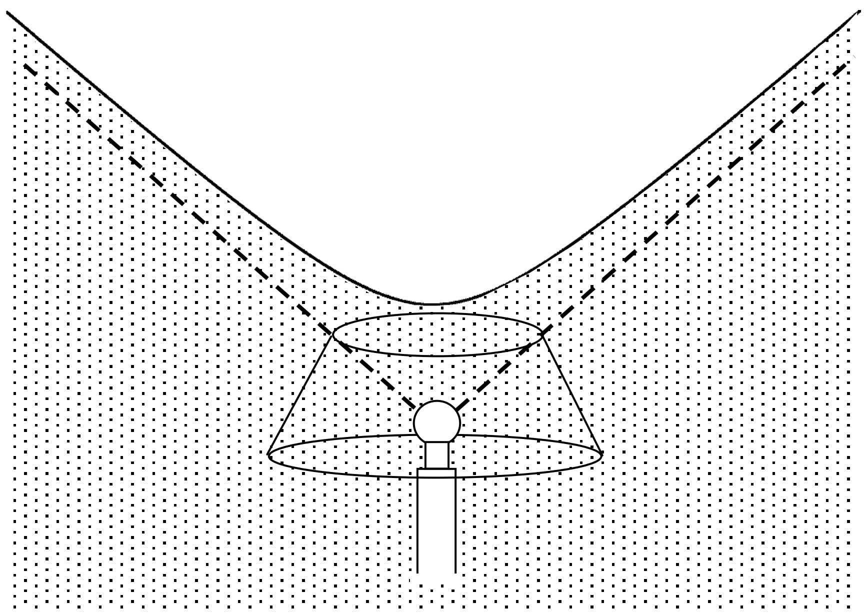
The graphs of rational functions of the form $y=\frac{A}{x}$ or $xy=A$, where A ≠ 0, are known as *rectangular hyperbolas*.

1.2 Examples of rational functions and hyperbolas

We often see hyperbolas in the shadows around us.

Example

The boundary of the shadow on a wall made by a reading lamp has a hyperbolic shape. The reflection of light off water on the inside of a glass can also be hyperbolic.



Example

Many natural phenomena obey relationships given by simple rational functions.

(a) The current (I amps) needed to run a light globe is related to the voltage (V volts) by the formula $IV = \text{constant}$ or $I = \dfrac{\text{constant}}{V}$.

(b) When a gas expands or is compressed, its pressure (P) is related to its volume (V) by the formula $PV = \text{constant}$ or $P = \dfrac{\text{constant}}{V}$.

Example

Rational functions frequently arise in economics. If it takes 10 shearers 20 days to shear some sheep, then 20 shearers would take 10 days to do the same job and 40 shearers would take 5 days. The relationship between the number of shearers (N) and the time needed (T) is given by the formula:

$$N \times T = 200 \quad \text{or} \quad N = \frac{200}{T}.$$

If the first day on the job is used for setting up equipment, then the relationship between the number of shearers (N) and the total time needed (TT = T + 1) is given by the formula:

$$N \times (TT - 1) = 200 \quad \text{or} \quad N = \frac{200}{TT - 1}\ .$$

If a cook is needed to cook for the shearers, then the relationship between the total number of employees (TN = N + 1) and the total time needed (TT = T + 1) is given by the mixed rational function:

$$TN = \frac{200}{TT - 1} + 1\ .$$

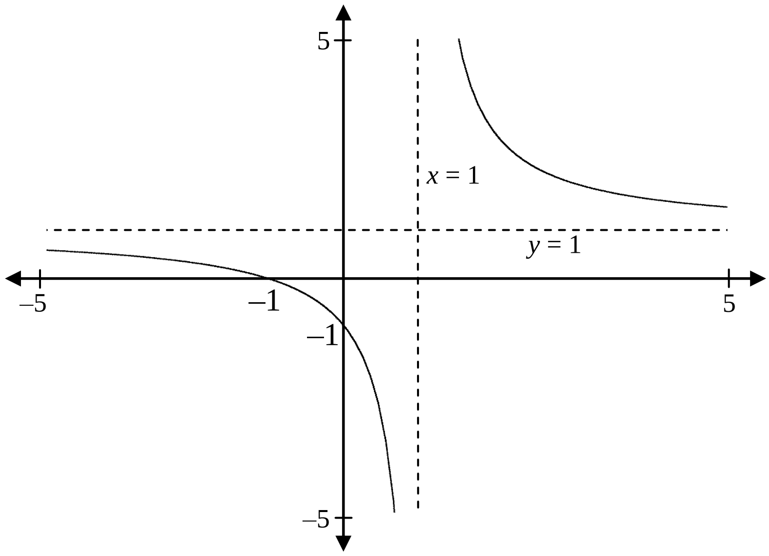
When we study the general properties of rational functions, we use *x* and *y* as the independent and dependent variables and place no restriction on the domain of the functions. The important characteristics of rational functions are found from their graphs. These are the *asymptotes* and the *x*- and *y-intercepts*.

Example

The graph of the rational function

$$y = 1 + \frac{2}{x - 1},\ x \neq 1$$

is shown below. This hyperbola has *x*-intercept (–1, 0) and *y*-intercept (0, –1). The two dotted lines *x* = 1 and *y* = 1 are called the *asymptotes* of the hyperbola. The hyperbola is called a *rectangular* hyperbola because its two asymptotes are right angles to each other.



The asymptotes of a hyperbola are important because the points on the hyperbola become very close to the asymptotes the further you move away from the origin (0, 0). If you 'zoom out' for the graph above, then the hyperbola would look like the pair of lines $x = 1$ and $y = 1$.

Asymptotes are traditionally drawn with dotted lines. They divide the plane into four regions that are important because *hyperbolas never cross their asymptotes*.

Problems 1

1. Complete the tables below, then use them to draw the hyperbolas $y = \frac{1}{x}$ and $y = -\frac{1}{x}$.

x	–3	–2	–1	0	1	2	3
$y = \frac{1}{x}$				–			
$y = -\frac{1}{x}$				–			

x	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
$y = \frac{1}{x}$				–			
$y = -\frac{1}{x}$				–			

2. What are the asymptotes for the hyperbolas in problem 1?

3. What are the domains and ranges of the rational functions below?

(a) $y = \frac{1}{x}$

(b) $-\frac{1}{x}$

(c) $f(x) = 1 + \frac{2}{x-1}$ (see page 3)

2
Sketching Hyperbolas

This section begins by using tables of values to sketch hyperbolas of the form $y = \frac{A}{x}, x \neq 0$ or $xy = A$. This will give a good understanding of the shape of a hyperbola and its symmetries. The section then continues by showing how to sketch general hyperbolas *without* using a table of values.

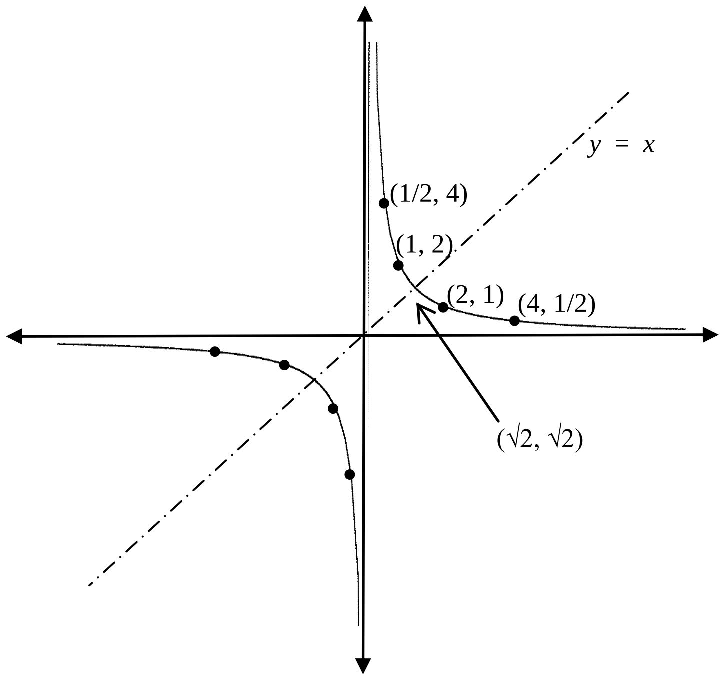
2.1 Hyperbolas of the form *xy* = *A*

Example

The hyperbola $y = \frac{2}{x}, x \neq 0$ can also be rewritten as $xy = 2$. A table of values is used to sketch it.

To find the shape of the curve near $x = 0$, use values of x near $x = 0$.

x	±1	±2	±4	±8	±1/2	±1/4
y	±2	±1	±1/2	±1/4	±4	±8



The asymptotes are $x = 0$ and $y = 0$, because the hyperbola becomes very close to these two lines as we move away from the origin.

If you compare the pairs of points (1, 2) & (2, 1), (1/2, 4) & (4, 1/2), (–1, –2) & (–2, –1), etc then you can see that they are symmetric across the line $y = x$, in the sense that this line divides the hyperbola into two parts, each of which is a mirror image of the other. The equation $xy = 2$ shows the reason for this: if (a, b) is a point on the curve, then so is the point (b, a). We say that *the hyperbola is symmetric about the line* $y = x$.

You can see from the graph that the points on the hyperbola that are closest to the origin are on the line $y = x$. Substituting $y = x$ into the equation $xy = 2$, gives

$$x \times x = 2$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$

So the points closest to the origin are $\left(\sqrt{2}, \sqrt{2}\right)$ and $\left(-\sqrt{2}, -\sqrt{2}\right)$.

The hyperbola has another symmetry – can you see it? The pairs of points (1, 2) & (–2, –1), (2, 1) & (–1, –2), etc are symmetric about the line $y = -x$. Can you find this line of symmetry? The equation $xy = 2$ shows the reason for this: if (a, b) is a point on the curve, then so is the point $(-b, -a)$. We say *the hyperbola is symmetric about the line* $y = -x$.

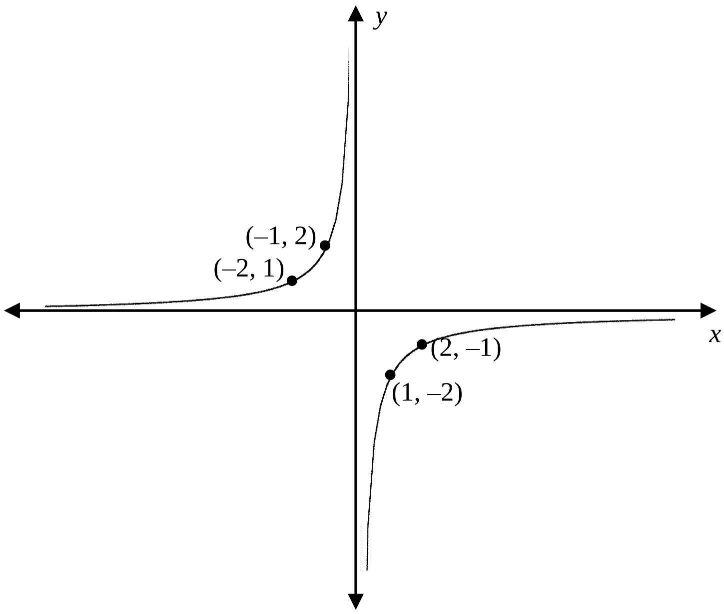
All hyperbolas have two asymptotes and two lines of symmetry.

Example

The hyperbola $y = -\dfrac{2}{x}$, $x \neq 0$ can also be rewritten as $xy = -2$. The hyperbola can be sketched using the table of values below.

x	± 1	± 2	± 4	± 8	$\pm 1/2$	$\pm 1/4$
y	∓ 2	∓ 1	$\mp 1/2$	$\mp 1/4$	∓ 4	∓ 8

x was given simple values to make the calculations easier.



The hyperbola has asymptotes $x = 0$ and $y = 0$, and is also symmetric about the lines $y = x$ and $y = -x$. (Sketch these lines on the diagram.)

The closest points to the origin are where the hyperbola meets the line $y = -x$. Substituting this into the equation $xy = -2$, gives

$$x \times (-x) = -2$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$

So the points closest to the origin are $\left(\sqrt{2}, -\sqrt{2}\right)$ and $\left(-\sqrt{2}, \sqrt{2}\right)$.

Problems 2.1

1. Sketch the hyperbolas below, using a table of values.

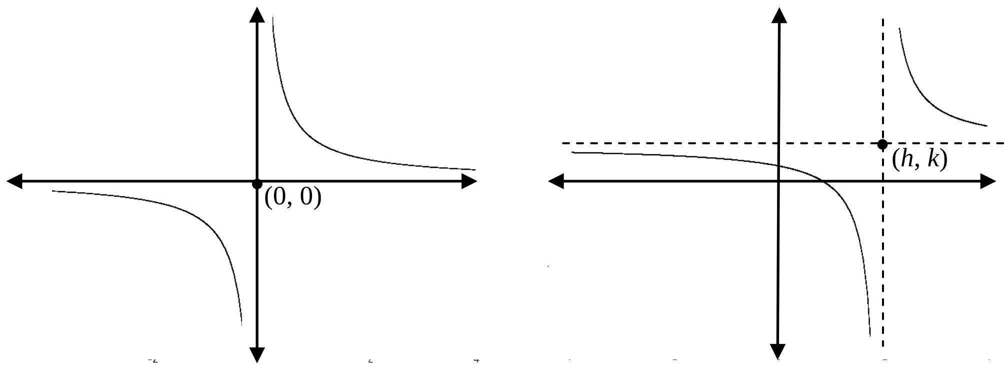
(a) $xy = 4$ (b) $xy = -4$ (c) $xy = 9$ (d) $xy = -9$ (e) $xy = 10$

2. Find the closest points to the origin on each of the hyperbolas.

2.2 Transforming the hyperbola *xy* = *A*

In module 2, we saw how transformations, such as *translations*, *reflections*, and *dilations*, could be used to sketch parabolas. They can also be used to sketch hyperbolas.

If the hyperbola $y = \dfrac{A}{x}$, $x \neq 0$ is shifted h units to the right and k units upwards, then its new equation is $y = \dfrac{A}{x-h} + k,\ x \neq h.$



We can use translations to sketch hyperbolas.

Example

(a) To sketch the hyperbola $y = \dfrac{4}{x-2} + 3,\ x \neq 2$:

- first sketch $y = \dfrac{4}{x},\ x \neq 0$, then
- shift it 2 units to the right and 3 units upwards.

$A = 4$, $h = -2$ and $k = -3$

(b) To sketch the hyperbola $y = \dfrac{4}{x+2} - 3,\ x \neq -2$

- sketch $y = \dfrac{4}{x},\ x \neq 0$, then
- shift it –2 units to the right and –3 units upwards

ie. 2 units to the left
and 3 units downwards

$A = -4$, $h = 2$ and $k = 3$

(b) To sketch the hyperbola $y = 3 - \dfrac{4}{x-2},\ x \neq 2$

- sketch $y = -\dfrac{4}{x}$ or $xy = -4, x \neq 0$, then
- shift it 2 units to the right and 3 units upwards.

Problems 2.2

Draw the hyperbola $y = \dfrac{4}{x},\ x \neq 0$, then use translations to sketch:

(a) $y = \dfrac{4}{x-1} + 2,\ x \neq 1$ (b) $y = \dfrac{4}{x-2} + 3,\ x \neq 2$ (c) $y = \dfrac{4}{x+1} + 3,\ x \neq -1$

(d) $y = \dfrac{4}{x-1} - 2,\ x \neq 1$ (e) $y = 2 - \dfrac{4}{x-1},\ x \neq 1$ (f) $(y-2)(x+3) = 4$

2.3 Sketching hyperbolas

A complete sketch of a hyperbola includes asymptotes and intercepts. Once you know the shape of the hyperbola $y = \frac{A}{x}$, $x \neq 0$ or $xy = A$, *there is no need to use a table of values*.

Example

Find the x- and y- intercepts of the hyperbola $y = \frac{4}{x-1} + 2,\ x \neq 1$.

Answer

(a) To find the y-intercept, put $x = 0$.

$$\begin{aligned} y &= \frac{4}{x-1} + 2 \\ &= \frac{4}{0-1} + 2 \\ &= -2 \end{aligned}$$

The y-intercept is (0, –2). (Check by substitution.)

(b) To find the x-intercept, put $y = 0$.

$$\begin{aligned} \frac{4}{x-1} + 2 &= 0 \\ \frac{4}{x-1} &= -2 \\ (x-1) \times \frac{4}{x-1} &= -2 \times (x-1) \\ 4 &= -2(x-1) \\ 4 &= -2x + 2 \\ x &= -1 \end{aligned}$$

Remove x – 1 from the denominator by multiplying both sides of the equation by x – 1.

The y-intercept is (–1, 0). (Check by substitution.)

Example

Find the x- and y- intercepts of the hyperbola $y = \frac{2x+2}{x-1},\ x \neq 1$.

Answer

(a) To find the y-intercept, put $x = 0$.

$$\begin{aligned} y &= \frac{2x+2}{x-1} \\ &= \frac{0+2}{0-1} \\ &= -2 \end{aligned}$$

The y-intercept is (0, –2). (Check by substitution.)

(b) To find the x-intercept, put $y = 0$.

Remove x – 1 from the denominator by multiplying both sides of the equation by x – 1.

$$\frac{2x+2}{x-1}=0$$

$$(x-1)\times\frac{2x+2}{x-1}=0\times(x-1)$$

$$2x+2=0$$

$$x=-1$$

The y-intercept is (–1, 0). (Check by substitution.)

If we wanted to sketch a hyperbola like $y=\dfrac{2x+2}{x-1}$, which is written in the form of an improper rational function, then we should rewrite it in the form of a mixed rational function like $y=\dfrac{A}{x-h}+k$.

Example

When we rewrite the improper fraction $\frac{7}{3}$ as a mixed fraction, we

(i) first ask how many multiples of 3 are there in 7, and how much is left over?

➔ $7=2\times3+1$.

(ii) then rewrite $\frac{7}{3}$ as $\frac{2\times3+1}{3}=\frac{2\times3}{3}+\frac{1}{3}=2+\frac{1}{3}$

Example

Write the equation of the hyperbola $y=\dfrac{2x+2}{x-1},\ x\neq1$ in the form $y=\dfrac{A}{x-h}+k,\ x\neq h$.

Answer

Firstly h = 1, since we have x – 1 in the denominator.

First find how many multiples of x – 1 are in 2x + 2, and how much is left over:
➔ 2x + 2 = 2(x – 1) + 2 + 2 = 2(x – 1) + 4

$$\frac{2x+2}{x-1}=\frac{2(x-1)+2+2}{x-1}$$

$$=\frac{2(x-1)+4}{x-1}$$

$$=\frac{2(x-1)}{x-1}+\frac{4}{x-1}$$

$$=2+\frac{4}{x-1}$$

The equation of the hyperbola is $y=\dfrac{4}{x-1}+2,\ x\neq1$.

Example

Write the equation of the hyperbola $y=\dfrac{3-2x}{x-2},\ x\neq2$ in the form $y=\dfrac{A}{x-h}+k,\ x\neq h$.

Answer

Firstly h = 2, because we have x – 2 in the denominator.

$$\begin{aligned}\frac{3-2x}{x-2} &= \frac{3-2(x-2)-4}{x-2}\\ &= \frac{-2(x-2)-1}{x-2}\\ &= -2-\frac{1}{x-2}\end{aligned}$$

First find how many multiples of $x - 2$ are in $3 - 2x$, and how much is left over:
➔ $3 - 2x = 3 - 2(x - 2) - 4 = -2(x - 2) - 1$

The equation of the hyperbola is $y = -\frac{1}{x-1} - 2,\ x \neq 1$.

Example

Sketch the hyperbola $y = \frac{3x+4}{x+2},\ x \neq -2$ showing the asymptotes and intercepts.

(Note. It doesn't matter whether you find the intercepts or the asymptotes first.)

Answer

(i) Intercepts

Put $x = 0$,

$$\begin{aligned}y &= \frac{3x+4}{x+2}\\ &= \frac{0+4}{0+2}\\ &= 2\end{aligned}$$

The y- intecept is (0, 2)

Put $y = 0$,

$$\begin{aligned}\frac{3x+4}{x+2} &= 0\\ (x+2)\times\frac{3x+4}{x+2} &= 0\times(x+2)\\ 3x+4 &= 0\\ x &= -\frac{4}{3}\end{aligned}$$

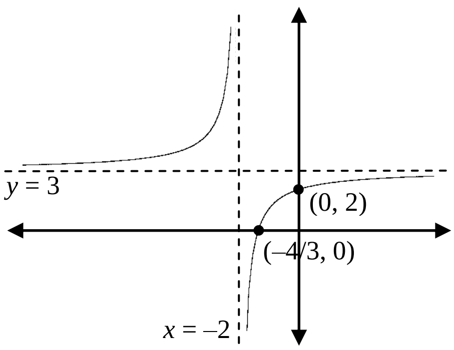
The x- intecept is (–4/3, 0)

(ii) Asymptotes

$$\begin{aligned}y &= \frac{3x+4}{x+2}\\ &= \frac{3(x+2)-6+4}{x+2}\\ &= \frac{3(x+2)-2}{x+2}\\ &= 3-\frac{2}{x+2}\end{aligned}$$

$A = -2$, $h = -2$, $k = 3$
➔ first sketch $y = -2/x$
then shift 2 units to left
and 3 units up.

(iii) Sketch



Note. The main features of a hyperbola are its general shape, and its asymptotes and intercepts. There is no need to use a table of values for sketching, unless you have a good reason for needing a more detailed sketch.

Problems 2.3

1. Find the *x*- and *y*-intercepts for the following hyperbolas.

(a) $y = \dfrac{6}{x-2} + 2,\ x \neq 1$ (b) $y = \dfrac{6}{x+2} - 3,\ x \neq -2$

(c) $y = 3 - \dfrac{6}{x-2},\ x \neq 2$ (d) $y = \dfrac{4}{x+3} + 2,\ x \neq -3$

2. Rewrite the equation of each hyperbola below in the form $y = \dfrac{A}{x-h} + k,\ x \neq h$.

(a) $y = \dfrac{2x+3}{x+1},\ x \neq -1$ (b) $y = \dfrac{2x+2}{x-1},\ x \neq 1$

(c) $y = \dfrac{3x+2}{x+1},\ x \neq -1$ (d) $y = \dfrac{2-3x}{x-1},\ x \neq 1$

3. Sketch the hyperbolas below, showing the asymptotes, and intercepts.

(a) $y = 3 + \dfrac{6}{x-3},\ x \neq 3$ (b) $y = \dfrac{3x+4}{x+2},\ x \neq -2$

2.4 Lines of Symmetry

While we do not normally draw the lines of symmetry of a hyperbola on the graph, it can sometimes be useful to know where they are. It was mentioned above that the lines of symmetry of the rational function $y = \dfrac{A}{x}, x \neq 0$ are $y = x$ and $y = -x$. We can use translations to find the lines of symmetry of the rational function $y = \dfrac{A}{x-h} + k,\ x \neq h$. This new function was obtained from the first by replacing y with $y - k$ and replacing x with $x - h$. If we do the same for the two lines of symmetry the equations become $y - k = x - h$ and $y - k = -(x - h)$, so these must be the lines of symmetry of the rational function $y = \dfrac{A}{x-h} + k,\ x \neq h$. Note that these are the lines of slope +1 and –1 that pass through the point (h, k).

3

Introduction to Limits

This section introduces concept of a *limit*, and shows how to use limits to sketch hyperbolas. You may prefer to use limits rather than translations for sketching.

3.1 Limits to Infinity

The table of values below shows that as x is given larger and larger values, the value of $\frac{1}{x}$ becomes smaller and smaller, . . . and closer to 0.

x	1	10	100	1000	10,000
y	1	0.1	0.01	0.001	0.000 1

This corresponds to the hyperbola $y = \frac{1}{x}$ having the x-axis as an asymptote: as the x-coordinates of points on the hyperbola become large, their y-coordinates becomes close to 0.

We can write this in *limit notation* as following:

$$\text{‘as } x \to +\infty,\ \frac{1}{x} \to 0\text{’.}$$

The symbols ‘as $x \to +\infty$’ are read as ‘*as x tends to positive infinity’*, and mean ‘ as x is given larger and larger values without limit’. The symbols ‘as $\frac{1}{x} \to 0$’ are read as ‘as $\frac{1}{x}$ *tends to 0’* and mean ‘as the value of $\frac{1}{x}$ becomes closer and closer to 0’.

When x is given larger and larger *negative* values, the value of $\frac{1}{x}$ also becomes smaller and smaller, . . . and closer to 0.

x	–1	–10	–100	–1000	–10,000
y	–1	–0.1	–0.01	–0.001	–0.000 1

We can write this using in *limit notation* as:

$$\text{‘as } x \to -\infty,\ \frac{1}{x} \to 0\text{’.}$$

Here the symbols ‘as $x \to -\infty$’ are read as ‘*as x tends to negative infinity’*, and mean ‘ as x is given larger and larger *negative* values without limit’.

These ideas can be used to find the asymptotes on hyperbolas.

Example

Consider the hyperbola $y = 1 + \frac{2}{x}$. We can see that

- if $x \to +\infty$, then $y \to 1$
- if $x \to -\infty$, then $y \to 1$

so the line $y = 1$ is an asymptote for the hyperbola.

Reason

As x is given larger and larger *positive* values without limit, then the value of $\frac{2}{x}$ becomes closer and closer to 0, and the value of $1 + \frac{2}{x}$ becomes closer and closer to 1 + 0 = 1.

The same is true when x is given larger and larger *negative* values without limit. From our knowledge of hyperbolas, we can see that the line $y = 1$ must be the asymptote.

Example

Consider the hyperbola $y = \frac{2x+3}{x-1}$. If the numerator and denominator are both divided by x, then this equation becomes $y = \frac{2+\frac{3}{x}}{1-\frac{1}{x}}$. Now you can see that

- if $x \to +\infty$, then $\frac{1}{x} \to 0$ and $y \to \frac{2+0}{1-0} = 2$
- if $x \to -\infty$, then $\frac{1}{x} \to 0$ and $y \to \frac{2+0}{1-0} = 2$,

so the line $y = 2$ is an asymptote for the hyperbola.

Here are some more examples.

Examples

(a) $y = 3 + \frac{4}{x+2}$
- if $x \to +\infty$, then $y \to 3$
- if $x \to -\infty$, then $y \to 3$

(b) $y = 3 - \frac{1.5}{2x+1}$
- if $x \to +\infty$, then $y \to 3$
- if $x \to -\infty$, then $y \to 3$

(c) $y = \frac{3x+1}{2x-1}$
- if $x \to +\infty$, then $y \to \frac{3}{2}$
- if $x \to -\infty$, then $y \to \frac{3}{2}$

(d) $y = \frac{3x+1}{1-2x}$
- if $x \to +\infty$, then $y \to \frac{3}{-2} = -\frac{3}{2}$
- if $x \to -\infty$, then $y \to \frac{3}{-2} = -\frac{3}{2}$

Problems 3.1

1. Find the limit as $x \to +\infty$ and as $x \to -\infty$ for each of the rational functions below.

(a) $y = 1.5 + \dfrac{3}{x+2}$ (b) $y = \dfrac{2}{x+1} + 12$ (c) $y = 4 + \dfrac{1}{1-2x}$ (d) $y = 2 - \dfrac{3}{2-3x}$

(e) $y = \dfrac{x+3}{x+2}$ (f) $y = \dfrac{2x+3}{x+4}$ (g) $y = \dfrac{1-2x}{1-x}$ (h) $y = \dfrac{4x+3}{2-2x}$

2. What are the horizontal asymptotes of the following hyperbolas.

(a) $y = 4 + \dfrac{1}{2x-3}, x \neq \dfrac{3}{2}$ (b) $y = \dfrac{2x+3}{2x-5}, x \neq \dfrac{5}{2}$

3.2 Using limits to sketch hyperbolas

When sketching, the important characteristics of hyperbolas are

- the asymptotes
- the general shape
- the intercepts

We know that hyperbolas never cross their asymptotes. This allows us to find the vertical asymptote easily.

Example

(a) The vertical asymptote of $y = \dfrac{2}{x+1} + 12, x \neq -1$ is the line $x = -1$, because we know that no point on on the hyperbola can have x-coordinate equal to -1.

(b) The vertical asymptote of $y = 4 + \dfrac{3}{2x-1}$ is the line $x = \dfrac{1}{2}$, as no point on the hyperbola can have x-coordinate equal to $\dfrac{1}{2}$.

(c) The vertical asymptote of $y = \dfrac{x+1}{x+2}$ is the line $x = -2$, as no point on on the hyperbola can have x-coordinate equal to -2.

Example

Sketch the hyperbola $y = \dfrac{2}{x+1} + 1, x \neq -1$, showing its asymptotes and intercepts.

Answer

(i) Asymptotes

The vertical asymptote is $x = -1$.

If $x \to \pm\infty$, then $y \to 1$ so the horizontal asymptote is $y = 1$.

(ii) Intercepts

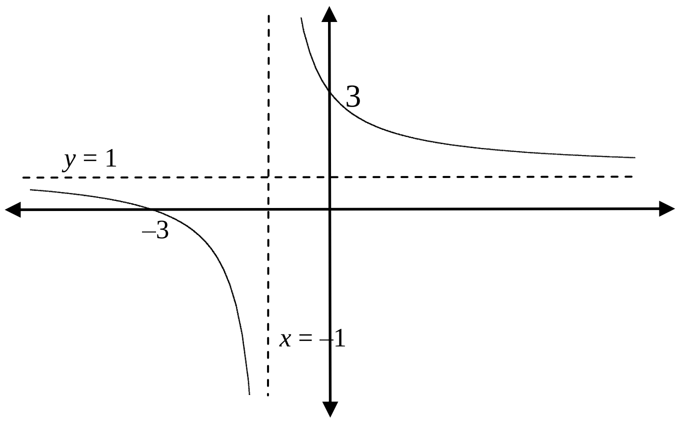
Put x = 0, then y = 2 so the y-intercept is (0, 3).

Put y = 0, then

$$\begin{aligned}
\frac{2}{x+1}+1&=0\\
\frac{2}{x+1}&=-1\\
(x+1)\times\frac{2}{x+1}&=-1\times(x+1)\\
2&=-(x+1)\\
2&=-x-1\\
x&=-3
\end{aligned}$$

So the x-intercept is (–3, 0).

(iii) Sketch



Problems 3.2

Redo problems 2.3(3) using this new approach.

A

Appendix: Answers

Answer 1

1. Complete the tables below, then use them to draw the hyperbolas $y = \frac{1}{x}$ and $y = -\frac{1}{x}$.

x	–3	–2	–1	0	1	2	3
$y = \frac{1}{x}$	$-\frac{1}{3}$	$-\frac{1}{2}$	–1	–	1	$\frac{1}{2}$	$\frac{1}{3}$
$y = -\frac{1}{x}$	$\frac{1}{3}$	$\frac{1}{2}$	1	–	–1	$-\frac{1}{2}$	$-\frac{1}{3}$

x	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
$y = \frac{1}{x}$	–2	–4	–8	–	8	4	2
$y = -\frac{1}{x}$	2	4	8	–	–8	–4	–2

2. The asymptotes are $x = 0$ and $y = 0$.

3. What are the domains and ranges of the rational functions below?

(a) domain $\{x : x \neq 0\}$, range $\{y : y \neq 0\}$

(b) domain $\{x : x \neq 0\}$, range $\{y : y \neq 0\}$

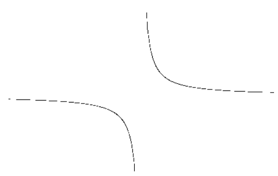
(c) domain $\{x : x \neq 1\}$, range $\{y : y \neq 1\}$

Answers 2.1

Hyperbola **Some points. . . .** **Shape** **Closest Points**

(a) $xy = 4$

x	±1	±2	±4
y	±4	±2	±1

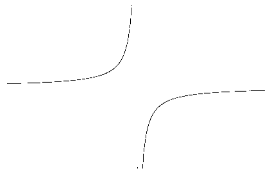


(2, 2)

(–2, –2)

(b) $xy = -4$

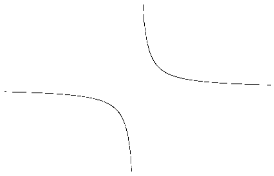
x	± 1	± 2	± 4
y	∓ 4	∓ 2	∓ 2



(2, –2)
(–2, 2)

(c) $xy = 9$

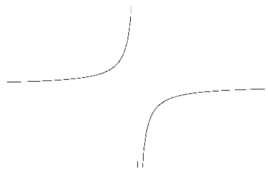
x	± 1	± 3	± 9
y	± 9	± 3	± 1



(3, 3)
(–3, –3)

(d) $xy = -9$

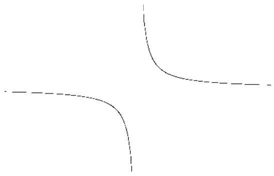
x	± 1	± 3	± 9
y	∓ 9	∓ 3	∓ 1



(3, –3)
(–3, 3)

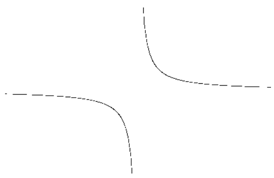
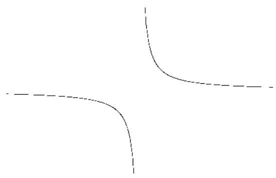
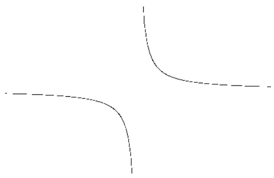
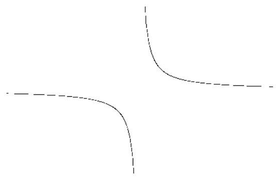
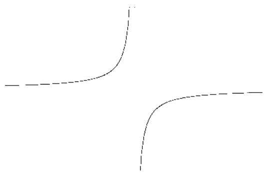
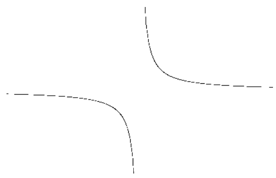
(e) $xy = 10$

x	± 1	± 2	± 5
y	± 10	± 5	± 2



$\left(\sqrt{10}, \sqrt{10}\right)$
$\left(-\sqrt{10}, -\sqrt{10}\right)$

Answers 2.2

	Shape	Asymptotes
(a)		$x = 1$ $y = 1$
(b)		$x = 2$ $y = 3$
(c)		$x = -1$ $y = 3$
(d)		$x = 1$ $y = -2$
(e)		$x = 1$ $y = 2$
(f)		$x = -3$ $y = 2$

Answers 2.3

1. (a) (0, –1) & (–1, 0) (b) (0, 0)
   (c) (4, 0) & (0, 6) (d) (–5, 0) & (0, 3.33)

2. (a)
$$y=\frac{2x+3}{x+1}=\frac{2(x+1)-2+3}{x+1}=\frac{2(x+1)-1}{x+1}=2-\frac{1}{x+1}=-\frac{1}{x+1}+2$$

(b)
$$y=\frac{2x+2}{x-1}=\frac{2(x-1)+2+2}{x-1}=\frac{2(x-1)+4}{x-1}=2+\frac{4}{x-1}=\frac{4}{x-1}+2$$

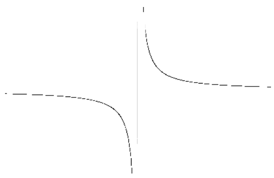
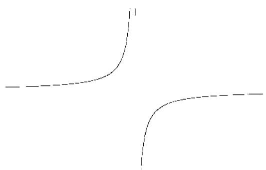
(c)
$$y=\frac{3x+2}{x+1}=\frac{3(x+1)-3+2}{x+1}=\frac{3(x+1)-1}{x+1}=3-\frac{1}{x+1}=-\frac{1}{x+1}+3$$

(d)
$$y=\frac{2-3x}{x-1}=\frac{2-3(x-1)-3}{x-1}=\frac{-3(x-1)-1}{x-1}=-3-\frac{1}{x-1}=-\frac{1}{x-1}-3$$

3. Sketch the hyperbolas below, showing the asymptotes, and intercepts.

(a) $y=3-\dfrac{6}{x-3},\ x\neq 3$ (b) $y=\dfrac{3x+4}{x+2},\ x\neq -2$

3.

Hyperbola	Shape	Asymptotes	Intercepts
$y=3+\dfrac{6}{x-3},\ x\neq 3$		$x = 3$ $y = 3$	(1, 0) (0, 1)
$y=\dfrac{3x+4}{x+2},\ x\neq -2$		$x = –2$ $y = 3$	(–4/3, 0) (0, 2)

Answers 3.1

1(a) 1.5 (b) 12 (c) 4 (d) 2 (e) 1 (f) 2 (g) 2 (h) –2

2(a) $y = 4$ (b) $y = 1$

Answers 3.2

See answers to 2.3(3)