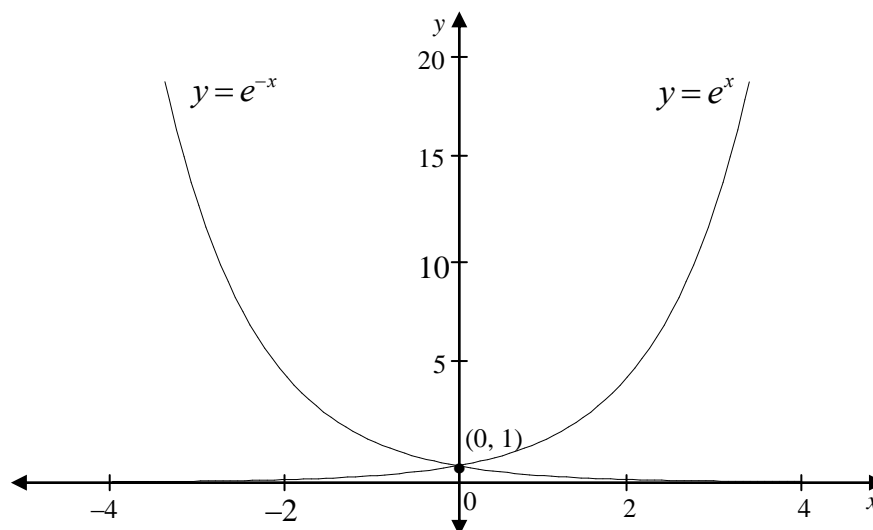


Topic 7

Exponential Functions



MATHS LEARNING CENTRE

Level 3, Hub Central, North Terrace Campus
The University of Adelaide, SA, 5005

TEL 8313 5862 | FAX 8313 7034 | EMAIL mathslearning@adelaide.edu.au
www.adelaide.edu.au/mathlearning/

This Topic...

This topic introduces exponential functions, their graphs and applications. Exponential functions are used to model growth and decay in many areas of the physical and natural sciences and economics. Further properties of exponential functions will be covered in Topic 8.

— Prerequisites

You will need a scientific calculator.

— Contents

Chapter 1 Powers.

Chapter 2 Exponential Functions.

Chapter 3 Growth and Decay.

Appendices

A. Answers

Printed: 18/02/2013

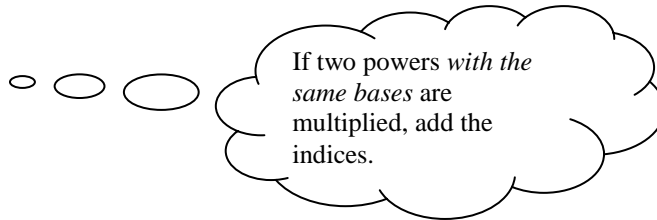
1

Powers

In the expression 2^5 , 2 is called the *base* and 5 is called the *index* or *power* or *exponent*.

We use the *index rules* to simplify expressions involving powers.

Rule 1: $a^n a^m = a^{n+m}$

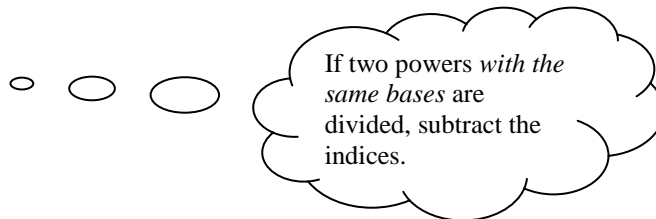


Example

(i) $2^2 \times 2^3 = (2 \times 2) \times (2 \times 2 \times 2) = 2^5$

(ii) $2^2 \times 7^2 \times 2^3 = 2^2 \times 2^3 \times 7^2 = 2^5 \times 7^2$

Rule 2: $\frac{a^n}{a^m} = a^{n-m}$



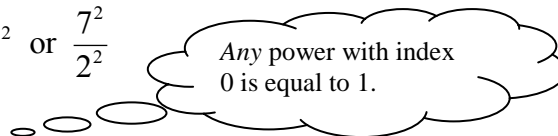
Example

(i) $\frac{2^5}{2^3} = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2^{5-3} = 2^2$

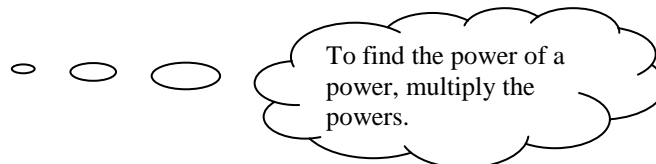
(ii) $\frac{2^3}{2^5} = \frac{2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = 2^{3-5} = 2^{-2}$ or $\frac{1}{2^2}$

(iii) $\frac{2^3 \times 7^2}{2^5} = 2^{3-5} \times 7^2 = 2^{-2} \times 7^2$ or $\frac{7^2}{2^2}$

(iv) $\frac{2^3}{2^3} = 2^{3-3} = 2^0$ or 1



Rule 3: $(a^n)^m = a^{nm}$



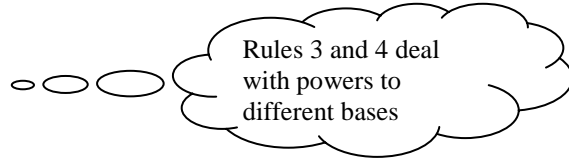
Example

(i) $(2^2)^3 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) = 2^6$

(ii) $(2^{-2})^3 = 2^{-6}$

(iii) $(2^2)^{-3} = 2^{-6}$

Rule 3: $(ab)^n = a^n b^n$

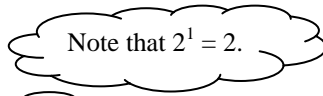


Example

(i) $(2 \times 3)^3 = (2 \times 3) \times (2 \times 3) \times (2 \times 3) = 2^3 \times 3^3$

(ii) $(2 \times 10^5)^3 = 2^3 \times (10^5)^3 = 8 \times 10^{15}$

Rule 4: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$



Example

$$2^3 \times \left(\frac{3}{2}\right)^2 = 2^3 \times \frac{3^2}{2^2} = 2 \times 3^2$$

Problems 1

1. Use the index rules to simplify the following:

(a) $b^3 b^4$ (b) $\frac{b^4}{b^3}$ (c) $\frac{b^4 \times b^3}{b^2 \times b^5}$ (d) $\frac{(b^3)^4}{b^4} + 10$

2. Write the following as simply as possible without brackets:

(a) $(2n^3)^4$ (b) $2w^2 \times (5w)^2$ (c) $\frac{(x^2 y)^3}{xy}$ (d) $\left(\frac{x}{y}\right)^2 \times xy^{-1}$

3. Simplify:

(a) $2^{x+1} \times 2^{x-1}$ (b) $2^{x+1} \times 4^{x-1}$ (c) $\frac{3^{2x}}{3^{x-1}}$ (d)

$$\left(\frac{3^x}{2^y}\right)^2 \times 3^{1-x} 2^{2y} + 1$$

2

Exponential Functions

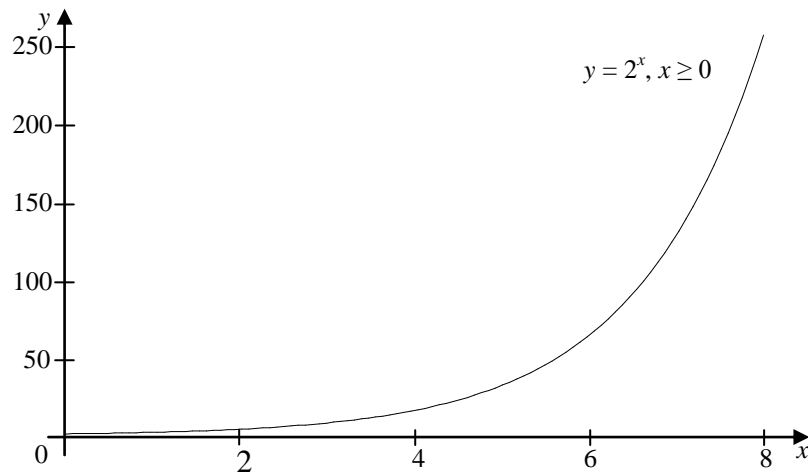
2.1 Exponential Functions and their Graphs

Exponential functions are functions like $f(x) = a^x$, where the base a is a fixed number and the index is given different values. When $a > 1$ the function increases rapidly as x increases, and when $a < 1$ the function decreases rapidly as x increases.

Example

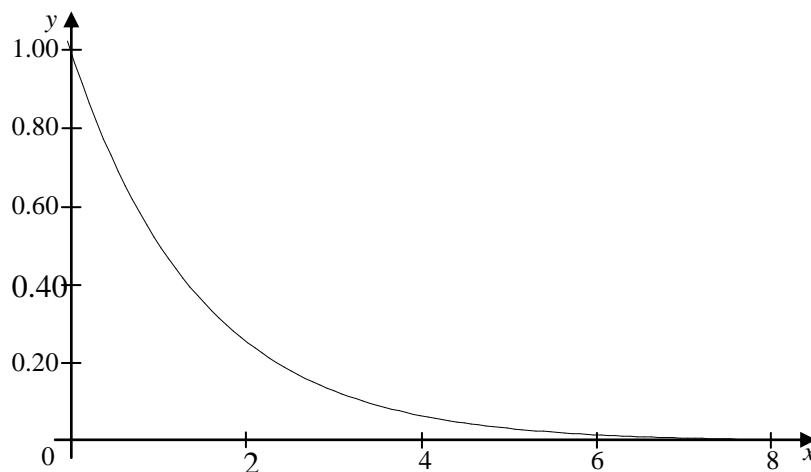
(a)

x	0	1	2	3	4	5	6	7	8
2^x	1	2	4	8	16	32	64	128	256



(b)

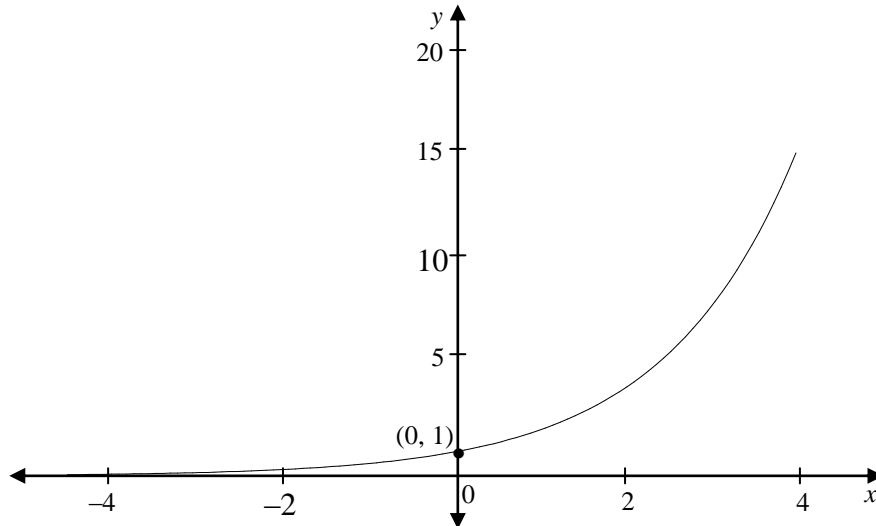
x	0	1	2	3	4	5	6	7
$(1/2)^x$	1	0.5	0.25	0.125	0.0625	0.03125	0.015625	0.0078125



Example

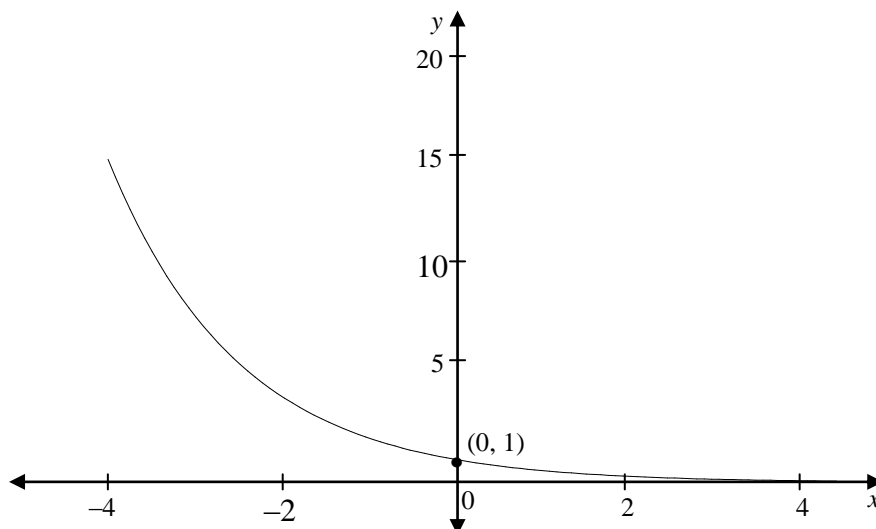
The graph of $y = 2^x$ combines the shapes of the two graphs above.

x	0	1	2	3	4	-1	-2	-3	-4
2^x	1	2	4	8	16	0.5	0.25	0.125	0.0625

*Example*

The graph of $y = 2^{-x}$ or $\left(\frac{1}{2}\right)^x$ is a reflection of $y = 2^x$ across the y-axis.

x	0	1	2	3	4	-1	-2	-3	-4
2^{-x}	1	0.5	0.25	0.125	0.0625	2	4	8	16



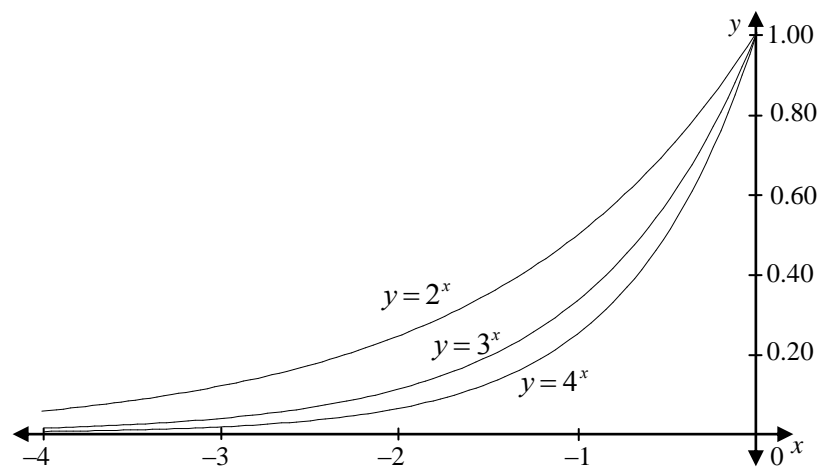
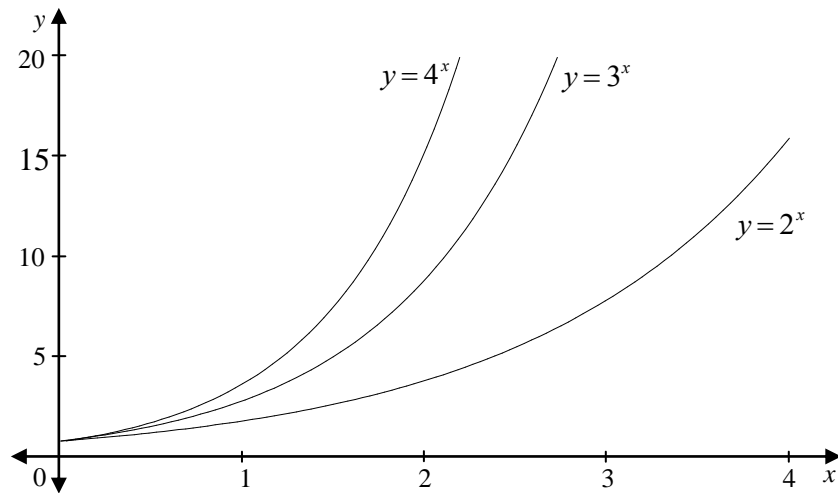
5 Exponential Functions

The graphs of exponential functions $y = a^x$ have the following properties:

- all are above the x -axis, as $a^x > 0$ for every value of x ,
- all have x -intercept $(0, 1)$, as $a^0 = 1$ for each value of a ,
- graphs with $a > 1$ increase rapidly as x increases,
- graphs with $a < 1$ decrease rapidly as x increases,
- all have the x -axis as a horizontal asymptote.

The example below compares exponential functions with different bases. If $a > b$, then the graph of $y = a^x$ is above (below) the graph of $y = b^x$ when x is positive (negative). Can you see why?

Example



2.2 Calculating Powers

General powers are calculated using the power key $\boxed{x^y}$ (or $\boxed{y^x}$ or $\boxed{x^\square}$ or $\boxed{\wedge}$) on a scientific calculator. This last one is used because when typing in a computer program or an email, you would write 3^2 as “3^2”.

Example

To calculate 3^2 , use: 3 $\boxed{x^y}$ 2 $\boxed{=}$

To calculate 3^{-2} , use: 3 $\boxed{x^y}$ $\boxed{(-)}$ 2 $\boxed{=}$

Problems 2.2

Calculate the following, giving your answers to 4 significant figures.

(a) 1.5^{10} (b) 2^{-5} (c) $\frac{2^{4.5} \times 1.5^3}{1.2^2 \times 2.7^3}$ (d) $(0.3^3)^2$ (e) 0.3^{3^2}

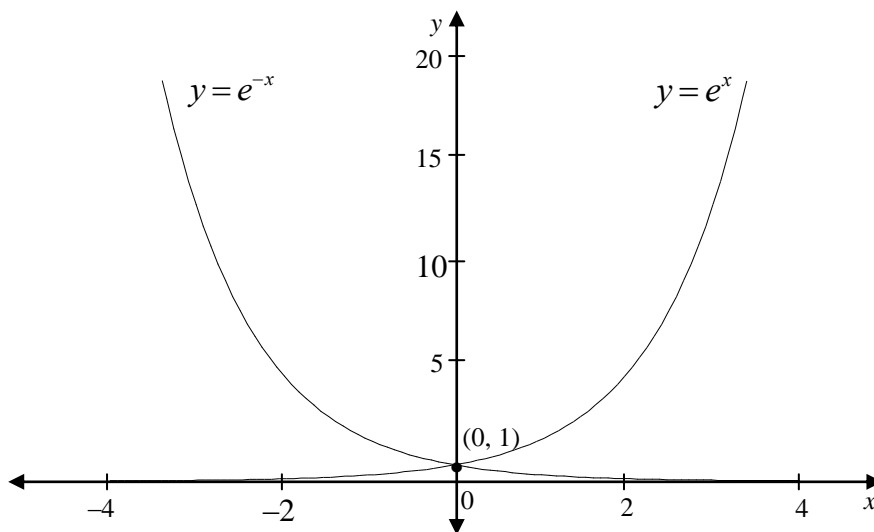
3

Growth and Decay

3.1 The number e

The most commonly used powers are powers to the bases 10, 2, and 2.718281828 Powers to the base 10 are used in scientific notation and also in chemistry (when describing the pH levels of solutions). Powers to the base 2 are used in computing and (occasionally) growth and decay models. Powers to the base 2.718281828 . . . are used commonly in many areas of mathematics and its applications. As it is so commonly encountered, it is always represented by the letter ‘ e ’, and the function $y = e^x$ is called *the exponential function*.

The graph of $y = e^x$ has a similar shape to the graphs in the previous section, and lies between the graphs of $y = 2^x$ and $y = 3^x$. Like those graphs, it cuts the y -axis at $(0, 1)$ and has the x -axis as an asymptote. The graph of $y = e^{-x}$ is the reflection of $y = e^x$ across the y -axis.



You can calculate powers to base e by using the exponential key e^x (or Exp) on a calculator.

Can you see it? It may need to be done by using SHIFT or 2ndF followed by ln .

Example

To calculate e^2 , use: e^x 2 =

To calculate e^{-2} , use: e^x (-) 2 =

To calculate e^{2x} for $x = 3.71$, use: e^x (2 × 3.71) =

Problems 3.1

1. Calculate the following, giving your answers to 4 significant figures.

(a) e^0 (b) e^1 (c) $e^{5.89}$ (d) $e^{-1.2}$ (e) $\frac{10e^{2.9}}{1+3e^{2.9}}$

2. The relative risk of a car accident after drinking alcohol is the probability of having an accident after drinking alcohol divided by the probability of having an accident without drinking alcohol.

The relative risk after having n standard drinks in one hour is given by the function $R(n) = e^{0.4n}$.

(a) Sketch the graph of the function for values of n between 0 and 6. (You can have a part of a drink.)

(b) What is the value of $R(0)$? What does it represent?

(c) The legal limit for blood alcohol corresponds to having three drinks in an hour. Interpret the value $R(3)$.

(d) How many time more likely are you to have an accident if you have 6 drinks in an hour compared to having two drinks in an hour?

3.2 Growth and Decay Models

A population that is growing at a constant rate will have

$$P(t) = P(0) e^{rt}$$

members after time t , where

- $P(0)$ is the initial population and
- r is the constant growth rate per unit time.

Example

The population of China was 850, 000, 000 in 1990 and was growing at the rate of 4% per year.

What would you predict the population to be in 2002?

9 Exponential Functions

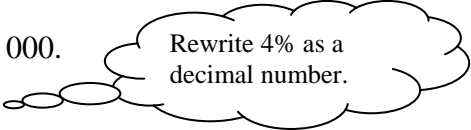
Answer

The initial population (in 1990) is $P(0) = 850,000,000$.

The growth rate is 0.04 per year.

We need to find the population in 2002, ie. $P(12)$.

$$\begin{aligned}P(12) &= P(0)e^{r \times 12} \\ &= 850,000,000 \times e^{0.04 \times 12} \\ &= 1,370,000,000 \text{ (3 sf)}\end{aligned}$$



Rewrite 4% as a decimal number.

A quantity which is decaying at a constant rate will have the amount

$$Q(t) = Q(0) e^{-rt}$$

left after time t , where

- $Q(0)$ is the initial amount and
- r is the constant *decay* rate per unit time.

The ‘quantity’ in the model above could refer to a decaying population or a decaying chemical. It is traditional to choose our own letters for these functions in growth and decay models. For example, $P(t)$ could be used with population growth, $B(t)$ with bacterial growth, and $M(t)$ with decaying mass.

Example

One kilogram of a radioactive isotope of iodine decays at a rate of 8.7% per day. How much would be left after one week, after 30 days.

Answer

The initial mass is $M(0) = 1$.

The decay rate is 0.087 per day.

We need to find $M(7)$ and $M(30)$

$$\begin{aligned}M(7) &= M(0)e^{-r \times 7} \\ &= 1 \times e^{-0.087 \times 7} \\ &= 0.544 \text{ (3 sf)} \\ M(30) &= M(0)e^{-r \times 30} \\ &= 1 \times e^{-0.087 \times 30} \\ &= 0.0735 \text{ (3 sf)}\end{aligned}$$

Problems 3.2

1. The population of the earth at the beginning of 1990 was 5 billion and is growing at the rate of 2% per year. What will the population be in 2040?
2. If three grams of a radioactive material decays at a constant rate of 50% per year, how much will be left after 5 years.
3. A culture of bacteria initially weighs 1 gm and is growing at a constant rate of 70% per hour. What will be its weight after 5 hours?

A

Appendix: Answers

Section 1.1

- 1(a) b^7 (b) b (c) 1 (d) $b^8 + 10$
2(a) $16n^{12}$ (b) $50w^4$ (c) x^5y^2 (d) x^3y^{-3}
3(a) 2^{2x} (b) 2^{3x-1} (c) 3^{x+1} (d) $3^{x+1} + 1$

Section 2.2

- (a) 57.67 (b) 0.03125 (c) 2.694 (d) 7.290×10^{-4} (e) 1.968×10^{-5}

Section 3.1

- 1(a) 1 (b) 2.718 (c) 361.4 (d) 0.3012 (e) 3.273

2(b) $R(0) = 1 \Rightarrow$ Relative risk is 1

(c) $R(3) = 3.32 \Rightarrow$ Having 3 standard drinks increases the chance of an accident 3.32 times.

(d) $R(6)/R(2) = 4.95 \Rightarrow$ five times more likely to have an accident.

Section 3.2

- 13.6 billion
- 0.25 gm
- 33.1 gm