MathsStart

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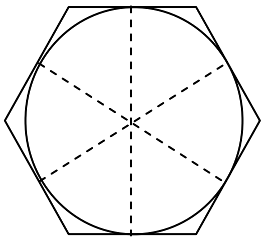
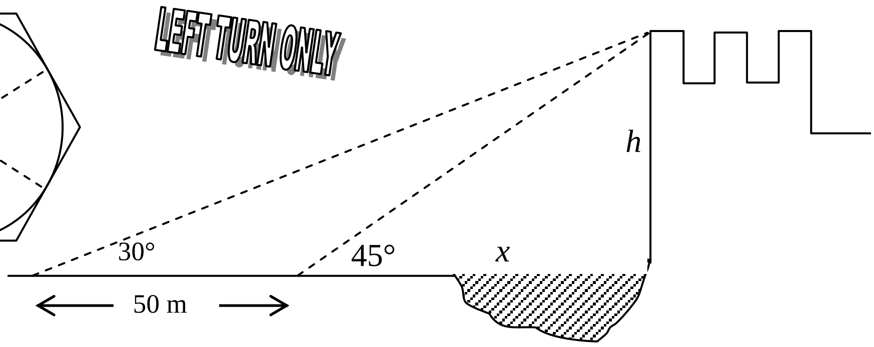
New books will be created during 2013 and 2014)

THE UNIVERSITY of ADELAIDE

Topic 5

Trigonometry I





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This Topic...

This topic introduces trigonometric ratios and their properties. Trigonometry was developed by the ancient Greeks and Hindus to find distances that were not easily accessible, for example the mathematician Eratosthenes used it to calculate the radius of the earth in the third century BC! Today, it is used extensively in engineering, surveying, architecture and astronomy. It is also used in many other mathematical applications.

Author of Topic 5: Paul Andrew

Prerequisites

You will need a scientific calculator.

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Chapter 1 Right-angled Triangles and Trigonometric Ratios.

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Chapter 3 Properties.

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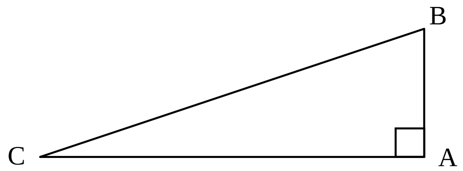
Printed: 18/02/2013

1

Right-angled Triangles and Trigonometric Ratios

1.1 Right-angled Triangles

A *right-angled triangle* is a triangle with one angle equal to 90^o (the *right angle*). A right angle in a triangle is indicated by a small square.



• The *vertices* of a triangle are labelled with upper case letters (generally in an anticlockwise direction). The triangle above is called 'the triangle ABC' and is written as Δ ABC (or Δ BCA or Δ CAB).

• The sides of a triangle are named from their *endpoints*, for example AB is the side having endpoints A and B. When there is no ambiguity, we can also use AB to mean the *length* of the side AB.

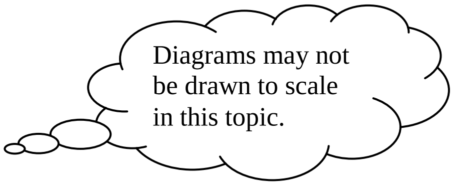
The *hypotenuse* of a triangle is the side opposite the right angle, and is the longest side in a right-angled triangle.

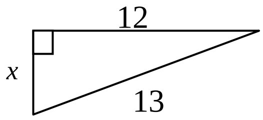
The ancient Egyptian surveyors used right-angled triangles with sides 3, 4 and 5 units to mark out accurate right angles. This is an example of *Pythagoras' theorem*:

> **Pythagoras' Theorem**
> In a right-angled triangle, the square of length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Example

Find x in the triangles below:





Answer

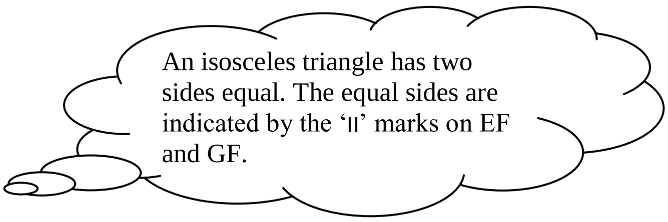
$$\begin{aligned} x^2 + 12^2 &= 13^2 \\ x^2 + 144 &= 169 \\ x^2 &= 25 \\ x &= 5 \quad \text{(as lengths are positive)} \end{aligned}$$

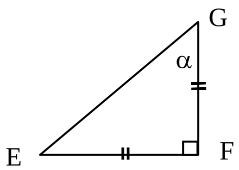
• An angle is formed when two sides meet at a vertex. The angle CAB in Δ ABC above is the angle with vertex A and sides CA and AB, and is written as ∠CAB. When there is no ambiguity, we can just call it 'the angle A'.

The sum of the angles in an triangle always add up to 180°.

Example

What is ∠ FGE in Δ EFG below?





Answer

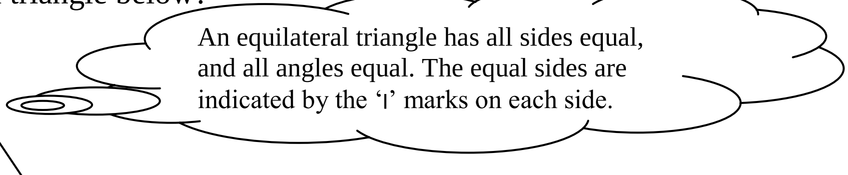
As Δ EFG is an isosceles triangle, the angles ∠ FGE and ∠ GEF are both equal.

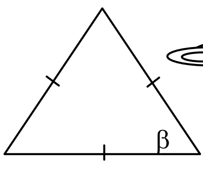
So $\alpha + \alpha + 90° = 180° \Rightarrow \alpha = 45°$.

It is very common to use the Greek alphabet for angles. Commonly used letters are α (alpha), β (beta), γ (gamma), δ (delta), θ (theta) and ω (omega).

Example

What is the angle β in the equilateral triangle below?





Answer

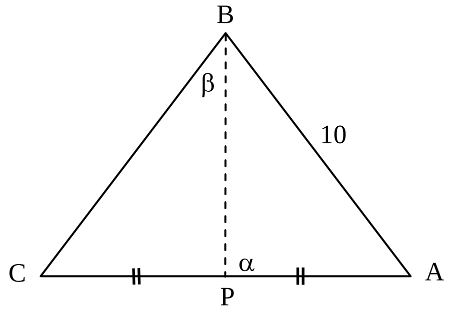
As the Δ is equilateral, all three angles are equal. So $\beta + \beta + \beta = 180° \Rightarrow \beta = 60°$.

Problems 1.1

In the equilateral triangle ABC below, side AB is 10 units long, find

(a) α and β

(b) the lengths of all sides in Δ APB and Δ APB.



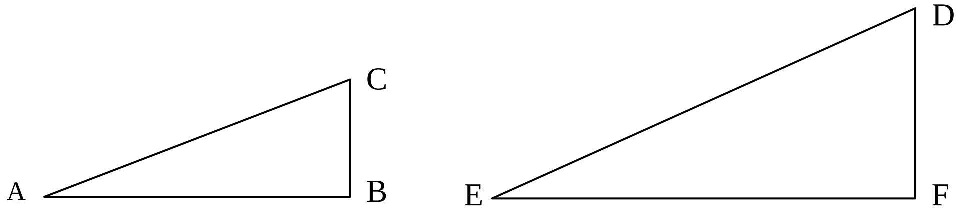
α = ‘alpha’
β = ‘beta’

1.2 Trigonometric Ratios

Triangles that have the same angles are called *similar triangles*. Similar triangles have the special property that the ratios of their sides are the same for each triangle.

Example

The triangles ABC and DEF below are similar



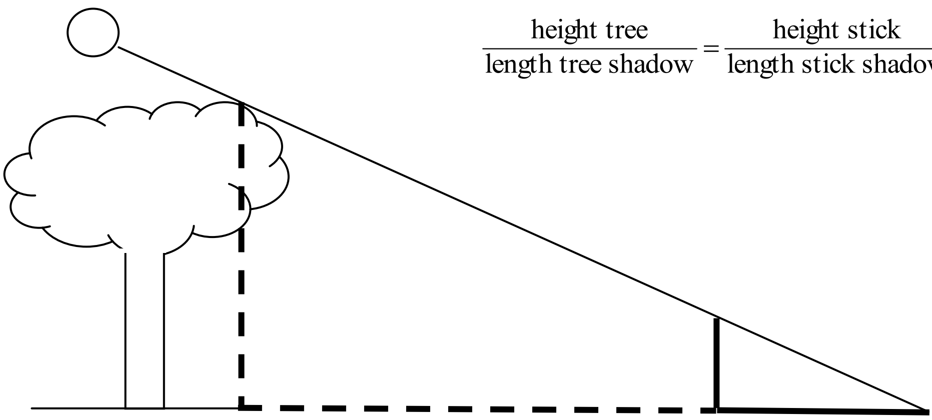
so $\frac{CB}{AB} = \frac{DF}{EF}, \frac{CB}{AC} = \frac{DF}{ED}, \frac{AB}{AC} = \frac{EF}{ED}$, etc.

Similar triangles can be used to measure distances that are not easily accessible.

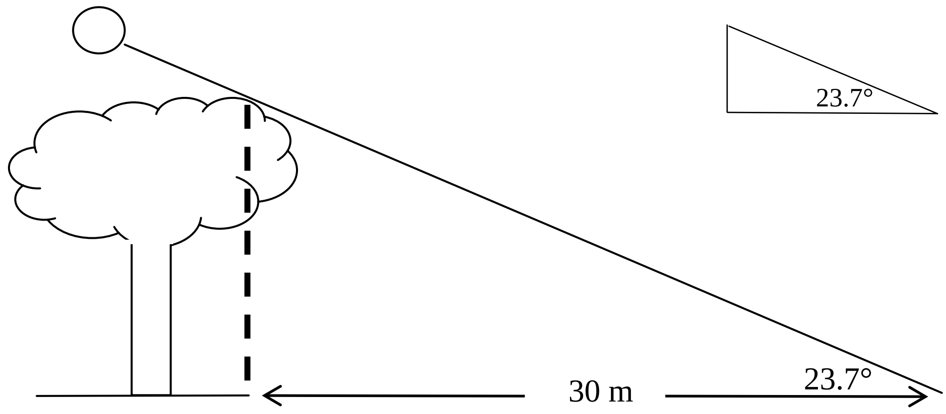
Example

The height of a tree can be found by comparing its shadow to the shadow thrown by a stick.

$$\frac{\text{height tree}}{\text{length tree shadow}} = \frac{\text{height stick}}{\text{length stick shadow}}$$



We can't rely on the sun being in a convenient position, so another idea is needed. This idea is to measure the *angle of elevation* of the tree from a known distance, say 30 m, and then to compare the ratio of tree height to distance with the ratio of the sides of a right-angled triangle having the same angle.



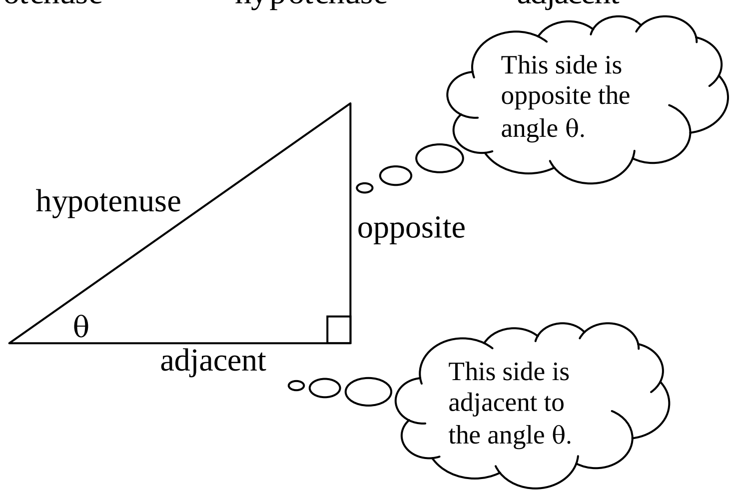
Trigonometry is based on this idea, and the fact that we can calculate the ratios of the sides of a right-angled triangle for any angle. These ratios are called *trigonometric ratios.* Fifty years ago the trigonometric ratios were published in books, but today they can be found from calculators.

The basic trigonometric ratios are the *sine*, the *cosine* and the *tangent*. They are often abbreviated to *sin* (read as 'sine'), *cos* (read as 'coz') and *tan*. (The tangent as a trigonometric ratio is different from the geometric idea of a tangent to a curve.)

These ratios are defined for the angle θ ('theta') by the rules

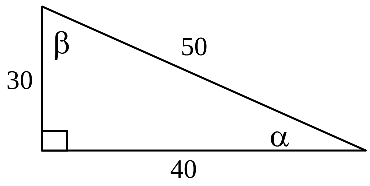
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

for the right-angled triangle:

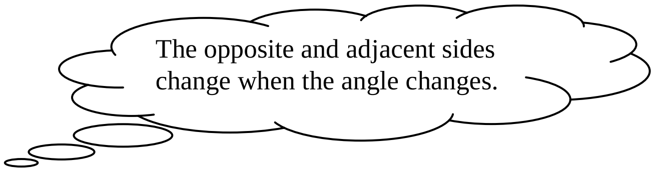


Example

Find sin α and sin β in the triangle below.



Answer

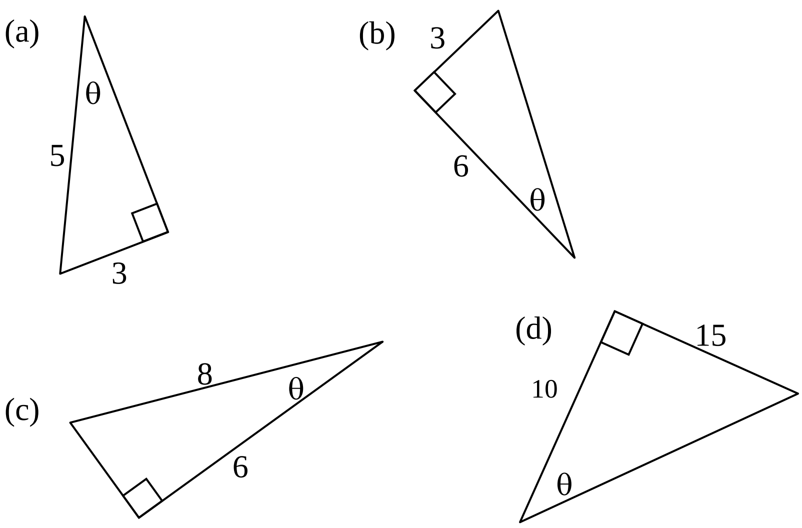


(i) $\sin\alpha = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{30}{50} = 0.6$

(ii) $\sin\beta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{40}{50} = 0.8$

Problems 1.2

Find sin θ, cos θ and tan θ for each of the triangles below.



2

Sine, Cosine and Tangent

2.1 Using Sine

The sine of an angle can be found by using the SIN key on a calculator.

Example

To find sin 30°, use SIN 30 = .

> Calculator Check: The answer should be 0.5. If you didn't get this, check if 'DEG' is displayed on your screen – this means the calculator is assuming the angles are measured in degrees. If it isn't displayed then you need to tell your calculator to work in degrees. Some calculators will have a DRG button to switch between the different ways to measure angles. Some will require you to press MODE either once or twice until DEG appears together with a digit, then press the digit.

In algebra we abbreviate write 2*x* instead of $2 \times x$. The same convention is used in trigonometry, for example we write $2\sin 30°$ instead of $2 \times \sin 30°$.

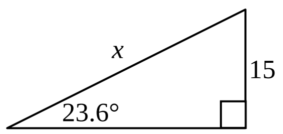
Example

To find 2sin 30° + 1, use 2 × SIN 30 + 1 = .

You can use trignometric ratios to find unknown sides of triangles.

Example

Find *x* in the triangle below.



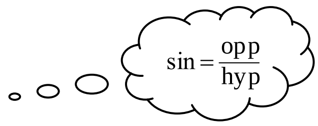
Answer

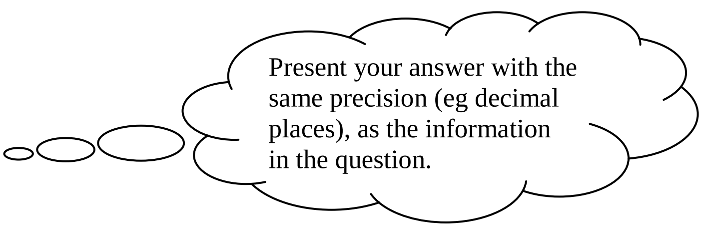
$$\sin 23.6° = \frac{15}{x}$$

$$x \sin 23.6° = 15$$

$$x = \frac{15}{\sin 23.6°}$$

$$= 37.5$$





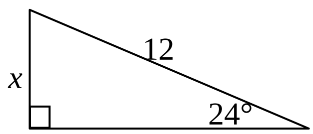
Problems 2.1A

1. Calculate the following

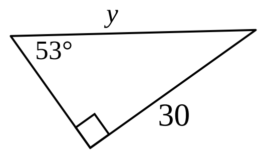
(a) $\sin 73.2°$ (b) $2 - 3\sin 42°$ (c) $\sqrt{\sin 81°}$ (d) $\dfrac{3}{2+\sin 15°}$

2. Find the unknown sides below:

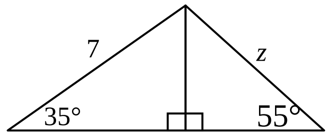
(a)



(b)



(c)



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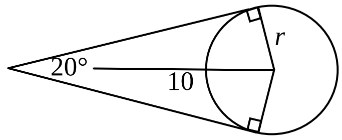


If the sin of an angle is known then the angle can be found by using the

- *inverse sine* key $\boxed{\text{SIN}^{-1}}$

on a calculator. On most calculators this is activated by using the primary keys

- $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$ or $\boxed{\text{SHIFT}}$ $\boxed{\text{SIN}}$ or $\boxed{\text{2ndF}}$ $\boxed{\text{SIN}}$,

depending on the brand of calculator.

Note

The name *inverse sine* and the notation *sin*$^{-1}$ are very misleading as they suggest that we using an inverse power like 2^{-1}. But this is not true. A more descriptive name would be *the reverse of sine* because:

➔ when we want to *find the sin of an angle* like 32° we use the calculator as follows:

$$32° \xrightarrow{\boxed{\text{SIN}}} 0.5299$$

➔ when we want to *find the angle which has sine equal to* 0.5299, then we *reverse the action* above or find the *inverse of the action of finding sine*, by using the *inverse sine* key $\boxed{\text{SIN}^{-1}}$.

$$32° \xleftarrow{\boxed{\text{SIN}^{-1}}} 0.5299$$

Example

Find the angle with sine is equal to 0.4.

Answer

$$\sin^{-1} 0.4 = 23.6°$$

Present your answer with the same precision (eg decimal places), as the information in the question.

Example

Solve the equation $\sin\theta = 0.8$.

θ = 'theta'

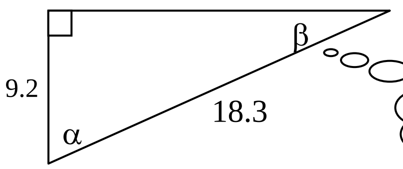
Answer

$$\begin{aligned} \sin\theta &= 0.8 \\ \theta &= \sin^{-1} 0.8 \\ &= 53.1° \end{aligned}$$

Did you expect to get twice the answer in the previous example? Remember that doubling the side in a triangle doesn't double the angle.

Example

Find the unknown angles below.



$\sin\beta = \frac{\text{opp}}{\text{hyp}}$

α = 'alpha'
β = 'beta'

Answer

(i) $$\begin{aligned} \sin\beta &= \frac{9.2}{18.3} \\ \beta &= \sin^{-1}\left(\frac{9.2}{18.3}\right) \\ &= 30.2° \end{aligned}$$

(*ii*) $$\begin{aligned} \alpha + 30.2° &= 90° \\ \alpha &= 90° - 30.2° \\ &= 59.8° \end{aligned}$$

Example

Solve the equation $2 - 3\sin\theta = 1.2$.

θ = 'theta'

Answer

$$\begin{aligned} 2 - 3\sin\theta &= 1.2 \\ 2 &= 1.2 + 3\sin\theta \\ 3\sin\theta &= 0.8 \\ \sin\theta &= \frac{0.8}{3} \\ \theta &= \sin^{-1}\left(\frac{0.8}{3}\right) \\ &= 15.5° \end{aligned}$$

To solve this equation, think of 'sin θ' as the unknown which has to be found first. After you find 'sin θ'you can then find θ.

Problems 2.1B

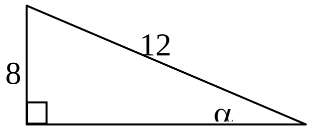
1. Solve the following equations

(a) $\sin\alpha = 0.75$ (b) $3\sin\beta = 1.23$ (c) $5.6 - 2.1\sin\gamma = 4.8$

α = ‘alpha’
β = ‘beta’
γ = ‘gamma’

2. Find the unknown angles below:

(a)



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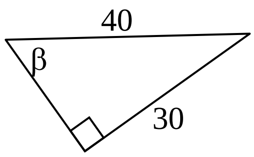
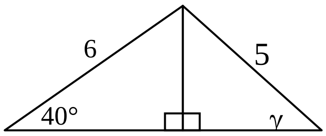


α = ‘alpha’
β = ‘beta
γ = ‘gamma’
δ = ‘delta

(c)



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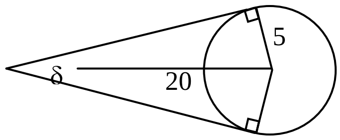


2.2 Using Cosine

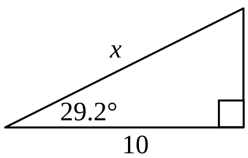
The cosine of an angle can be found by using the $\boxed{\text{COS}}$ key on a calculator.

Example

To find cos 30°, use $\boxed{\text{COS}}$ 30 $\boxed{=}$.

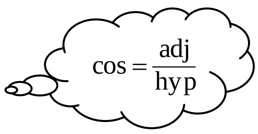
Example

Find *x* in the triangle below.



Answer

$$\cos 29.2° = \frac{10}{x}$$
$$x\cos 29.2° = 10$$
$$x = \frac{10}{\cos 29.2°}$$
$$= 11.5$$

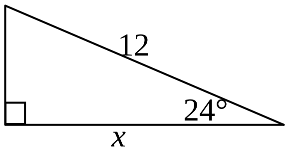


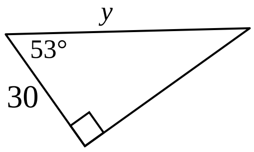
Problems 2.2A

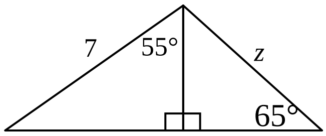
1. Calculate the following

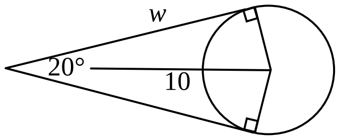
(a) cos 57.9° (b) 3.7 – 2.9cos 27° (c) $10.5\sqrt{\cos 14°}$ (d) $\dfrac{2}{3 - \cos 25°}$

2. Find the unknown sides below:

(a) 

(b) 

(c) 

(d) 

If the sin of an angle is known then the angle can be found by using the

- *inverse cosine* key $\boxed{\text{COS}^{-1}}$.

This is activated by using the primary keys

- $\boxed{\text{INV}}$ $\boxed{\text{COS}}$ or $\boxed{\text{SHIFT}}$ $\boxed{\text{COS}}$ or $\boxed{\text{2ndF}}$ $\boxed{\text{COS}}$.

Note

To *find the cos of an angle* like 32° we use:

$$32^\circ \xrightarrow{\boxed{\text{COS}}} 0.8480$$

To *find the angle which has cos equal to* 0.5299, then we *reverse the action* or find the *inverse of the action*, by using the *inverse sine* key $\boxed{\text{COS}^{-1}}$.

$$32^\circ \xleftarrow{\boxed{\text{COS}^{-1}}} 0.8480$$

Example

Solve the equation $\cos\theta = 0.6$.

θ = 'theta'

Answer

$$\begin{aligned}\cos\theta &= 0.6\\ \theta &= \cos^{-1} 0.6\\ &= 53.1^\circ\end{aligned}$$

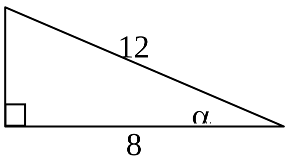
Problems 2.2B

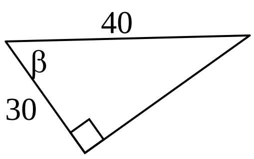
1. Solve the following equations

α = 'alpha'
β = 'beta
γ = 'gamma'

(a) $\cos\alpha = 0.43$ (b) $3.1\cos\beta = 2.3$ (c) $1.2 + 3.9\cos\gamma = 2.7$

2. Find the unknown angles below:

(a) 

(b) 

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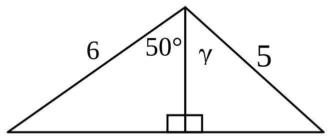

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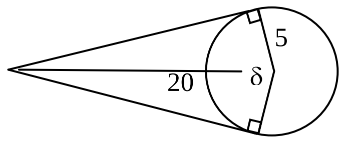


δ = 'delta'

We usually say and write 'tan' rather than 'tangent'.

2.3 Using Tan

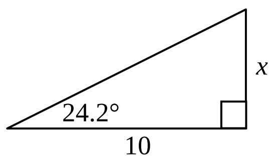
The tan of an angle can be found by using the $\boxed{\text{TAN}}$ key on a calculator.

Example

To find tan 30°, use $\boxed{\text{TAN}}$ 30 $\boxed{=}$.

Example

Find x in the triangle below.



Answer

$$\tan 24.2° = \frac{x}{10}$$
$$x = 10\tan 24.2°$$
$$= 4.5$$

$$\tan = \frac{\text{opp}}{\text{adj}}$$

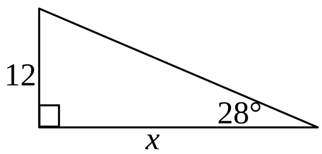
Problems 2.3A

1. Calculate the following

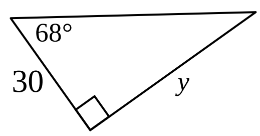
(a) tan 89.8° (b) 5.1 – 2.9tan 27° (c) $10.5(\tan 36°)^3$ (d) $\dfrac{1+\tan 10°}{1-\tan 10°}$

2. Find the unknown sides below:

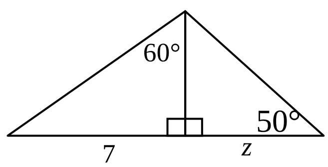
(a)



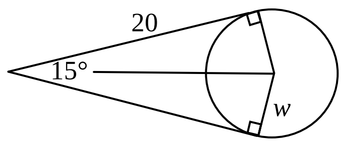
(b)



(c)



(d)



If the tan of an angle is known then the angle can be found by using the

- *inverse tan* key $\boxed{\text{TAN}^{-1}}$.

This is activated by using the primary keys

- $\boxed{\text{INV}}$ $\boxed{\text{TAN}}$ or $\boxed{\text{SHIFT}}$ $\boxed{\text{TAN}}$ or $\boxed{\text{2ndF}}$ $\boxed{\text{TAN}}$.

Example

Solve the equation $\tan\theta = 5$.

θ = 'theta'

Answer

$$\begin{aligned} \tan\theta &= 5 \\ \theta &= \tan^{-1} 5 \\ &= 79° \end{aligned}$$

Problems 2.3B

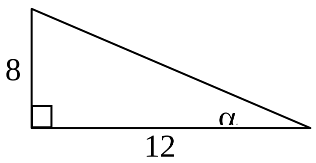
1. Solve the following equations

α = 'alpha'
β = 'beta
γ = 'gamma'
δ = 'delta'

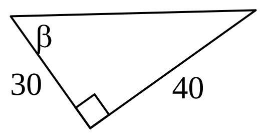
(a) $\tan\alpha = 0.38$ (b) $20.7 - 2.1\tan\beta = 2$ (c) $\dfrac{1}{2 - \tan\gamma} = 3$

2. Find the unknown angles below:

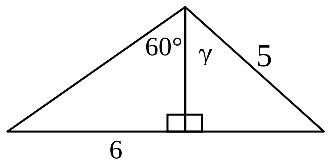
(a)



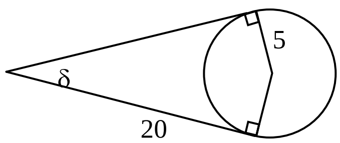
(b)



(c)



(d)



2.4 More Problems

1. Australian building regulations specify that ramps for wheelchair access should rise no more than 1m for each 20m along the surface of the ramp.

(a) What is the maximum angle of a ramp?

(b) If you needed to build a ramp with height 0.5m, what would its horizontal length be?

2. A ship's anchor chain is 120m long and the current forces the ship to drift until the angle from the chain and surface of the water is 40°.

(a) How deep is the water?

(b) If 20m of chain was taken in, what is the horizontal distance of the ship from the anchor?

3. When the moon is half full, the triangle made by the moon, sun and earth is a right-angled triangle with the moon at the right angle. The Greek astronomer Aristarchus observed that when the moon is just half full, then the angle between the lines of sight from the earth to the moon and to the sun is 3° less than a right angle.

(a) How far is the sun from the earth if the distance from the earth to the moon is taken to be one unit?

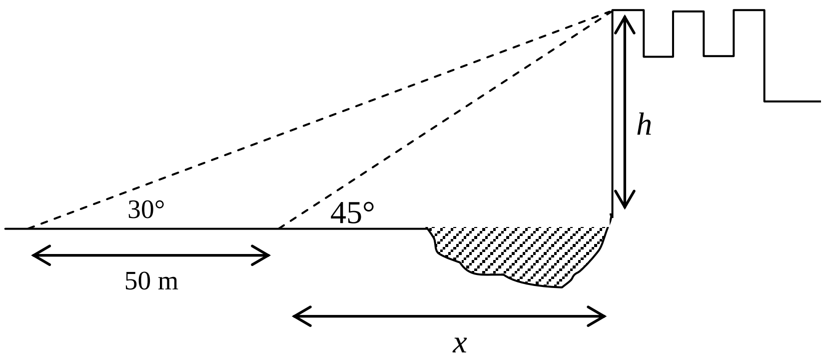
(b) If measurements show that the distance from the Earth to the sun is about 400 times the distance of the of the Earth to the moon, what would be the actual angle between the lines of sight from the earth to the moon and to the sun.

4. The largest moon of the planet Neptune is Triton. When *Voyager 2* was 38, 000 km from Triton, the angle *subtended* by Triton to *Voyager 2* was 4°.

(a) Estimate the diameter of Triton.

(b) If the angle subtended by Triton was 5°, estimate the distance of *Voyager 2* from Triton.

5. Estimate the height (h) of a castle surrounded by a moat, using the information below.



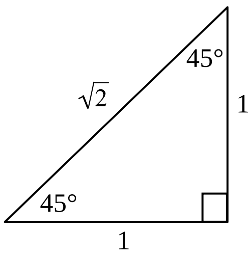
3
Properties

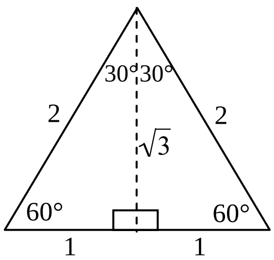
3.1 Exact Values

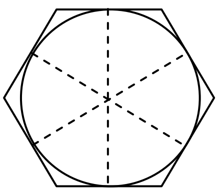
Calculators give approximate values for trigonometric ratios rather than exact values - though they are accurate enough for all practical applications. However, the exact values are known for some special angles such as 30°, 45° and 60°. These exact values are part of the history of trigonometry and are only occasionally used in mathematical modelling.

θ	**sin θ**	**cos θ**	**tan θ**
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

You can check these values using the equilateral and isosceles triangles below.

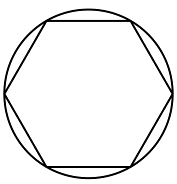






Problems 3.1

1. A hexagon fits exactly over a circle of radius 1 unit. Show that the perimeter of the hexagon is $\frac{12}{\sqrt{3}}$ units.

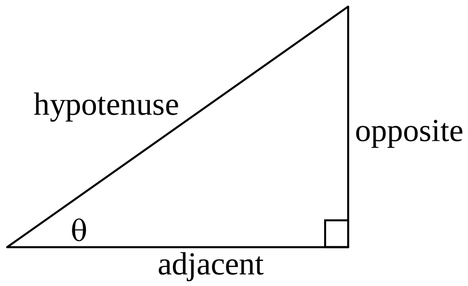


2. Show that a hexagon fitting exactly inside a circle of radius 1 unit has perimeter 6 units.

3.2 Relationships

There are six trigonometric ratios in a right- angled triangle. The three main ratios are sine, cosine and tangent:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$



The other trigonometric ratios are secant, cosecant and cotangent. These are given by the rules:

$$\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite}}, \quad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}, \quad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}.$$

As you can see

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

These new trig ratios don't extend our set of problem solving tools very much, and we will not continue with them here.

The three main trigonometric ratios, sine, cosine and tangent, are each very useful. However, they are all connected:

$$\frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{\text{opp}}{\text{hyp}}\right)}{\left(\frac{\text{adj}}{\text{hyp}}\right)} = \frac{\left(\frac{\text{opp}}{\text{hyp}}\right)\times\text{hyp}}{\left(\frac{\text{adj}}{\text{hyp}}\right)\times\text{hyp}} = \frac{\text{opp}}{\text{adj}} = \tan\theta.$$

This shows that if we know the sin and cos of an angle, then we can find tan by division.

Example

From section 3.1, sin 30° = $\frac{1}{2}$ and cos 30° = $\frac{\sqrt{3}}{2}$, so

$$\frac{\sin 30°}{\cos 30°} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{\left(\frac{1}{2}\right)\times 2}{\left(\frac{\sqrt{3}}{2}\right)\times 2} = \frac{1}{\sqrt{3}},$$

which agrees with the value of tan 30°.

The sine and cosine ratios are also connected:

$$
\begin{aligned}
(\sin\theta)^2 + (\cos\theta)^2 &= \left(\frac{\text{opp}}{\text{hyp}}\right)^2 + \left(\frac{\text{adj}}{\text{hyp}}\right)^2 \\
&= \frac{\text{opp}^2}{\text{hyp}^2} + \frac{\text{adj}^2}{\text{hyp}^2} \\
&= \frac{\text{opp}^2 + \text{adj}^2}{\text{hyp}^2} \\
&= \frac{\text{hyp}^2}{\text{hyp}^2} \\
&= 1
\end{aligned}
$$

Pythagoras' theorem tells us that: $\text{opp}^2 + \text{adj}^2 = \text{hyp}^2$

This relationship is very important and usually written as:

$$(\sin\theta)^2 + (\cos\theta)^2 = 1.$$

Note that some people will write $\sin^2\theta$ as an abbreviation for $(\sin\theta)^2$, even though it is not strictly correct.

Example

It was shown in section 3.1 that sin 30° = $\frac{1}{2}$ and cos 30° = $\frac{\sqrt{3}}{2}$, so

$$
\begin{aligned}
(\sin 30°)^2 + (\cos^2 30°)^2 &= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \\
&= \frac{1^2}{2^2} + \frac{\left(\sqrt{3}\right)^2}{2^2} \\
&= \frac{1}{4} + \frac{3}{4} \\
&= 1
\end{aligned}
$$

This is an example of the relationship $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

Problems 3.2

Use the exact values in section 3.1 to check that

(a) $(\sin 45°)^2 + (\cos 45°)^2 = 1$

(b) $(\sin 60°)^2 + (\cos 60°)^2 = 1$.

A
Answers

Answers are rounded to one decimal place, except when there are exact answers or very large numbers, or when this makes checking difficult.

Section 1.1

(a) $\alpha = 90°$ and $\beta = 30°$

(b) CB = 10, CP = PA = 5, BP = $\sqrt{75}$ = 8.7

Section 1.2

(a) 0.6, 0.8, 0.75 (b) 0.45, 0.89, 0.5

(c) 0.66, 0.75, 0.88 (d) 0.83, 0.55, 1.5

Section 2.1A

1(a) 0.96 (b) –0.007 (c) 0.99 (d) 1.3

2(a) 4.9 (b) 37.6 (c) 4.9 (d) 1.7

Section 2.1B

1(a) 48.6° (b) 24.2° (c) 22.4°

2(a) 41.8° (b) 48.6° (c)50.5° (d)28.96°

Section 2.2A

1(a) 0.53 (b) 1.1 (c) 10.3 (d) 0.96

2(a) 10.96 (b) 49.8 (c) 4.4 (d) 9.85

Section 2.2B

1(a) 64.5° (b) 42.1° (c) 67.4°

2(a) 48.2° (b) 41.4° (c) 39.5° (d) 151.0°

Section 2.3A

1(a) 286.5 (b) 3.6 (c) 4.0 (d) 1.4

2(a) 22.6 (b) 74.3 (c) 3.4 (d) 2.6

Section 2.3B

1(a) 20.8° (b) 83.6° (c) 59.0°

2(a) 33.7° (b) 53.1° (c) 46.1° (d) 28.1°

Section 2.4

1(a) 2.9° (b) 9.987 m,

2(a) 77.1 m (b) 63.6 m

3(a) 19.1 (b) 89.86°

4(a) 2,654 km (b) 30, 393 km

5. 68.7 m